Wednesday, May 5, 2021 10:03 AM

* 3.2 System $\underline{Ax=b.}$

$A_{m\times n} \longrightarrow Ax=b$ / $Ax=0$.

* If $x_0$ is a solution to $\boxed{Ax=b}$ and $x_1$ is a solution to $Ax=0$, then $\boxed{x_0+x_1}$ is a solution to $\boxed{Ax=b}$.

$$A\underline{(x_0+x_1)} = \underbrace{Ax_0}_{=b} + \underbrace{Ax_1}_{=0} = b+0=b.$$

* Any solution to $Ax=b$ is of the form $y=x_0+z$, $x_0$ a solution to $Ax=b$ and $z$ is a solution to $Ax=0$.

Th: Let $A$ be $m\times n$-matrix. If $x_0$ is a particular solution to $Ax=b$ (so $Ax=b$ is consistent), then $y$ is a solution to $Ax=b$ if and only if $\boxed{y=x_0+z}$, where $z\in N(A)$ ($z$ is a solution to $Ax=0$)

Proof: Now if $x_0$ is a solution to $Ax=b$ and $y$ is any solution to $Ax=b$

$\Rightarrow Ax_0=b$ and $Ay=b$

$\Rightarrow Ay-Ax_0=0 \Rightarrow A(y-x_0)=0$.

Let $\boxed{z=y-x_0} \Rightarrow Az=0 \Rightarrow$ $\boxed{z \text{ is a solution to } Ax=0}$

$\Rightarrow z\in N(A)$.

and $z=y-x_0 \Rightarrow \boxed{y=x_0+z}$, $z\in N(A)$.

Ex: 15/3.2) $A_{4\times 3}$, $\boxed{c=2a_1+1a_2+1a_3}$, $a_i$: columns of $A$.

1) If $N(A)=\{0\}$ How many solutions does $\boxed{Ax=c}$ have?

1) If $N(A)=\{0\}$ & How many solutions does $Ax=c$ have?

$Ax=c$ consistent ( $c$ is a linear comb. of columns of $A$ $\Rightarrow$ $Ax=c$ is consistent.)

$\boxed{\text{a solution to } Ax=c \text{ is } x_0=\begin{pmatrix}2\\1\\1\end{pmatrix}}$ $\quad A_{4\times 3}\ x_{3\times 1}$

Are there other solutions ( No )

If $y$ is a solution to $Ax=c$ $\Rightarrow$ $\boxed{y=x_0+z}$, $z\in N(A)$.

$\Rightarrow y=\begin{pmatrix}2\\1\\1\end{pmatrix}+z$, $z\in N(A)=\{0\}$.

$\Rightarrow z=0$

$\Rightarrow y=\begin{pmatrix}2\\1\\1\end{pmatrix}+0=\begin{pmatrix}2\\1\\1\end{pmatrix}$

so $Ax=c$ has a unique solution!

2) If $N(A)\neq\{0\}$

$\Rightarrow$ $Ax=c$ has infinite number of solutions.

$y=x_0+z$, $z\in N(A)$.

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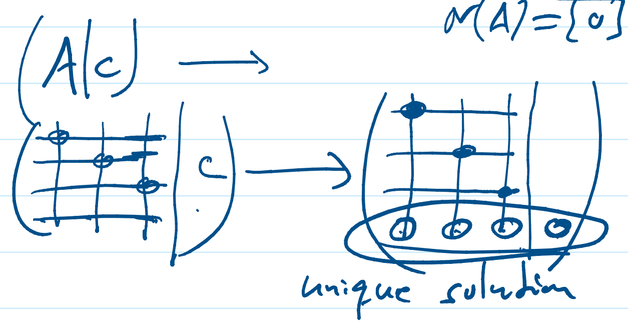
$A_{4\times 3}$, $Ax=c$ has unique solution. $\quad N(A)=\{0\}$.

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3.3 Linear independence.

