Monday, February 22, 2021 9:56 AM

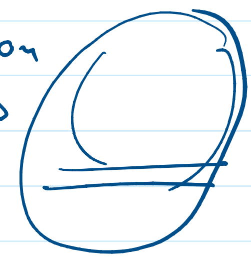
$\boxed{1.1 + 1.2}$

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\}$$

$m \times n$-system.

augmented matrix

$$(A|b) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right) \xrightarrow{\text{reduction}}$$



Def: A matrix M is said to be in Row echelon form if

1) The first nonzero element in each nonzero row is 1.
2) For each nonzero row k the number of leading zeros in row $k+1$ is greater than the number of leading zeros in row k.
3) If there are zero rows, they are below other rows.

Ex: $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ in REF.

$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ REF. Cond 2: rows 1, 2

$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & \boxed{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ not REF.

$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ not REF.

$\begin{pmatrix} 1 & 0 & 2 & -3 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ not REF.

Remark: Any matrix can be transformed to a matrix in REF using elementary row operations.

1) $R_{i \leftrightarrow j}$. — Row operation I.
2) αR_i, $\alpha \neq 0$. — " " II
3) $(cR_i + R_j) \leftrightarrow R_j$. — " " III.

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Ex: $A = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right)$ tranform A into REF. (find the REF of A)

→ pivot element.

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix} \xrightarrow{-R_1+R_2} \begin{pmatrix} \boxed{1} & 1 & -1 & 0 \\ 0 & \boxed{-2} & 1 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & \boxed{1} & -1 & 1 \\ 0 & -2 & 1 & 1 \end{pmatrix} \xrightarrow{2R_2+R_3} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & \boxed{1} & -1 & 1 \\ 0 & 0 & \boxed{-1} & 3 \end{pmatrix}$$

$$\xrightarrow{-R_3} \boxed{\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -3 \end{pmatrix}}$$

in REF

is called the REF of A.

Remark: Given $(A|b) \xrightarrow{E.R.O's} \boxed{(U|d)}$ in REF.

The system $(A|b)$ is equivalent to $(U|d)$.

Method: Gauss Elemination method.

Ex: solve

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 2 \\ 4x_1 + 3x_2 + 3x_3 &= 4 \\ 3x_1 + x_2 + 2x_3 &= 3. \end{aligned}$$

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{array}\right) \xrightarrow[\substack{-4R_1+R_3 \\ -3R_1+R_4}]{-2R_1+R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & \boxed{-5} & -1 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \end{array}\right)$$

$$\xrightarrow[-R_2+R_4]{-R_2+R_3} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{-1}{5}R_2} \boxed{\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)} \Big\} \text{REF.}$$

Is there a row of the form $\boxed{(0 \;\; 0 \cdots 0 \,|\, c \neq 0)}$? No

so the system is consistent. (one solution or more?)

leading ones $\longrightarrow$ leading variables: x_1, x_2 (2 L.V.)

rest of variables: free variables: x_3.

The system has infinite number of solution.

(If there are no free variables, then the system has only one solution).

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$$\longrightarrow \quad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

has inf. # of solutions

x_3 : free variables

Let $x_3 = \alpha$

$(2) \Rightarrow \quad x_2 + \frac{1}{5} x_3 = 0 \quad \Longrightarrow \quad \boxed{x_2 = -\frac{1}{5}\alpha}$

$(1) \Rightarrow \quad x_1 + 2x_2 + x_3 = 1 \quad \Longrightarrow \quad x_1 = 1 - 2\left(-\frac{1}{5}\alpha\right) - \alpha$

$= 1 + \frac{2}{5}\alpha - \alpha = 1 - \frac{3\alpha}{5}$

{write the leading variables $\{x_1, x_2\}$ in terms of free variables $\{x_3\}$ }

so any solution has the form $\boxed{x = \begin{pmatrix} 1 - \frac{3\alpha}{5} \\ -\frac{1}{5}\alpha \\ \alpha \end{pmatrix}}$, $\alpha \in \mathbb{R}$

α is a scalar

Let $\alpha = 5 \longrightarrow x = \begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix}$ is a sol.

$\alpha = 0 \longrightarrow x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is a sol.

$\alpha = 1 \longrightarrow x = \begin{pmatrix} 2/5 \\ -\frac{1}{5} \\ 1 \end{pmatrix}$ is a sol.

$\vdots$

Remark: (G.E.M)

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$(A|b)$

$\downarrow$ reduction to REF.

$\boxed{(U|d)}$

1) If there is a row of the form $\boxed{(0\ 0 \cdots 0 | c^{\neq 0})}$, then the system is <u>inconsistent</u> (No solutions).

2) <u>If not</u>, $\left\{ (U|d) \text{ has no row of the form } (0\ 0 \cdots 0 | c^{\neq 0}) \right\}$

then the system is <u>consistent</u>.

- there are free variables, the inf. # of solutions
  - → write the leading variables in terms of free variables.
  - general form of the solutions.
- there are no free variables, the only one sol.
  - → find it. (back substitution)

<u>Ex.</u>

$$-x_2 - x_3 + x_4 = 0$$
$$x_1 + x_2 + x_3 + x_4 = 6$$
$$2x_1 + 4x_2 + x_3 - 2x_4 = -1$$

$3x_1 + x_2 - 2x_3 + 2x_4 = 3.$

$$\left(\begin{array}{cccc|c} 0 & -1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 6 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right) \longrightarrow \cdots \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 13/3 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

cons., 4 leading

no free var.

so one solution.

$x = \boxed{}$