3.2: Cont.

linear combination of $v_1, v_2, \dots, v_n$:

If $v_1, v_2, \dots, v_n \in V$, a sum of the form

$\boxed{\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n}$, where α_i are scalars, is called a linear combination of $v_1, v_2, \dots, v_n$.

$$\boxed{\begin{gathered} v_1 = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \dots, v_n = \begin{pmatrix} \quad \end{pmatrix} \in \mathbb{R}^n \\ \alpha_1 v_1 + \dots + \alpha_n v_n. \end{gathered}}$$

Ex. If $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \in \mathbb{R}^3$.

a) A linear comb. of v_1, v_2, v_3 is $2v_1 + 3v_2 - 4v_3 = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \begin{pmatrix} 6 \\ -3 \\ 0 \end{pmatrix} - \begin{pmatrix} 12 \\ 4 \\ -4 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix}$.

$\therefore \begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix}$ is a linear comb. of v_1, v_2, v_3.

b) Is $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ a linear comb. of v_1, v_2, v_3?

solve $\boxed{\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}}$ for $\alpha_1, \alpha_2, \alpha_3$.

$\to$ system

consistent $\to$ Yes $\therefore$ is a l.c.

inconsistent $\to$ No. is not a l.c.

$\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \to$ solve $\boxed{\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}}$

consistent or incons.

Consistent or incons.

Yes No.

* $\boxed{\text{The set of all linear combinations of } v_1, \dots, v_n.}$

$$= \{ \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n : \alpha_i \in \mathbb{R} \}.$$

is called span of $v_1, v_2, \dots, v_n$.

$\text{Span}(v_1, v_2, \dots, v_n) =$ The set of all linear combinations of $v_1, \dots, v_n$

$$\boxed{\text{Span}(v_1, \dots, v_n) = \{ \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n : \alpha_i \in \mathbb{R} \}.}$$

* Is $v_1 \in \text{Span}(v_1, \dots, v_n)$? (Yes)

$$v_1 = \boxed{(1) v_1 + (0) v_2 + \dots + (0) v_n} \; ? \Rightarrow v_1 \in \text{Span}(v_1, \dots, v_n)$$

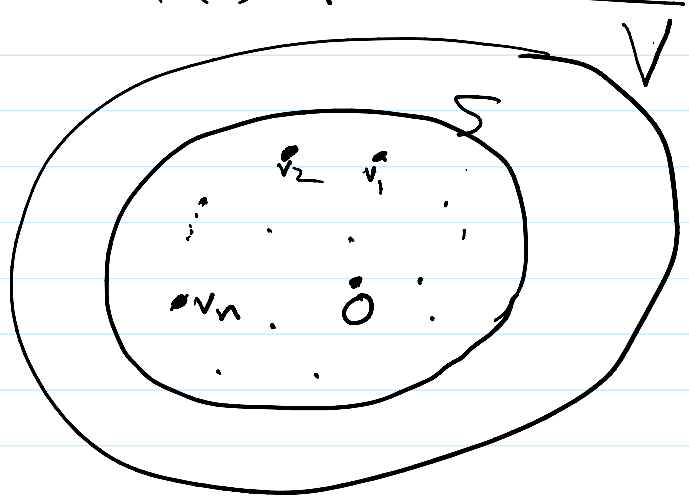
$v_2 \in \text{Span}(v_1, \dots, v_n)$ since $v_2 = 0 v_1 + 1 \cdot v_2 + 0 v_3 + \dots + 0 v_n$.

$v_i \in \text{Span}(v_1, \dots, v_n)$, $\forall \, i = 1, \dots, n$.

* Is $0 \in \text{Span}(v_1, \dots, v_n)$??

$$0 = (0) v_1 + (0) v_2 + \dots + (0) v_n \Rightarrow 0 \in \text{Span}(v_1, \dots, v_n)$$

* $S = \text{span}(v_1, \dots, v_n)$



Th. Let $v_1, \dots, v_n \in V$, then $\text{span}(v_1, \dots, v_n)$ is a subspace of V.

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proof: $S = \text{Span}(v_1, \dots, v_n) = \{ \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n : \alpha_i \text{ scalars} \}$.

1) $\text{Span}(v_1, \dots, v_n) \neq \phi$ since $0 \in \text{Span}(v_1, \dots, v_n)$

$0 = 0v_1 + 0v_2 + \cdots + 0v_n$.

