Wednesday, March 3, 2021 10:00 AM

Matrices:

operations:

1) $A = B$

2) $\underset{m\times n}{A} \pm \underset{m\times n}{B}$

3) αA

4) $\underset{m\times n}{A}\ \underset{n\times s}{B} = \underset{m\times s}{C}$

$\boxed{AB} \neq \boxed{BA}$ (in general).

Properties: Let A, B, C are matrices such that the indicated operations are defined

1) $\boxed{A+B = B+A}$ {commutative law for addition} $a_{ij}+b_{ij} = b_{ij}+a_{ij}$

2) $\boxed{(A+B)+C} = A+(B+C)$ {associative law for addition)

3) $\boxed{A(BC)} = (AB)C$ (" " " multiplication)

$\underset{m\times n}{A}(\underset{n\times r}{B}\ \underset{r\times s}{C})_{n\times s}$ $\quad$ $(\underset{m\times n}{A}\ \underset{n\times r}{B})_{m\times r}\ \underset{r\times s}{C}$

Remark: $(A+B)+C = B+(A+C)$

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ -1 & 1 \\ 2 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$.

check $A(BC) \stackrel{?}{=} (AB)C$.

$$A(BC) = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix}\left(\begin{pmatrix} 2 & 1 \\ -1 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ -1 & 4 \\ 4 & -2 \end{pmatrix} = \boxed{\begin{pmatrix} 15 & 3 \\ -1 & -3 \end{pmatrix}}$$

$$= \begin{pmatrix} \cdots \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cdots \\ -1 & 4 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} \cdots \\ -1 & -3 \end{pmatrix}$$

$$(AB)C = \left(\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 1 \\ 2 & 0 \end{pmatrix} \right) \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 3 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} = \boxed{\begin{pmatrix} 15 & 3 \\ -1 & -3 \end{pmatrix}}$$

4) $\boxed{\underset{m\times n}{A}(\underset{n\times r}{B} + \underset{n\times r}{C})} = AB + AC$

5) $(A+C)B = AB + CB.$

Distributive law.

$AB \neq BA$

6) $(\alpha\beta)A = \alpha(\beta A)$

7) $\alpha(AB) = (\alpha A)B = A(\alpha B).$

8) $(\alpha+\beta)A = \alpha A + \beta A.$

9) $\alpha(A+B) = \alpha A + \alpha B$

$xy = yx.$

$a(b+c) = ab+ac.$

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Transpose of a matrix.

Def: Let A be $m\times n$-matrix, we define the transpose of A as the matrix C, where

$A^T = C = (c_{ij})_{n\times m}$, where $c_{ij} = a_{ji}$.

Ex. $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix}_{2\times 3}$, $A^T = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 1 \end{pmatrix}_{3\times 2}$

Ex: $A = \begin{pmatrix} 3 & 4 & 5 \\ 4 & -1 & 6 \\ 5 & 6 & 2 \end{pmatrix}_{3\times 3}$ main diagonal

$A^T = \begin{pmatrix} 3 & 4 & 5 \\ 4 & -1 & 6 \\ 5 & 6 & 2 \end{pmatrix} = A$

$B = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & 0 \\ 3 & 4 & 5 \end{pmatrix}_{3\times 3}$

$B^T = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 4 \\ -1 & 0 & 5 \end{pmatrix} \neq B$

B not symmetric.

Def: An $n\times n$-matrix is called symmetric if $A^T = A$.

properties of transpose:

1) $(A^T)^T = A$

2) $(A \pm B)^T = A^T \pm B^T$

3) $(\alpha A)^T = \alpha (A^T)$

4) $\boxed{(AB)^T = B^T A^T}$

$A^T B^T$

$A^T B^T$

Ex: Let A be $\boxed{m\times n\text{-matrix}}$, let $\underbrace{C}_{m\times m} = \underbrace{A}_{m\times n}\underbrace{A^T}_{n\times m}$

Is C symmetric?

$$\boxed{C^T} = \left((A)(A^T)\right)^T = (A^T)^T A^T$$
$$= AA^T = C$$

$\therefore$ C is symmetric

Also A^TA is symmetric $\left\{ (A^TA)^T = A^T \cdot (A^T)^T = A^T \cdot A \right.$.

