Monday, March 15, 2021 9:55 AM

Ex: $A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$. $\boxed{\text{Is } A \text{ nonsingular.}}$

try to find $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ s.t. $\boxed{AB = I} = BA$?

$$AB = I \Longleftrightarrow \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\Rightarrow \begin{pmatrix} 2b_1 + b_3 & 2b_2 + b_4 \\ 6b_1 + 3b_3 & 6b_2 + 3b_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\begin{cases} 2b_1 + b_3 = 1 \\ 2b_2 + b_4 = 0 \\ 6b_1 + 3b_3 = 0 \\ 6b_2 + 3b_4 = 1. \end{cases}$$

4x4 - system.

If inconsistent $\Rightarrow$ A is singular.

If consistent $\Rightarrow$ A is nonsingular.

$$\left(\begin{array}{cccc|c} 2 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \\ 6 & 0 & 3 & 0 & 0 \\ 0 & 6 & 0 & 3 & 1 \end{array}\right) \xrightarrow{-3R_1 + R_3} \left(\begin{array}{cccc|c} 2 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 6 & 0 & 3 & 1 \end{array}\right) \to$$

The system is inconsistent

$$\boxed{\text{so } A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix} \text{ is singular}} \text{ (has no inverse)}$$

Remark: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $ad - bc \neq 0$

then A is nonsingular and inverse of A is $\boxed{\dfrac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$

then $A$ is nonsingular and inverse of $A$ is $\boxed{\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$

{ If $ad-bc=0$, then $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is singular }.

If $\boxed{ad-bc\neq 0}$

consider $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} =$

$$= \begin{pmatrix} \frac{ad}{ad-bc} - \frac{bc}{ad-bc} & \frac{-ab}{ad-bc} + \frac{ab}{ad-bc} \\ \frac{cd}{ad-bc} - \frac{dc}{ad-bc} & \frac{-bc}{ad-bc} + \frac{ad}{ad-bc} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{ad-bc}{ad-bc} & \frac{ab-ab}{ad-bc} \\ \frac{cd-dc}{ad-bc} & \frac{ad-bc}{ad-bc} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$= I$.

Also $\begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = I$.

Ex: $A=\begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix}$ nonsingular ?

$ad-bc = 2\times 5 - 0\times 3 = 10 \neq 0$

$\begin{pmatrix} 2 & 3 \end{pmatrix}$

$ad-bc = 2\times 5 - 0\times 3 = 10 \neq 0$

$\therefore A = \begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix}$ is nonsingular

and inverse of $A$ is $\frac{1}{10}\begin{pmatrix} 5 & -3 \\ 0 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} \frac{5}{10} & \frac{-3}{10} \\ 0 & \frac{2}{10} \end{pmatrix}}$

Remarks:

① If $\underset{n\times n}{A}$ is nonsingular (has an inverse), then

$\boxed{\text{inverse of } A \text{ is unique.}}$

Proof: Assume $A$ is nonsingular (has inverse)

to show inverse of $A$ is unique assume $B$ and $C$ are both inverses of $A$. $\{\boxed{\text{show } B = C}\}$

$B$ is inverse of $A \implies \boxed{AB = I} = BA$. —— (1)

$C$ " " " $A \implies AC = \boxed{I = CA}$ —— (2)

Now $B = IB = (CA)B = C(AB) = CI = C$

$\therefore B = C$.

so inverse is unique.

* Notation: If $A$ is nonsingular (inverse of $A$ exists)

we denote it by $\boxed{A^{-1}}$

$\left( A A^{-1} = I = A^{-1} A \right)$

AB

Theorem

Theorem
Remark 2: If $A, B$ are nonsingular $n\times n$-matrices, then $AB$ is nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$

Proof: Assume $A, B$ are nonsingular. ($A^{-1}$ exists, $B^{-1}$ exists)

consider $\underbrace{\boxed{(AB)}}_{C}\underbrace{(B^{-1}A^{-1})}_{D} = A(BB^{-1})A^{-1}$

$= AIA^{-1} = AA^{-1} = I$

Also $(B^{-1}A^{-1})(AB) = I$

so $AB$ is nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$.

