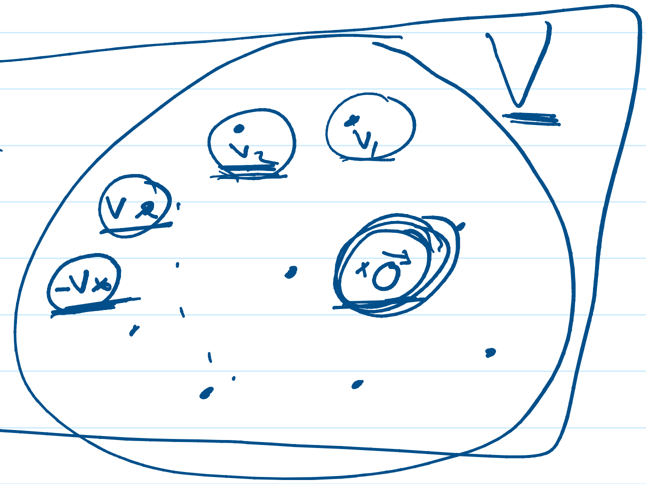
Wednesday, April 21, 2021 9:54 AM

× If V is a vector space.

$v_1+v_2 \in V$: $+$

$\alpha v \in V$: $\cdot$

$v+(-v)=\vec{0}$

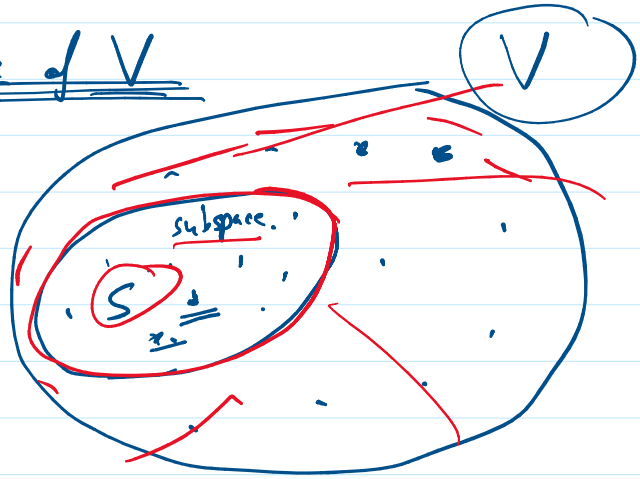


S subspace of V

1) $S \neq \emptyset$
2) if $s_1, s_2 \in S$, we have $s_1+s_2 \in S$
3) if $s_1 \in S$, $\alpha \in \mathbb{R}$ then $\alpha s_1 \in S$.

$\Downarrow$

S is a vector space.



Ex: Let V be a vector space, ① $S=\{\vec{0}\}$. Is S a subspace.

1) $S \neq \emptyset$ since $0 \in S$
2) Let $\boxed{s_1, s_2 \in S} \Rightarrow \boxed{s_1=0,\ s_2=0.}$
   Now $s_1+s_2 = 0+0 = 0 \in S$ ✓
3) Let $s_1 \in S$, $\alpha \in \mathbb{R}$
   $\Rightarrow s_1 = 0.$
   Now $\alpha s_1 = \alpha 0 = 0 \in S$ ✓

$\therefore\ S = \{\vec{0}\}$ is a subspace of V.

Ex. Is $S = V$ is a subspace of V.

1) $V \neq \phi \qquad 0 \in V$

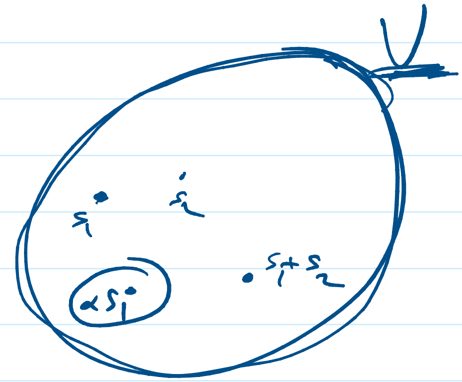
2) Let $s_1, s_2 \in V$

$s_1 + s_2 \in V$

3) Let $s_1 \in V$, $\alpha \in \mathbb{R}$

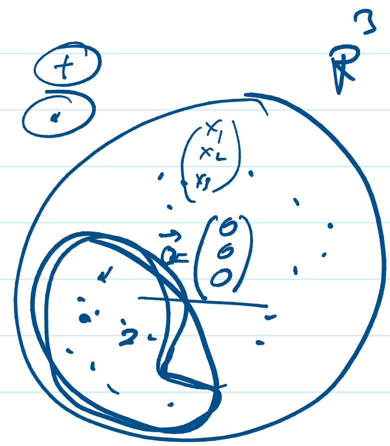
$\alpha s_1 \in V$

$\therefore\ V$ is subspace of V.



Ex. $S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;;\; x_1 = 2x_2 \right\}$

Is S a subspace of $\mathbb{R}^3$.



