Wednesday, February 17, 2021 9:54 AM

1.1 Linear Systems

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

unknowns: $x_1, \ldots, x_n$.

$m \times n$ linear system

m: # equations

Remark: Any $m\times n$-system either has no solution (inconsistent)

or exactly one solution.

or infinite # of solutions.

consistent.

Def: An $m\times n$-system is called inconsistent if it has no solutions. It is called consistent if it has solution(s) {one or infinite solutions}.

$m\times n$-system: Can be written as

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

augmented matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix}$$ coefficient matrix.

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$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$ constants.

* $m \times n$ - system $(A|b)$

Ex:
$$\begin{cases} x_1 - x_2 + x_4 = 2 \\ -x_1 + 3x_2 + x_3 + x_4 = 1 \\ 4x_1 - x_3 + 3x_4 = 5 \end{cases}$$
3×4 - system.

Augmented matrix $(A|b)$ =
$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ -1 & 3 & 1 & 1 & 1 \\ 4 & 0 & -1 & 3 & 5 \end{array}\right)$$

$$-x_1 + 3x_2 + x_3 + x_4 = 1$$

Elementary Row Operations: (matrix)

① Row operation I: Interchange two rows. $(R_i \Longleftrightarrow R_j)$ cto

" " II: Multiply a row by nonzero constant. (cR_i)

" " III: Replace a row by its sum with a multiple of a nother row. $\{(kR_i + R_j) \Longleftrightarrow R_j\}$

$$3x_1 = \cancel{-2} - 2(3) + 2 = -6 \Rightarrow x_1 = -2$$

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Ex: (a)

$$3x_1 + 2x_2 - x_3 = -2$$
$$x_2 = 3 \rightarrow x_2 = 3$$
$$2x_3 = 4 \rightarrow x_3 = 2$$

back substitution

solution (a) $x = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$

(b)

$$3x_1 + 2x_2 - x_3 = -2$$
$$-3x_1 - x_2 + x_3 = 5$$
$$3x_1 + 2x_2 + x_3 = 2$$

(b) has the same solutions as (a).

so $x = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$ solution to (b)

Def: Two systems with the same unknowns are called equivalent if they have the same solution set.

Remark: Given an $m \times n$-system $(A|b)$

Applying E.R.O's $\{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$ on $(A|b)$ produces an equivalent system $(U|d)$. → equivalent.

→ easy to solve.

Ex:

$$\left.\begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ -3x_1 - x_2 + x_3 &= 5 \\ 3x_1 + 2x_2 + x_3 &= 2 \end{aligned}\right\} (*)$$

$x_1 = -2$

$3x_1 + 2x_2 + x_3 = 2$

$$\left(\begin{array}{ccc|c} 3 & 2 & -1 & -2 \\ -3 & -1 & 1 & 5 \\ 3 & 2 & 1 & 2 \end{array}\right) \xrightarrow[(-R_1+R_3)\to R_3]{(1R_1+R_2)\to R_2} \left(\begin{array}{ccc|c} 3 & 2 & -1 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 4 \end{array}\right) \begin{array}{l} \to 3x_1 = -2-2(3)+2 \\ \to x_2 = 3 \\ \to x_3 = 2 \end{array}$$

$x_1 = -2$

solution $x = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$ to $(*)$

Def: An $m\times n$-system is said to be in strict triangular form if in kth equation, the coefficients of $x_1, \dots, x_{k-1}$ are all zero and the coefficient of x_k is nonzero.

$$\left.\begin{aligned} 2x_1 + x_2 - x_3 &= 1 \\ -x_2 + 3x_3 &= 2 \\ 2x_3 &= 6. \end{aligned}\right\}$$

in strict triangular form.

back substitution

So it has one solution.

2×2-system

3×3?

Ex:

$$\left.\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 1 \\ -x_1 - x_2 + x_5 &= -1 \\ -2x_1 - 2x_2 + 3x_5 &= 1 \\ x_2 + x_3 + x_4 + 3x_5 &= -1 \\ x_1 + x_2 + 2x_3 + 2x_4 + 4x_5 &= 1. \end{aligned}\right\}$$

5×5

pivot element.

$\boxed{1}$ $\quad$ $\boxed{1}$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 1 & 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 2 & 4 & 1 \end{array}\right) \xrightarrow[\substack{2R_1+R_3 \\ -R_1+R_5}]{1R_1+R_2} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 1 & 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_4} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 3 & -1 \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right) \xrightarrow{R_3 \leftrightarrow R_4} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right)$$

pivot element.

$$\xrightarrow[-R_3+R_5]{-2R_3+R_4} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{-R_4+R_5} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{array}\right)$$

$0=0$

$0x_1+0x_2+0x_3+0x_4+0x_5=-3$

The system is $\boxed{\text{inconsistent}}$