Monday, March 1, 2021 10:11 AM

* $M \longrightarrow$ RREF.

Gauss-Elem. Method

$(A|b) \longrightarrow (U|d)$ REF.

* **Gauss-Jordan Elemination Method.**

$$(A|b) \xrightarrow[\text{E.R.O's}]{\text{reduction}} \boxed{(U|d) \text{ in RREF.}}$$

* row of the form $(0\ 0 \cdots 0 | c^{\neq 0}) \Rightarrow$ the system is inconsistent.
* If not $\longrightarrow$ free variables?
  - No free $\longrightarrow$ One solution
  - There are free var. $\longrightarrow$ inf. # of solutions.

Ex:

$$\begin{aligned} -x_1 + x_2 - x_3 &= 1 \\ 3x_1 + x_2 - x_3 &= 0 \\ 2x_1 - x_2 - 2x_3 &= 2. \end{aligned}$$

use **Gauss-Jordan El. Method.**

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & 1 \\ 3 & 1 & -1 & 0 \\ 2 & -1 & -2 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} \boxed{1} & -1 & 1 & -1 \\ 3 & 1 & -1 & 0 \\ 2 & -1 & -2 & 2 \end{array}\right)$$

$$\xrightarrow[-2R_1+R_3]{-3R_1+R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & \boxed{4} & -4 & 3 \\ 0 & 1 & -4 & 4 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & \boxed{1} & -4 & 4 \\ 0 & 4 & -4 & 3 \end{array}\right)$$

$$\xrightarrow{R_2+R_1} \left(\begin{array}{ccc|c} 1 & 0 & -3 & 3 \\ 0 & 1 & -4 & 4 \end{array}\right) \xrightarrow{\frac{1}{12}R_3} \left(\begin{array}{ccc|c} 1 & 0 & -3 & 3 \end{array}\right)$$

$$\xrightarrow[-4R_2+R_3]{R_2+R_1} \left(\begin{array}{ccc|c} 1 & 0 & -3 & 3 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 12 & -13 \end{array}\right) \xrightarrow{\frac{1}{12}R_3} \left(\begin{array}{ccc|c} 1 & 0 & -3 & 3 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & \boxed{1} & \frac{-13}{12} \end{array}\right)$$

$$\xrightarrow[3R_3+R_1]{4R_3+R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{-13}{12} \end{array}\right)$$

in **RREF.**

back subs $\frac{-13}{4}+3$ $\quad \cancel{\frac{-13}{3}+4}$

Consistent! one solution.

$x_3 = \frac{-13}{12}$

$x_2 = -\frac{1}{3}$

$x_1 = -\frac{1}{4}$

solution $x = \begin{pmatrix} -1/4 \\ -1/3 \\ -13/12 \end{pmatrix}$.

3x4 system

Ex: Use G.J.E.M.

$$\begin{aligned} -x_1 + x_2 - x_3 + 3x_4 &= 0 \\ 3x_1 + x_2 - x_3 - x_4 &= 0 \\ 2x_1 - x_2 - 2x_3 - x_4 &= 0. \end{aligned}$$

homg. and underdetermined
$\Downarrow$
has inf # of solutions.

$$\left(\begin{array}{cccc|c} -1 & 1 & -1 & 3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{array}\right)$$

$$\xrightarrow[-2R_1+R_3]{-3R_1+R_2} \left(\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & \boxed{4} & -4 & 8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -4 & 5 & 0 \\ 0 & 4 & -4 & 8 & 0 \end{array}\right)$$

$$\xrightarrow[-4R_2+R_3]{R_2+R_1} \left(\begin{array}{cccc|c} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & \boxed{12} & -12 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & \boxed{1} & -1 & 0 \end{array}\right)$$

$$\begin{pmatrix} 0 & 0 & \boxed{12} & -12 & \big| & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & \boxed{1} & -1 & \big| & 0 \end{pmatrix}$$

$$\xrightarrow[3R_3+R_1]{4R_3+R_2} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \leftarrow \text{RREF} \qquad \begin{array}{l} \to x_1 = \alpha \\ \to x_2 = -\alpha \\ \to x_3 = \alpha \end{array}$$

Consistent, $x_4 = \alpha$ free $\longrightarrow$ inf. # of solutions.

Any solution $x = \begin{pmatrix} \alpha \\ -\alpha \\ \alpha \\ \alpha \end{pmatrix}$, $\alpha \in \mathbb{R}$.

Ex: Consider the system

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ 2x_1 + x_2 - x_3 &= 5 \\ x_1 - x_2 + a x_3 &= b. \end{aligned}$$

① For what values of a, b does the system have no solution

② " " " " a, b " " " " one solution.