1) $S \neq \phi$, $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in S$ $\Big\{ x_1 = 0 = 2\, x_2 = 0$

2) Let $s_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $s_2 = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in S$.

$\Longrightarrow x_1 = 2x_2$, $y_1 = 2y_2$.

$\boxed{\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}}$, $z_1 = 2z_2$

Now $s_1 + s_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{pmatrix} \overset{?}{\in} S$

Now $x_1 + y_1 = 2x_2 + 2y_2 = 2(x_2 + y_2)$

so $\boxed{s_1 + s_2 \in S.}$

3) Let $s_1 \in S$, $\alpha \in \mathbb{R}$

$\Rightarrow s_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $\boxed{x_1 = 2x_2.}$

Now $\alpha s_1 = \alpha \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{pmatrix} \overset{?}{\in} S$

Now $\alpha x_1 = \alpha(2x_2) = 2(\alpha x_2)$ $\Rightarrow$ $\alpha S_1 = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{pmatrix} \in S$

(1), (2) & (3) $\Rightarrow$ S is a subspace of $\mathbb{R}^3$.

Ex: $S = \{ A \in \mathbb{R}^{2\times 2} : a_{21} = -a_{12} \}$.

Is S a subspace of $\mathbb{R}^{2\times 2}$.

① ② 3X

$A = \begin{pmatrix} & - \\ - & \end{pmatrix}$

1) $S \neq \phi$, $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S$

$0 = a_{21} = -0 = -a_{12}$

2) Let $A, B \in S$ $\Rightarrow$ $\boxed{a_{21} = -a_{12}}$ and $\boxed{b_{21} = -b_{12}}$.

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$.

Now $A + B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$

$c_{21} = a_{21} + b_{21} = -a_{12} - b_{12} = -(a_{12} + b_{12}) = -c_{12}$.

so $A + B \in S$.

3) Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in S$, $\alpha \in \mathbb{R}$.

$\Rightarrow a_{21} = -a_{12}$

Now $\alpha A = \alpha \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{pmatrix} \in S$

$\alpha a_{21} = \alpha(-a_{12}) = -(\alpha a_{12})$ $\Rightarrow$ $\alpha A \in S$.

$$\alpha a_{21} = \alpha(-a_{12}) = -(\alpha a_{12}) \Rightarrow \alpha A \in S.$$

(1),(2) & (3) $\Rightarrow$ S is a subspace of $\mathbb{R}^{2\times 2}$.

Ex: $S = \{p(x) \in P_4 : \deg(p(x)) \text{ is even}\}$.

1) $S \neq \phi$, $0 \in S$

2) $p(x), q(x) \in S \Rightarrow \deg(p(x))$ is even, $\deg(q(x))$ is even.

$p(x)+q(x)$ $\boxed{\overset{?}{\in} S}$

$\deg(p+q)$:

take: $\boxed{p(x) = x + x^2 \in S}$

$\boxed{q(x) = x - x^2 \in S}$

$\boxed{p+q = 2x \notin S}$

So $\boxed{S \text{ is not a subspace.}}$

3) Let $p(x) \in S$, $\alpha \in \mathbb{R}$.

$\deg(p(x))$ is even.

$\deg(\alpha p(x))$ is even. $\Rightarrow$ $\alpha p(x) \in S$,

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$S = \{p(x) \in P_4 : p(1) = 0\}$ Is S a subspace of P_4.

1) $S \neq \phi$ since $\boxed{0 \in S}$ $\{ 0(1) = 0$

2) Let $p(x), q(x) \in S \Rightarrow p(1) = 0, q(1) = 0.$

Now $\boxed{p+q \overset{?}{\in} S}$ $(p+q)(1) = p(1)+q(1) = 0+0 = 0$

$\therefore p+q \in S.$

3) Let $\boxed{p(x) \in S}$, $\alpha \in \mathbb{R}$

$\therefore\ p+q \in S.$

3) let $\boxed{p(x) \in S}$, $\alpha \in \mathbb{R}$

$\Rightarrow p(1) = 0.$

Now $\boxed{\alpha p \overset{?}{\in} S}$: $(\alpha p)(1) = \alpha(p(1)) = \alpha(0) = 0$

$\therefore\ \alpha p \in S.$

(1, 2, 3) $\Rightarrow$ S is a subspace.

$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$

$p(1) = \boxed{a_0 + a_1 + a_2 + a_3 = 0}$

$p(x) = 2 + x + x^3 \overset{?}{\in} S$

$p(1) = 2+1+1 = 4 \neq 0 \Rightarrow p(x) = 2 + x + x^3 \notin S,$

$q(x) = 1 - x + 2x^2 - 2x^3 \overset{?}{\in} S$

$q(1) = 1 - 1 + 2 - 2 = 0 \Rightarrow q(x) \in S.$

Ex. $S = \{ f(x) \in C^{2}[-1,1] : f''(x) + f(x) = 0 \}$. Is S a subspace of $C[-1,1]$.

