Wednesday, February 24, 2021 10:07 AM

* $(A|b)$ $m\times n$-system → no solution } inconsistent.
→ one solution } consistent
→ inf. # of solutions } consistent

**Def:** • An $m\times n$-system is called $\boxed{\text{overdetermined}}$ if $\boxed{m>n}$
• " " " " " underdetermined if $m<n$.
• " " " " " square if $m=n$.

**Ex:**

$$\left.\begin{aligned} x_1+x_2&=1\\ x_1-x_2&=3\\ -x_1+2x_2&=-2\end{aligned}\right\}$$

3x2-system
overdetermined.

$$\left(\begin{array}{cc|c}1&1&1\\1&-1&3\\-1&2&-2\end{array}\right)\longrightarrow\left(\begin{array}{cc|c}1&1&1\\0&-2&2\\0&3&-1\end{array}\right)\longrightarrow\left(\begin{array}{cc|c}1&1&1\\0&1&-1\\0&3&-1\end{array}\right)$$

$$\longrightarrow\left(\begin{array}{cc|c}1&1&1\\0&1&-1\\0&0&2\end{array}\right)\longrightarrow$$ The system is inconsistent.

solve
**Ex:**

$$\left(\begin{array}{ccc|c}1&2&1&1\\2&-1&1&2\\4&3&3&4\\2&-1&3&5\end{array}\right)$$

4x3-system
overdetermined.

$$\longrightarrow\left(\begin{array}{ccc|c}1&2&1&1\\0&1&\frac{1}{5}&0\\0&0&1&\frac{3}{2}\\0&0&0&0\end{array}\right)$$

$\to x_1=1-2x_2-x_3=1-2\left(\frac{-3}{10}\right)-\frac{3}{2}=\frac{1}{10}$

$\to x_2=-\frac{1}{5}x_3=-\frac{1}{5}\left(\frac{3}{2}\right)=\frac{-3}{10}$

$\to x_3=\frac{3}{2}$

consistent: { leading variables: $x_1, x_2, x_3$
No free variables.
$\Rightarrow$ $\boxed{\text{one solution.}}$

sol. $x=\begin{pmatrix}1/10\\-3/10\\3/2\end{pmatrix}$

**Ex:**

$$\left.\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 2 \\ 4x_1 + 3x_2 + 3x_3 &= 4 \\ 3x_1 + x_2 + 2x_3 &= 3 \end{aligned}\right\}$$

$4\times 3$ – system
overdetermined.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \to x_1 = 1 - 2\left(\frac{-\alpha}{5}\right) - \alpha \\ \to x_2 = -\frac{1}{5}\alpha \\ \\ \\ \end{array}$$

$x_1 = 1 - \frac{3}{5}\alpha$

Consistent.

$\boxed{x_3 = \alpha}$ free variable $\Longrightarrow$ $\boxed{\text{Inf. \# of sol.}}$

Any solution has the form $\boxed{x = \begin{pmatrix} 1 - \frac{3\alpha}{5} \\ \frac{-\alpha}{5} \\ \alpha \end{pmatrix}}$

* overdetermined $\begin{cases} \xrightarrow{\text{or}} \text{no sol.} \\ \xrightarrow{\text{or}} \text{one sol.} \\ \xrightarrow{\text{or}} \text{inf. \# of sol.} \end{cases}$

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**Ex:**

$$\left.\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 + 4x_2 + 2x_3 &= 3 \end{aligned}\right\}$$

$2\times 3$ – system.
undetermined.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

The system is inconsistent (No sol.)

**Ex:**

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{array}\right)$$

$3\times 5$ – system.
undetermined.

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \end{array}\right) \to x_1 = \cancel{2} - \alpha - \beta - \cancel{2} + 1 = 1 - \alpha - \beta$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right) \begin{array}{l} \to x_1 = \cancel{2} - \alpha - \beta - \cancel{2} + 1 = 1-\alpha-\beta \\ \to x_4 = 1-(-1) = 2 \\ \to \boxed{x_5 = -1} \end{array}$$

Consistent: free variables: $x_2 = \alpha$, $x_3 = \beta$. has inf. # of sol.