③ " " " " a, b " " " " inf. # of sol.

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 2 & 1 & -1 & 5 \\ 1 & -1 & a & b \end{array}\right) \xrightarrow[-R_1+R_3]{-2R_1+R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 3 & -3 & 1 \\ 0 & 0 & \boxed{a-1} & \boxed{b-2} \end{array}\right)$$

1) $\boxed{\text{No solution}}$ (inconsistent) $\iff$ $a-1=0$ and $b-2\neq 0$

$(0 \quad 0 \quad 0 \mid \neq 0)$ $\iff$ $\boxed{a=1 \text{ and } b\neq 2.}$

2) one solution $\iff$ $a-1\neq 0$ $\iff$ $\boxed{a\neq 1.}$, $\boxed{b \text{ any Value.}}$

3) inf. # of solutions $\iff$ $a-1=0$ and $b-2=0$.

$\iff$ $\boxed{a=1 \text{ and } b=2}$

1.2 ✓

1.3 + 1.4 Matrices

Def: An $m \times n$-matrix has the form

$$\underset{m \times n}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

m = # of rows
n = # of columns

1st row $(a_{11} \ a_{12} \ \cdots \ a_{1n})$

1st column $= \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}$

$\vec{a}_i$ = ith row $= (a_{i1} \ a_{i2} \ \cdots \ a_{in})$

a_j = jth column $= \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} \longrightarrow$ column vector.

a_{ij} : element in ith row and jth column.

size of A is $m \times n$.
rows, columns.

Ex: $A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & -1 & \boxed{1} & 3 \end{pmatrix}$ 2×4.

$$\underset{2\times 4}{)}$$

$$a_{\cdot 2} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}_{2\times 1}, \quad \vec{a}_{2} = \begin{pmatrix} 2 & -1 & 1 & 3 \end{pmatrix}_{1\times 4}.$$

$$a_{23} = 1.$$

Operations of Matrices:

1) Equality:

Two matrices A, B are called equal $(A = B)$ if they have the same size and $a_{ij} = b_{ij}$, for all i, j.

Ex. $A = \begin{pmatrix} 2 & 3 & 4 \\ -1 & 0 & 1 \end{pmatrix}_{2\times 3}$, $B = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}_{2\times 3}$

$A = B \iff a = 2,\ b = 3,\ c = 4,\ d = -1,\ e = 0,\ f = 1.$

2) Addition: $(+)$

Let A, B are $m\times n$-matrices (same size), we define

$$A + B = (c_{ij})_{m\times n}, \quad c_{ij} = a_{ij} + b_{ij}.$$

Ex. $$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix}_{2\times 3} + \begin{pmatrix} 2 & -1 & 1 \\ 3 & 4 & -5 \end{pmatrix}_{2\times 3} = \begin{pmatrix} 3 & 1 & 4 \\ 2 & 4 & -4 \end{pmatrix}_{2\times 3}$$

Remark: $$A_{m\times n} - B_{m\times n} = (a_{ij} - b_{ij})_{m\times n}$$

③ Scalar Multiplication:

If $A = (a_{ij})_{\boxed{m\times n}}$ and α is a scalar {real number}

we define $\alpha A = (\alpha a_{ij})_{m\times n}$

Ex: $5\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix}_{2\times 3} = \begin{pmatrix} 5 & 10 & 15 \\ -5 & 0 & 5 \end{pmatrix}_{2\times 3}$.

4) Matrix Multiplication.

If $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{n\times s}$, then we define

$\underset{m\times n}{A}\,\underset{n\times s}{B} = (c_{ij})_{m\times s}$, $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$.

$\vec{a}_i \; b_j$

ith row of A

jth column of B.

Ex: $A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 2 \end{pmatrix}_{2\times 3}$, $B = \begin{pmatrix} 2 & -1 \\ 1 & 3 \\ 4 & 5 \end{pmatrix}_{3\times 2}$.

$\underset{2\times 3}{A}\,\underset{3\times 2}{B} = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 3 \\ 4 & 5 \end{pmatrix}$

$$= \begin{pmatrix} 3 & 2 \\ 10 & 9 \end{pmatrix}$$

2×2

$$\underset{3\times 2}{B}\ \underset{2\times 3}{A} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 2 \end{pmatrix}.$$

$$= \begin{pmatrix} 3 & 6 & -4 \\ 5 & 3 & 5 \\ 13 & 12 & 6 \end{pmatrix}$$

3×3

but $\boxed{AB \neq BA.}$

in general.

$\underset{2\times 3}{A}\ \underset{3\times 4}{B}$ defined

$\underset{3\times 4}{B}\ \underset{2\times 3}{A}$ not defined

$xy = yx.$