2) If $s_1, s_2 \in \text{Span}(v_1, \dots, v_n)$

$\Rightarrow s_1 = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$, $\alpha_i \in \mathbb{R}$.

and $s_2 = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$, $\beta_i \in \mathbb{R}$.

Now $\boxed{s_1 + s_2} = \alpha_1 v_1 + \cdots + \alpha_n v_n + \beta_1 v_1 + \cdots + \beta_n v_n$

$= (\alpha_1 + \beta_1) v_1 + (\alpha_2 + \beta_2) v_2 + \cdots + (\alpha_n + \beta_n) v_n$.

$= \gamma_1 v_1 + \gamma_2 v_2 + \cdots + \gamma_n v_n$.

$\therefore s_1 + s_2 \in \text{Span}(v_1, \dots, v_n)$.

3) Let $s_1 \in \text{Span}(v_1, \dots, v_n)$ and α is a scalar.

$\Rightarrow s_1 = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

Now $\alpha s_1 = \alpha(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = (\alpha\alpha_1) v_1 + (\alpha\alpha_2) v_2 + \cdots + (\alpha\alpha_n) v_n$

$\Rightarrow \alpha s_1 \in \text{Span}(v_1, \dots, v_n)$.

(1), (2) and (3) $\Rightarrow$ $\text{Span}(v_1, \dots, v_n)$ is a subspace of V.

Ex: Is $p(x) = 3 + 4x - 5x^2$ a linear combination of $p_1(x) = 1 + x + x^2$, $p_2(x) = 1 - x$, $p_3(x) = 1 + x$.

solve: $p(x) = \alpha_1 p_1(x) + \alpha_2 p_2(x) + \alpha_3 p_3(x)$.

$3 + 4x - 5x^2 = \alpha_1(1 + x + x^2) + \alpha_2(1 - x) + \alpha_3(1 + x)$

$$\boxed{3+4x-5x^2=\alpha_1(1+x+x^2)+\alpha_2(1-x)+\alpha_3(1+x).}$$

$$\begin{array}{ll} \text{Coeff } x^2: & \\ \text{Coeff } x: & \\ \text{Const}: & \end{array} \boxed{\begin{aligned} -5 &= \alpha_1 \\ 4 &= \alpha_1-\alpha_2+\alpha_3 \\ 3 &= \alpha_1+\alpha_2+\alpha_3 \end{aligned}}$$

Consistent or inconst.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 1 & -1 & 1 & 4 \\ 1 & 1 & 1 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & -1 & 1 & 9 \\ 0 & 1 & 1 & 8 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & -1 & 1 & 9 \\ 0 & 0 & 2 & 17 \end{array}\right)$$

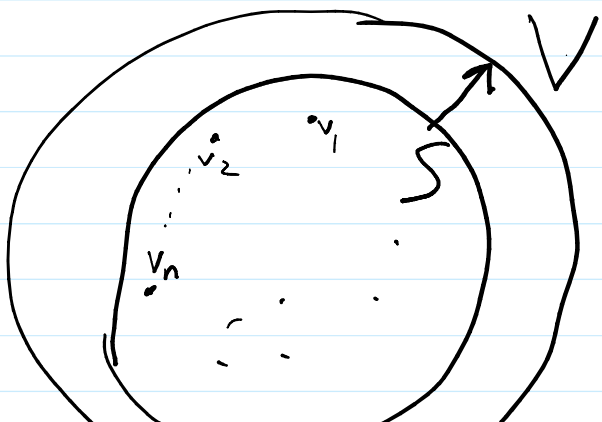
Consistent.

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & \neq 0 \end{array}\right)$$

$\therefore$ $p(x)=3+4x-5x^2$ is a linear comb. of $p_1(x)$, $p_2(x)$, $p_3(x)$.

Def: Let $v_1, v_2, \dots, v_n \in V$, we say $v_1, v_2, \dots, v_n$ is a spanning set for V

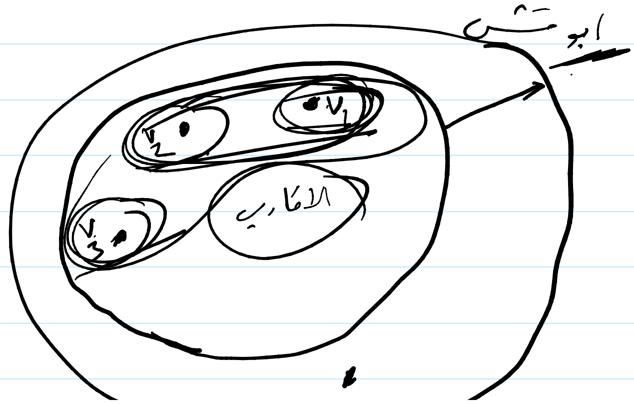
if $\boxed{\boxed{\text{Span}(v_1, \dots, v_n)} = V}$