Any solution has the form
$$\boxed{x = \begin{pmatrix} 1-\alpha-\beta \\ \alpha \\ \beta \\ 2 \\ -1 \end{pmatrix}}$$

For any $m \times n$-system: $\boxed{\#\text{ of leading variables} \le m}$ ($m$ = # of rows)

**Th.** If an $m\times n$-system is $\boxed{\text{consistent}}$ and $\boxed{\text{underdetermined}}$ then it has $\boxed{\text{infinite number of solutions.}}$

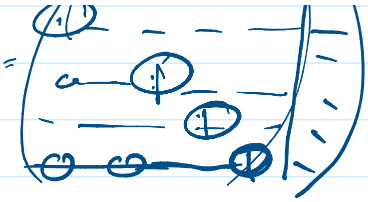
Proof: Given an $m \times n$-system $\boxed{(A|b)}$ and assume it is $\boxed{\text{consistent}}$ and $\boxed{m < n}$ {underdetermined}.

Use G.E.M to get $(U|d)$ in R.E.F.

The system is $\boxed{\text{consistent}}$ $\Rightarrow$ $(U|d)$ has no row of the form $(0\ 0 \cdots 0 \mid c^{\neq 0})$.

Now in $\boxed{(U|d)}$ we have # of leading variables $\le$ # of rows

# of leading variables $\le m < n$

$\boxed{\#\text{ of leading variables}} \leq m < n \ (\text{UA})$ = 

but $m < \boxed{n = \#\text{ of variables.}}$

$\Rightarrow$ # of leading variables $<$ $n$ = # of variables.

$\Rightarrow$ $\boxed{\#\text{ of variables} - \#\text{ of leading variables}} > 0.$

$\Rightarrow$ $\parallel$ # of free variables $> 0$.

$\Rightarrow$ there are free variables so the system has inf. # of solutions.

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- **Homogenuous system:**

An $\boxed{m \times n}$ system is called homogenuous if $b_1 = b_2 = \cdots = b_m = 0$.

$$\left.\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0. \end{cases}\right\} \text{Homog. system.}$$

- **Remark!** $\boxed{x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}$ is a solution to Homog. system.
is called zero solution (trivial solution)

1) Any $\boxed{\text{homogenous}}$ system is consistent.

or

only one solution

has only the zero solution

or

inf # of solutions.

has nonzero solutions {inf. # of solutions}

$x = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

Ex:

$$\boxed{\begin{aligned} -x_1 + x_2 - x_3 + 3x_4 &= 0 \\ 3x_1 + x_2 - x_3 - x_4 &= 0 \\ 2x_1 - x_2 - 2x_3 - x_4 &= 0. \end{aligned}}$$

solve

$3\times 4$ system

Consistent. (Homog.)

and underdetermined.

$\Downarrow$

has inf. # of solutions.

$$\left(\begin{array}{cccc|c} +1 & -1 & +1 & -3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{array}\right) \xrightarrow[-2R_1+R_3]{-3R_1+R_2} \left(\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -4 & 5 & 0 \\ 0 & 4 & -4 & 8 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & 12 & -12 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \begin{array}{l} \to x_1 = \alpha - \alpha + 3\alpha = \alpha \\ \to x_2 = 4\alpha - 5\alpha = \alpha \\ \to x_3 = \alpha \end{array}$$

$$\boxed{\begin{array}{c} \boxed{(0\ 0\ 0\ 0 \mid 1)} \\ 0 = 1 \quad \text{No solution.} \end{array}}$$

consistent, $\boxed{x_4 = \alpha}$ is free.

$\longrightarrow$ the system has inf # of solutions

~~$(0\ 0 \cdots 0 \mid c \neq 0)$~~

Any solution has the form $\boxed{x = \begin{pmatrix} \alpha \\ -\alpha \\ \alpha \\ \vdots \end{pmatrix}}$

$(0\ 0\ 0\ 1 \mid 0) \to x_4 = 0$

Any solution has the form $x = \begin{pmatrix} -\alpha \\ \alpha \\ \alpha \\ \alpha \end{pmatrix}$

$\alpha \in \mathbb{R}$.

$\left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \end{array}\right) \Rightarrow x_4 = 0$

$x_4 = 0$.

(take $\alpha = 0 \Rightarrow x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is a sol.)

$\alpha = 1 \Rightarrow x = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$ is a sol.

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**Def.** A matrix $M$ is called in reduced row echelon form {RREF} if

REF

1) $M$ is in REF.

2) each leading one is the only nonzero element in its column.

Ex. $A = \begin{pmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ REF ✓ in RREF.

$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ REF. ✓ not RREF.

* Any matrix can be transformed into a matrix in RREF. (How?).