Ex: $v_1=\begin{pmatrix}1\\-1\\2\end{pmatrix}$, $v_2=\begin{pmatrix}-2\\3\\1\end{pmatrix}$, $v_3=\begin{pmatrix}-1\\3\\8\end{pmatrix}$

① Is $v_1$ a linear comb. of $v_2, v_3$
② Is $v_2$ a linear comb of $v_1, v_3$
③ Is $v_3$ a " " of $v_1, v_2$.

Can we write one of them as a linear combination of the other 2 vectors?

① $v_1=\alpha_2 v_2+\alpha_3 v_3$
solve → Yes / No.

② —

& a linear combination of the other vectors.

solve $\boxed{\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0}$.

has nonzero solutions | has only the zero solution.

none of $v_1, v_2, v_3$ can be written as a l. comb. of the others.

$x = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ is a solution to $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$.

$\Rightarrow \boxed{1 v_1 + 0 v_2 + 2 v_3 = 0} \Rightarrow v_1 = 0 v_2 - 2 v_3.$

$\Rightarrow v_3 = -\frac{1}{2} v_1 + 0 v_2.$

$\Rightarrow$ one of them can be written as a l.c. of the others.

the one with nonzero coefficient.

solve ... No.

② —

③ —

**Def:** Let $v_1, v_2, \ldots, v_n \in V$. If the system $\boxed{c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0}$ has only the zero solution, we say $v_1, \ldots, v_n$ are **Linearly independent.**

If $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$ has a nonzero solution, we say $v_1, v_2, \ldots, v_n$ are **Linearly dependent.**

Remarks: ① The vectors $v_1, \ldots, v_n$ are Linearly dependent iff they are not linearly independent.

② $v_1, v_2, \ldots, v_n$ are linearly dependent if and only if one of them can be written as a linear combination of the other vectors. ($\Longleftrightarrow$ the system $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$ has nonzero solutions).

③ $v_1, v_2, \ldots, v_n$ are linearly independent if and only if none of them can be written as a linear combination of the other vectors.

$\boxed{\mathbb{R}^n}$

$\mathbb{R}^n$

Ex: $v_1 = \begin{pmatrix}1\\2\end{pmatrix}, \; v_2 = \begin{pmatrix}4\\8\end{pmatrix}$ L.I or L.D.

solve $\boxed{c_1 v_1 + c_2 v_2 = 0}$ — (*) solve: $c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}4\\8\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 4 & 0 \\ 2 & 8 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

(*) has nonzero solutions.

$\Longrightarrow v_1 = \begin{pmatrix}1\\2\end{pmatrix}, \; v_2 = \begin{pmatrix}4\\8\end{pmatrix}$ are L.D.

or $v_1 = \frac{1}{4} v_2$.

* $v_2 = 4 v_1 \Longrightarrow$ $v_1, v_2$ are L.D.

Remark: If $v_1, v_2 \in V$. $v_1, v_2$ are L.D $\Longleftrightarrow$ one of them can be written as a scalar multiple of the other.

Ex: $p_1(x) = x^2 + x$, $\quad p_2(x) = x + 1$ L.I or L.D.

(none of them can be written as a constant multiple of the other)

so L.I.

Method (2) solve $c_1 p_1(x) + c_2 p_2(x) = 0$.

$$c_1(x^2 + x) + c_2(x+1) = 0.$$

Coefficients:

$x^2$ : $c_1 = 0$

$x$ : $c_1 + c_2 = 0$.

Const. : $c_2 = 0$

$\Longrightarrow c_1 = 0 = c_2$.

only the zero solution.

$\therefore$ $p_1(x), p_2(x)$ are L.I.