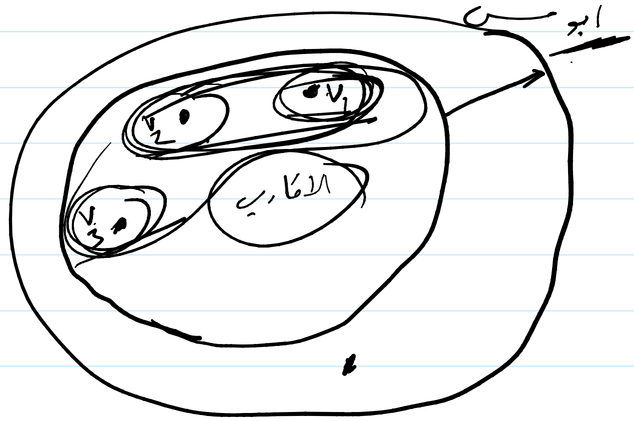


$\iff$ any $v \in V$ is a linear combination of $v_1, \dots, v_n$.

$\iff$ for any $v \in V$, there exist scalars $\alpha_1, \dots, \alpha_n$ s.t. $v=\alpha_1 v_1+\alpha_2 v_2+\dots+\alpha_n v_n$.

$\iff$ For any $v \in V$, the system $v=\alpha_1 v_1+\alpha_2 v_2+\dots+\alpha_n v_n$ is consistent.





How check if $v_1, \dots, v_n \in V$ is a spanning set?

Ex: Is $\left\{ v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ a sp. set for $\boxed{\mathbb{R}^3}$

Let $\boxed{v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3}$ $\rightarrow$ check if the system $\boxed{\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}}$ is consistent for **all** a, b, c.

solve: $\alpha_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & 1 & 0 & b \\ 1 & 0 & 1 & c \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 0 & -1 & b-a \\ 0 & -1 & 0 & c-a \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & -1 & 0 & c-a \\ 0 & 0 & -1 & b-a \end{array}\right)$$

Consistent for all a, b, c.

$\therefore \left\{ v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a sp. set for $\mathbb{R}^3$.

Ex. Is $\left\{ v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \right\}$ a sp. set for $\boxed{\mathbb{R}^3}$?

Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ and solve $\boxed{\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}}$

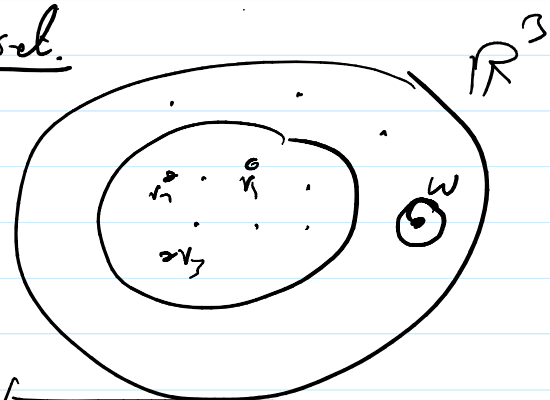
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & 2 & 3 & b \\ 1 & 3 & 5 & c \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 2 & b-a \\ 0 & 2 & 4 & c-a \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 2 & b-a \\ 0 & 0 & 0 & \boxed{c+a-2b} \end{array}\right)$$

$\boxed{\text{Consistent}}$ if $\boxed{c+a-2b=0}$ $\boxed{0 = c+a-2b}$

$\boxed{\text{Consistent}}$ if $\boxed{c+a-2b=0}$ $\boxed{0 = c+a-2b}$

$\therefore \{v_1, v_2, v_3\}$ is not a spanning set.

take $w = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $w \notin \mathrm{Span}(v_1, v_2, v_3)$.



take $w = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$ s.t. $c+a-2b \neq 0$

inconsistent.

$2+2-2(3) = -2 \neq 0$

$\therefore w = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \notin \mathrm{Span}(v_1, v_2, v_3)$

$w = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$: $c+a-2b=0$

$\Rightarrow w \in \mathrm{Span}(v_1, v_2, v_3)$

Ex: Is $\{p_1(x) = x^2+x+1,\ p_2(x) = x+3,\ p_3(x) = x^2-x+2\}$ a sp. set for $\boxed{P_3}$?

