Wednesday, March 10, 2021 10:02 AM

$$x \in \mathbb{R}^n \iff x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n \times 1}$$

Def: Let $a_1, a_2, \ldots, a_k$ be column vectors in $\boxed{\mathbb{R}^n}$

a sum of the form $\boxed{\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_k a_k}$ is called

a linear combination of $a_1, a_2, \ldots, a_k$.

Ex: $a_1 = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$, $a_2 = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$, $a_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^3$.

1) find a linear combination of a_1, a_2, a_3

L.c.: $2a_1 + 3a_2 - 4a_3 = 2\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + 3\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} - 4\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 15 \\ 18 \\ 0 \end{pmatrix}$

$\therefore \begin{pmatrix} 15 \\ 18 \\ 0 \end{pmatrix}$ is a linear combination of a_1, a_2, a_3.

2) Is $\begin{pmatrix} 3 \\ 5 \\ 9 \end{pmatrix}$ a linear comb. of a_1, a_2, a_3? Yes / No

$\iff$ solve $\begin{pmatrix} 3 \\ 5 \\ 9 \end{pmatrix} = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3$

solve the system $\begin{pmatrix} 3 \\ 5 \\ 9 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \alpha_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. consistent or inconsistent.

matrix form

$$\begin{pmatrix} 3 \\ 5 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 4 & 0 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \quad (*)$$

$$\begin{pmatrix}3\\5\\9\end{pmatrix}$$ is a linear comb of a_1, a_2, a_3 $\iff$ the system (*) is consistent.

Remark ① Theorem: The system $Ax=b$ is consistent if and only if b can be written as a linear combination of columns of A.

② $x=\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}$ is a solution to $Ax=b \iff$

$$b = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n.$$

Ex: Let A be 5×3 matrix and $\boxed{b = a_1 + a_2} = a_2 + a_3$.

what can we conclude about the number of solutions of $Ax=b$, $\{a_1, a_2, a_3$ columns of $A\}$. $\boxed{\in \mathbb{R}^5}$

1) $Ax=b$ consistent ?

since $\boxed{b = a_1 + a_2 = \boxed{1\cdot a_1 + 1\cdot a_2 + 0\cdot a_3}} \Rightarrow x=\begin{pmatrix}1\\1\\0\end{pmatrix}$ is a solution to $Ax=b$

$\Rightarrow$ so $Ax=b$ is consistent.

Now since $b = a_2 + a_3 = 0\cdot a_1 + 1\, a_2 + 1\, a_3$. $\Rightarrow$ $y=\begin{pmatrix}0\\1\\1\end{pmatrix}$ is a solution to $Ax=b$.

so $Ax=b$ has infinite # of solutions

so $Ax=b$ has infinite # of solutions.

Check! $\begin{pmatrix}2\\2\\0\end{pmatrix}=2x$ solution to $Ax=b$. ?

we know $Ax=b$.

$$A\begin{pmatrix}2\\2\\0\end{pmatrix}=A(2x)=2(Ax)=2b\neq b$$

so $\begin{pmatrix}2\\2\\0\end{pmatrix}$ is not a solution.

Another solution is
$$\frac{1}{4}\begin{pmatrix}1\\1\\0\end{pmatrix}+\frac{3}{4}\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}1/4\\1\\3/4\end{pmatrix}$$

$\alpha\begin{pmatrix}1\\1\\0\end{pmatrix}+\beta\begin{pmatrix}0\\1\\1\end{pmatrix}$ with $\alpha+\beta=1$.

If y and z are solutions to $\boxed{Ax=b}$, $b\neq 0$

$\Rightarrow Ay=b$, $Az=b$

1) Is $(y+z)$ a solution to $Ax=b$. $\boxed{b\neq 0}$

Consider $A(y+z)=Ay+Az=b+b=2b\neq b$

$\Rightarrow y+z$ is not a solution to $Ax=b$.

2) Is $\frac{1}{4}y+\frac{3}{4}z$ a solution to $Ax=b$

Consider $A(\frac{1}{4}y+\frac{3}{4}z)=A(\frac{1}{4}y)+A(\frac{3}{4}z)$

$$=\frac{1}{4}Ay+\frac{3}{4}Az=\frac{1}{4}b+\frac{3}{4}b=b$$

$\therefore$ $\frac{1}{4}y+\frac{3}{4}z$ is a solution to $Ax=b$.

3) $\alpha y+\beta z$ is a solution $\Leftrightarrow$ $\alpha+\beta=1$.

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* If y, z are solutions to $Ax=0$:

* If $\underline{y, z}$ are solutions to $Ax=0$:
$\Rightarrow Ay=0, Az=0$

1) Is $y+z$ a solution to $Ax=0$.
consider $A(y+z) = Ay + Az = 0+0=0$

so $y+z$ is a solution to $Ax=0$.