Ex: $v_1 = \begin{pmatrix}1\\-1\\1\end{pmatrix}, \; v_2 = \begin{pmatrix}1\\0\\1\end{pmatrix}, \; v_3 = \begin{pmatrix}1\\1\\0\end{pmatrix}$ in $\mathbb{R}^3$. L.I. or L.D ?

solve $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

(3×3 − system).

solve $c_1v_1 + c_2v_2 + c_3v_3 = 0$

$$\boxed{c_1\begin{pmatrix}1\\-1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\0\\1\end{pmatrix} + c_3\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}} \longleftarrow (*) \quad (3\times3\text{-system}).$$

$$\boxed{\begin{pmatrix}1 & 1 & 1\\ -1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}} \longrightarrow \left(\begin{array}{ccc|c}1 & 1 & 1 & 0\\ 0 & 1 & 2 & 0\\ 0 & 0 & -1 & 0\end{array}\right) \begin{array}{l}\to c_1 = 0\\ \to c_2 = 0\\ \to c_3 = 0.\end{array} \Rightarrow (*) \ \boxed{\text{has only the zero solution.}}$$

so $v_1, v_2, v_3$ are L.I. (none of them can be written as a linear combination of the other vectors).

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**Th.** If $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$. Then the vectors $v_1, v_2, \ldots, v_n$ are <u>linearly independent</u> if and only if the matrix $X = (v_1\ v_2\ \cdots\ v_n)_{n\times n}$ is nonsingular ($\Longleftrightarrow \det(X) \neq 0$).

$\propto$ $v_1, \ldots, v_n$ are L.D. $\Longleftrightarrow$ $X$ is singular.

**Ex:** $\boxed{v_1 = \begin{pmatrix}1\\-1\\1\\2\end{pmatrix}, v_2 = \begin{pmatrix}1\\0\\2\\1\end{pmatrix}, v_3 = \begin{pmatrix}-1\\0\\0\\1\end{pmatrix}, v_4 = \begin{pmatrix}2\\-2\\4\\6\end{pmatrix}}$ $\in \mathbb{R}^4$ L.I or L.D.

$$X = \begin{pmatrix}1 & 1 & -1 & 2\\ -1 & 0 & 0 & -2\\ 1 & 2 & 0 & 4\\ 2 & 1 & 1 & 6\end{pmatrix}$$

→ singular $\Longrightarrow$ $v_1, v_2, v_3, v_4$ are L.D.

or → nonsingular $\Longrightarrow$ $v_1, v_2, v_3, v_4$ are L.I.

$$|X| = \begin{vmatrix}1 & 1 & -1 & 2\\ -1 & 0 & 0 & -2\\ 1 & 2 & 0 & 4\\ 2 & 1 & 1 & 6\end{vmatrix} = \begin{vmatrix}1 & 1 & -1 & 2\\ 0 & 1 & -1 & 0\\ 0 & 1 & 1 & 2\\ 0 & -1 & 3 & 2\end{vmatrix} = \begin{vmatrix}1 & 1 & -1 & 2\\ 0 & 1 & -2 & 0\\ 0 & 0 & 2 & 2\\ 0 & 0 & 2 & 2\end{vmatrix}$$

$$= \begin{vmatrix}1 & 1 & -1 & 2\\ 0 & 1 & -2 & 0\\ 0 & 0 & 2 & 2\\ 0 & 0 & 0 & 0\end{vmatrix} = 0$$

$\therefore$ $X$ is singular

$\Longrightarrow$ $v_1, v_2, v_3, v_4$ are L.D.

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$\boxed{P_n}$: **Ex:** $p_1(x) = x^2 - 2x + 3$, $p_2(x) = 2x^2 + x + 8$, $p_3(x) = x^2 + 8x + 7$.

L.D or L.I ?

solve: $c_1(x^2 - 2x + 3) + c_2(2x^2 + x + 8) + c_3(x^2 + 8x + 7) = 0.$

Coef $x^2$: $c_1 + 2c_2 + c_3 = 0$

$x$: $-2c_1 + c_2 + 8c_3 = 0$

Const: $3c_1 + 8c_2 + 7c_3 = 0$.

$(*)$ solve

$$\begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 3 & 8 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 5 & 10 \\ 0 & 2 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$\boxed{\text{free } c_3}$

$(*)$ has nonzero solution

$\Longrightarrow$ $\boxed{p_1(x),\ p_2(x),\ p_3(x) \text{ are L.D.}}$

$\Longleftrightarrow$ $\boxed{\text{one of them can be written as a l.c. of the other 2}}$

which one?

find solution?

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$