Let $p(x) = ax^2+bx+c \in P_3$: solve $p(x) = \alpha_1 p_1(x) + \alpha_2 p_2(x) + \alpha_3 p_3(x)$

solve $\boxed{ax^2+bx+c = \alpha_1(x^2+x+1) + \alpha_2(x+3) + \alpha_3(x^2-x+2)}$

Coef x^2: $a = \alpha_1 + \alpha_3$

Coef x: $b = \alpha_1 + \alpha_2 - \alpha_3$

Const: $c = \alpha_1 + 3\alpha_2 + 2\alpha_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 1 & 1 & -1 & b \\ 1 & 3 & 2 & c \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & -2 & b-a \\ 0 & 3 & 1 & c-a \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & -2 & b-a \\ 0 & 0 & 7 & c+2a-3b \end{array}\right)$$

consistent for any a, b, c.

$\therefore \{p_1(x) = x^2+x+1,\ p_2(x) = x+3,\ p_3(x) = x^2-x+2\}$ is a sp. set for P_3.

Ex: Is $\left\{E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ E_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},\ E_3 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\right\}$ is a sp. set for $\mathbb{R}^{2\times 2}$

Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$ and solve

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha_1+\alpha_2+\alpha_3 & \alpha_2 \\ \alpha_3 & \alpha_1 \end{pmatrix}$$

System:

$$\boxed{\begin{aligned} a &= \alpha_1+\alpha_2+\alpha_3 \\ b &= \alpha_2 \\ c &= \alpha_3 \\ d &= \alpha_1 \end{aligned}}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 1 & 0 & 0 & d \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & \boxed{1} & 0 & b \\ 0 & 0 & 1 & c \\ 0 & -1 & -1 & d-a \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & -1 & d-a+b \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & d-a+b+c \end{array}\right)$$

Consistent if $\boxed{d-a+b+c=0.}$

(inconsistent if $d-a+b+c\neq 0$)

$\therefore \{E_1, E_2, E_3\}$ is not a sp. set for $\mathbb{R}^{2\times 2}$.

Notation: in $\mathbb{R}^3$: $\boxed{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \to i\text{th} \quad \in \mathbb{R}^n$$

$e_1 \in \mathbb{R}^2 \Longrightarrow e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$e_2 \in \mathbb{R}^2 \longrightarrow e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Ex. Find $\operatorname{span}(e_1, e_2)$ in $\mathbb{R}^3$.

$$\operatorname{span}(e_1, e_2) = \{ \alpha_1 e_1 + \alpha_2 e_2 : \alpha_1, \alpha_2 \in \mathbb{R} \}$$

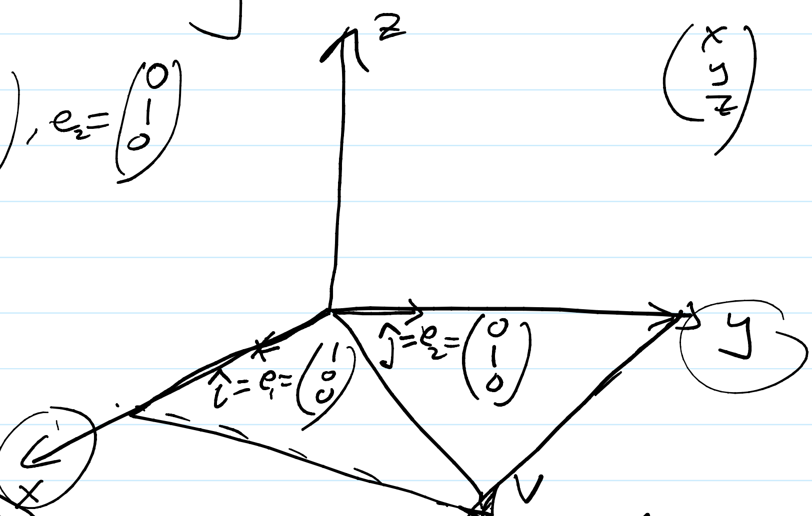
$$= \left\{ \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} : \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$

$$\operatorname{Span}(e_1, e_2) = \left\{ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix} : \alpha_1, \alpha_2 \in \mathbb{R} \right\} \quad \text{xy-plane.}$$

$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix}$



$v = \alpha_1 \hat{i} + \alpha_2 \hat{j}$

$v \in$ xy-plane.

- in $\mathbb{R}^3$: $\operatorname{span}(v_1, v_2)$: plane containing v_1, v_2.

not parallel