2) Is $\alpha y+\beta z$ a solution to $Ax=0$

$$A(\alpha y+\beta z) = \alpha Ay + \beta Az = \alpha(0)+\beta(0)=0.$$

$\therefore$ $\alpha y+\beta z$ is a solution to $Ax=0$
for any $\alpha,\beta \in \mathbb{R}$.

* If y is a solution to $\boxed{Ax=b}$, z is a solution to $\boxed{Ax=0}$.

$y+z$ solution to ?

consider $A(y+z) = \underline{Ay} + \underline{Az} = b+0=b$.

so $y+z$ is a sol. to $Ax=b$.

Q 12) 1.3

$A_{3\times 4}$, $\boxed{b = a_1+a_2-a_3+a_4.}$

$Ax=b$?

$\therefore$ $x=\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}$ is a solution to $Ax=b$.

so $Ax=b$ is $\boxed{\text{consistent.}}$

and since $Ax=b$ is underdetermined, so it has inf. # of solutions.

1.3 + 1.4 Some special Matrices:

*) zero matrix $O = (0)_{m\times n}$

$$O = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

real numbers

$\cdot, +, -,$

0 (zero)

$0+x=x$

$0\cdot x=0$

1) $\underset{m\times n}{A} + \underset{m\times n}{O} = \underset{m\times n}{A}$.

2) $\underset{m\times n}{O}\ \underset{n\times r}{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nr} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{pmatrix}$

$\therefore\ OA = O$

Also $BO = O$

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3) let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq O$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \neq O$

but $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\boxed{AB = O}$, $A \neq O$, $B \neq O$.

real numbers

$xy=0$

$\Downarrow$

$x=0$ or $y=0$.

$x\neq 0$, $y\neq 0$ and $\boxed{xy=0}$ not possible.

(If A, B matrices, and $AB = O$ ~~$\Rightarrow$~~ $A=O$ or $B=O$)

$xy=0$

$\Downarrow$

$A=0$ or $B=0$

* Identity matrix $= I_n = (\alpha_{ij})_{n\times n}$ $n\times n$-matrix

where $\alpha_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j. \end{cases}$

$$I_n = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}_{n\times n}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

real numbers:

$1 \in \mathbb{R}$

*) $1.x = x = x.1$

*) for any $x \neq 0$,

$x\left(\frac{1}{x}\right) = 1.$

$\frac{1}{x} \in \mathbb{R}$

$\frac{1}{x}$: multiplicative inverse of x.

1) Let A be $m\times n$-matrix, then

$$A_{m\times n} I_n = A. \quad \text{and} \quad I_m A_{m\times n} = A.$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \\ 0 & 0 & & \cdots 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & & \cdots & a_{mn} \end{pmatrix}$$

*) Given $A = (a_{ij})_{n\times n}$. Is there a matrix $B = (a_{ij})_{n\times n}$ s.t. $AB = I = BA$?

$x \neq 0$, there exists $\frac{1}{x} \in \mathbb{R}$ s.t. $x \cdot \left(\frac{1}{x}\right) = 1.$

$x \cdot \left(\frac{1}{x}\right) = 1$

$\left(\frac{1}{x}\right) x$

Def: Let $A = (a_{ij})_{n\times n}$, we say A is nonsingular (invertible) if there exists a matrix $B = (b_{ij})_{n \times n}$ such that $\boxed{AB = I = BA.}$

(B (if exists) is called an inverse of A)

Ex: $\boxed{A = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}}$, $B = \begin{pmatrix} -1/10 & 2/5 \\ 3/10 & -1/5 \end{pmatrix}$.

Is B an inverse of A?

consider $AB = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} -1/10 & 2/5 \\ 3/10 & -1/5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$

and check $\boxed{BA} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$

In general $AB \neq BA$

So B is an inverse of A.

and so A is nonsingular. (has an inverse).

Is A an inverse of B? $\Longleftarrow\!\!\Longrightarrow$ $AB = I = BA$.

$\boxed{\begin{array}{l} \text{If } C, D \text{ are } n\times n\text{-matrices and } CD = I = DC \\ \text{① } C \text{ and } D \text{ are nonsingular. and} \\ \text{② inverse of } C \text{ is } D \text{ and inverse of } D \text{ is } C. \end{array}}$

⊛ If C, D are $n\times n$-matrices and $CD = I = DC$

① C and D are nonsingular. and
② inverse of C is D and inverse of D is C.

⊛ Ex: $A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$. Is A nonsingular.

try to find $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ s.t. $AB = I = BA$?