$f(x) = \sin x \overset{?}{\in} S$: $f'(x) = \cos x$

$f''(x) = -\sin x.$

$f''(x) + f(x) = -\sin x + \sin x = 0.$

$\therefore\ \sin x \in S.$ $\cos x \in S.$

$e^x \notin S$ $f(x) = x + x^2 + x^3 \notin S$ (check)

$C^n[a,b]$: all functions $f(x)$ such that $f, f', f'', \dots, f^{(n)}$ are cont. on $[a,b]$.

1) $S \neq \emptyset$ since $0 \in S$ ✓

2) let $f(x), g(x) \in S \Rightarrow f''(x) + f(x) = 0$ and $g''(x) + g(x) = 0.$

Now $\boxed{f+g \overset{?}{\in} S}$ consider $(f+g)'' + (f+g) = f''(x) + g''(x) + f(x) + g(x)$

$= 0 + 0 = 0$

Now $\boxed{f+g \overset{?}{\in} S}$ consider $(f+g)''+(f+g)=f''(x)+g''(x)+f(x)+g(x)$

$=0+0=0.$

$\therefore\ f+g\in S.$

3) Let $f(x)\in S$, $\alpha\in\mathbb{R}$ $\Rightarrow$ $f''(x)+f(x)=0.$

Now $\boxed{\alpha f \overset{?}{\in} S}$ consider $(\alpha f)''+(\alpha f)=\alpha f''(x)+\alpha f(x)$

$$=\alpha\left(f''(x)+f(x)\right)$$

$$=\alpha(0)=0.$$

$\therefore\ \alpha f\in S.$

So S is a subspace of $C[-1,1]$.

Ex. Null space of a matrix.

Let $A=(a_{ij})_{m\times n}$ be a matrix.

and let $\boxed{N(A)}=\{x\in\mathbb{R}^n : x \text{ is a solution to } \boxed{Ax=0}\}$. (Nullspace of A)

Is $N(A)$ a subspace of $\mathbb{R}^n$.

1) $N(A)\neq\phi$ ✓ since $0=\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}$ is a solution to $Ax=0$.

2) Let $\boxed{x, y\in N(A)}$ $\Rightarrow$ $\boxed{Ax=0}$, $\boxed{Ay=0.}$

Now $\boxed{x+y \overset{?}{\in} N(A)}$ $A(x+y)=Ax+Ay=0+0=0.$

$\therefore\ x+y\in N(A).$

3) Let $x\in N(A)$, $\alpha\in\mathbb{R}$. $\Rightarrow$ $Ax=0.$

Now is $\alpha x \overset{?}{\in} N(A)$: consider $A(\alpha x)=\alpha(Ax)=\alpha(0)=0.$

$\therefore\ \alpha x\in N(A).$

(1),(2),(3) $\Longrightarrow$ $N(A)$ is a subspace of $\mathbb{R}^n$.

$$\boxed{\text{Nullspace of } A = N(A) : \text{the set of all solutions of } \underline{Ax=0}.}$$

Ex: Find the nullspace of $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & -1 & 1 & 2 \\ 1 & -1 & 0 & 1 \end{pmatrix}_{3\times 4}$

{Find all solutions of $Ax=0$}

$\boxed{\text{solve } Ax=0.}$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 2 & -1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 1 & 0 \end{array}\right) \longrightarrow \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -5 & 3 & 0 \\ 0 & -3 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{-\frac{3}{5}R_2+R_3} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & -5 & 3 & 0 & 0 \\ 0 & 0 & -\frac{4}{5} & 0 & 0 \end{array}\right)$$

$\to -5x_2 + 3x_3 = 0 \Longrightarrow \underline{x_2 = 0}.$

$\to -\frac{4}{5}x_3 = 0 \Longrightarrow \boxed{x_3 = 0.}$

$x_4 = \alpha$ free

$x_1 + 2x_2 - x_3 + x_4 = 0$

$x_1 + \alpha = 0 \Longrightarrow \boxed{x_1 = -\alpha.}$

$$\therefore \underline{N(A)} = \left\{ \begin{pmatrix} -\alpha \\ 0 \\ 0 \\ \alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

$$\boxed{N(A) = \left\{ \alpha \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}.}$$

Ex: $A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$, Find $\underline{N(A)}$.

$$A \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & \\ 2 & 1 & 0 & 1 & \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & \\ 0 & -1 & -2 & 1 & \end{array}\right)$$

$\to -x_2 - 2x_3 + x_4 = 0$

$x_2 = -2\alpha + \beta.$

$x_2 = -2\alpha + \beta.$

$x_3 = \alpha, \quad x_4 = \beta,$

$x_1 = -x_2 - x_3 = 2\alpha - \beta - \alpha$

$\boxed{x_1 = \alpha - \beta.}$

$$N(A) = \left\{ \begin{pmatrix} \alpha - \beta \\ -2\alpha + \beta \\ \alpha \\ \beta \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}.$$

$$N(A) = \left\{ \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}.$$