Linear Systems:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

$$\boxed{(A|b)}$$

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n}$ coefficients matrix.

$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1}$ unknowns, $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}_{m\times 1}$

$$\underset{m\times n}{A}\,\underset{n\times 1}{x} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}_{m\times 1} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = b.$$

so $(*)$ {the $m\times n$-system} can be written as $\boxed{Ax=b}$.

Remark:

$$Ax = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{m1}x_1 \end{pmatrix} + \begin{pmatrix} a_{12}x_2 \\ a_{22}x_2 \\ \vdots \\ a_{m2}x_2 \end{pmatrix} + \cdots + \begin{pmatrix} a_{1n}x_n \\ a_{2n}x_n \\ \vdots \\ a_{mn}x_n \end{pmatrix}$$

$$= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}.$$

$b = Ax = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$, where a_i are the columns of A.

$\boxed{\therefore\ b = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n}$ $\rightarrow$ another method to write $m\times n$-system $Ax=b$.

Ex:

$$\begin{aligned} x_1 + x_2 - x_3 &= 2 \\ -x_1 + x_2 + x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 3. \end{aligned}$$

3×3-system.

$$(A|b): \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ -1 & 1 & 1 & 1 \\ 2 & -1 & 1 & 3 \end{array}\right)$$

~~Gauss~~

$$\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

method for sol.

$$\cancel{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{3\times 1} \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix}_{3\times 3}}$$

$$x_1\begin{pmatrix}1\\-1\\2\end{pmatrix}+x_2\begin{pmatrix}1\\1\\-1\end{pmatrix}+x_3\begin{pmatrix}-1\\1\\1\end{pmatrix}=\begin{pmatrix}2\\1\\3\end{pmatrix}.$$

$$\begin{pmatrix}x_3\end{pmatrix} \quad x_2 \quad -1 \quad 1 \qquad 3\times 1 \qquad 3\times 3$$

$$\begin{pmatrix}1 & 1 & -1\\-1 & 1 & 1\\2 & -1 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

$$\begin{pmatrix}x_1\\-x_1\\2x_1\end{pmatrix}+\begin{pmatrix}x_2\\x_2\\-x_2\end{pmatrix}+\begin{pmatrix}-x_3\\x_3\\x_3\end{pmatrix}=\begin{pmatrix}2\\1\\3\end{pmatrix}$$

$$\begin{pmatrix}x_1+x_2-x_3\\-x_1+x_2+x_3\\2x_1-x_2+x_3\end{pmatrix}=\begin{pmatrix}2\\1\\3\end{pmatrix}$$

Remark: An $m\times n$-system can be written in the following forms:

1) $\boxed{(A|b)}$ ← augmented matrix.

2) $\boxed{A\underline{x}=\underline{b}.}$ ⟶ matrix form

3) $\boxed{x_1a_1+x_2a_2+\cdots+x_na_n=b}$, where a_i : columns of A. { columns of A form }.

Ex: Let $a=\begin{pmatrix}1\\2\\3\end{pmatrix}$, $b=\begin{pmatrix}3\\4\\5\end{pmatrix}$, $c=\begin{pmatrix}2\\-1\\1\end{pmatrix}$. { columns }

find $3a-4b+5c=\begin{pmatrix}3\\6\\9\end{pmatrix}-\begin{pmatrix}12\\16\\20\end{pmatrix}+\begin{pmatrix}10\\-5\\5\end{pmatrix}=\begin{pmatrix}1\\-15\\-6\end{pmatrix}.$

$\begin{pmatrix}1\\-15\\-6\end{pmatrix}=\boxed{3\underline{a}-4\underline{b}+5\underline{c}.}$

$\boxed{\text{linear combination of } a, b, c.}$

linear combination of a, b, c.