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Remark 3: If $A_1, A_2, \dots, A_k$ are nonsingular $n\times n$-matrices, then $A_1A_2\cdots A_k$ is nonsingular and $(A_1A_2\cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}$

* $((A_1A_2\cdots A_{k-1})A_k)^{-1} = A_k^{-1}((A_1\cdots)A_{k-1})^{-1}$

$= A_k^{-1}[A_{k-1}^{-1}(A_1\cdots A_{k-2})^{-1}$

$\vdots$

$= A_k^{-1}A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}$.

Remark 4: Notation: If $A$ is $n\times n$-matrix, then

$\underbrace{A\cdot A\cdots A}_{} := A^k, \quad k\in\mathbb{Z}^+.$

$$\underbrace{A \cdot A \cdots A}_{k\text{-times}} := A^k, \quad k \in \mathbb{Z}^+.$$

True or False?

1) If $A$, $B$ are $\boxed{\text{nonsingular}}$ $n\times n$-matrices, then $A+B$ is nonsingular. (False).

Ex: $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ nonsingular.

$I + I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ nonsingular.

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}$ nonsingular

$I_n$ nonsingular $\boxed{I \cdot I = I}$

$\therefore \; I_n^{-1} = I_n$

Ex: $A$, $B$ nonsingular and $A+B$ is singular

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ are nonsingular.

$A + B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is singular

$O_{n\times n}$ zero matrix is singular.

$OB = O \neq I$

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ nonsingular

$A + B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is singular

2) The sum of two $\boxed{\text{singular matrices}}$ is singular? (If $A$, $B$ are singular, then $A+B$ is singular) False

Ex: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are singular but $A+B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is nonsingular.

③ If $A$ is singular, $B$ is nonsingular, then $A+B$ is singular

False Ex:

④ If $A$ is singular, $B$ is nonsingular, then $A+B$ is nonsingular (False)

Ex:

$\{ x^2-y^2 = (x-y)(x+y) = x^2 - \cancel{xy} + \cancel{yx} - y^2$

⑤ If $A, B$ are $n \times n$-matrices, then $A^2 - B^2 = (A-B)(A+B)$.

(False)

Consider $(A-B)(A+B) = A^2 + \boxed{AB - BA} - B^2$

⑥ If $A, B$ are $n \times n$-matrices, $\boxed{AB = 0}$, then $A=0$ or $B=0$.

(False)

Ex: ⓐ $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq 0$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \neq 0$ but $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

• $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \neq 0$, $B = \begin{pmatrix} 6 & 6 \\ -3 & -3 \end{pmatrix} \neq 0$

$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

⑦ If $A^2 = 0$, then $A = 0$ (False)

7) If $A^2=0$, then $A=0$ (False)

Ex. $A = \begin{pmatrix} & \\ & \end{pmatrix}$

8) If $AB=AC$, then $B=C$. (False).

(~~True~~: multiply by $A^{-1}$ from left $\Rightarrow (A^{-1}A)B=(A^{-1}A)C$

$\Rightarrow IB=IC$

$\Rightarrow B=C$.

NEED NOT exists

$0.2 = 0.4$

$(\frac{1}{0} \cdot 0) \times 2 = \frac{1}{0} \cdot 0 \cdot 4$

$2 = 4$

9) If $A$ is nonsingular and $AB=AC$, then $B=C$ (True)

$A^{-1}$ exists: so $A^{-1}(AB)=A^{-1}(AC)$

$\vdots$

$B=C$ ✓

$A^{-1}$ exists?

10) If $A$ is $n\times n$-matrix and $A^2=A$ then $A=I$ (False)

11) If $A$ is $n\times n$-matrix and $A^2=A$, then $A=0$ or $A=I$.

$A^2=A \Rightarrow A^2-A=0$

$\boxed{A(A-I)=0}$

number

$x^2=x$

$x^2-x=0$

$x(x-1)=0$

$x=0$ or $x=1$

$\cancel{\Rightarrow}$ $A=0$ or $A-I=0$

$\to A=I$

$AB=0$

$\cancel{\Rightarrow}$ $A=0$ or $B=0$

Ex: $\boxed{A=\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}}$, $A^2=A$.

but $A\neq 0$, $A\neq I$

$0^2=0$

$I^2=I$

12) If $A_{n\times n}$ such that $A^2=A$, then $(A+I)$ is nonsingular and $(A+I)^{-1}=I-\frac{1}{2}A$. (True).

Proof:

13) If $A_{n\times n}$, $A^2=0$, then $I-A$ is nonsingular and $(I-A)^{-1}=I+A$. (True)

Proof: