Wednesday, March 17, 2021 9:56 AM

12) If $A_{n\times n}$ such that $\boxed{A^2 = A}$, then $\boxed{(A+I)}$ is nonsingular and $(A+I)^{-1} = I - \frac{1}{2}A$. (True).

Proof:

Consider $(A+I)(I - \frac{1}{2}A) = A - \frac{1}{2}A^2 + I - \frac{1}{2}A$

$= A - \frac{1}{2}A - \frac{1}{2}A + I$

$= A - A + I = 0 + I = I$

Also $(I - \frac{1}{2}A)(A+I) = \dots = I$.

So $(A+I)$ is nonsingular and $(A+I)^{-1} = I - \frac{1}{2}A$.

13) If $A_{n\times n}$, $\boxed{A^2 = 0}$, then $I - A$ is nonsingular and $(I-A)^{-1} = \boxed{I + A}$. (True)

Proof:

Consider $\boxed{(I-A)}(I+A) = I + A - A - A^2$

$= I + 0 - 0 = I$.

Also $(I+A)(I-A) = I$.

So $I - A$ is nonsingular and $(I-A)^{-1} = I + A$.

$A = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}_{2\times 2} ?$

$\neq 0$

$A^2 = 0$

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$

$A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$(0 \quad 0)$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So $I-A$ is nonsingular and $(I-A)^{-1} = I+A$.

(so $I+A$ is nonsingular and $(I+A)^{-1} = I-A$)

* If A is nonsingular, then A^{-1} is nonsingular and $(A^{-1})^{-1} = A$.

Proof: Assume A is nonsingular.

Consider $(A^{-1})(A) = \boxed{I} = A\,A^{-1}$

so A^{-1} is nonsingular and $(A^{-1})^{-1} = A$.

* $A_{n\times n}$: Is A is nonsingular? If yes, find A^{-1}

$A_{2\times 2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\boxed{\alpha = ad - cb}$

If $\alpha \neq 0 \Rightarrow$ A is nonsingular and $A^{-1} = \frac{1}{\alpha}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

If $\alpha = 0 \Rightarrow A$ is singular.

1.5 Elementary matrices:

Def: An $n\times n$-matrix E is called elementary if E can be obtained from I_n by applying one $\boxed{\text{row operation on } I}$ once.

Ex: ① $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \xleftarrow{R_2 \leftrightarrow R_3} I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (row operation I.)

$\longrightarrow$ is elementary of type I.

$\begin{matrix} 0 & 1 & 0 \end{matrix}$ → is elementary of type I.

3 Types of elementary matrices.

② $E = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xleftarrow{\boxed{-2R_3+R_1}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ row operation III

is elementary of type III.

③ $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -3 & 0 & 1 \end{pmatrix} \longleftarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

not elementary

$E = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ not elementary.

$\not\uparrow$

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

row operations:

① row operation I: $R_i \longleftrightarrow R_j$

⇓

type I.

② row operation II: αR_i, $\alpha \neq 0$

⇓

type II.

③ row operation III: $cR_i + R_j$.

⇓

type III.

Ex: $E = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xleftarrow{\boxed{2R_1+R_2}} I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

is elementary of type III.

→ row operation related to E is $\boxed{2R_1+R_2}$

Ex: $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$ ← elementary of type III $(-3R_3+R_2)$

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow$

(column operation of E: $-3C_2+C_3$.

① $EA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}-3a_{31} & a_{22}-3a_{32} & a_{23}-3a_{33} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$-3R_3+R_2$

② $AE = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & -3a_{12}+a_{13} \\ a_{21} & a_{22} & -3a_{22}+a_{23} \\ a_{31} & a_{32} & -3a_{32}+a_{33} \end{pmatrix}$

column operations

$-3C_2+C_3$

Remark: 1) left multiplication of a matrix A by an elementary matrix E $\{EA\}$ has the same effect as performing the row operation of E on A.

2) right multiplication of a matrix A by an elementary matrix E $\{AE\}$ has the same effect as performing the column operation of E on A.

Ex: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \boxed{\begin{pmatrix} 7 & 8 & 9 \\ 2 & 4 & 3 \\ -1 & 5 & -2 \end{pmatrix}} = \begin{pmatrix} 7 & 8 & 9 \\ 10 & 20 & 15 \\ -1 & 5 & -2 \end{pmatrix}$

elementary of type II. $5R_2$ $\quad$ $\xrightarrow{5R_2}$

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* I_n nonsingular. $(I^{-1} = I)$

E elementary $\quad$ Is E nonsingular?
If yes, E^{-1}??

Theorem: If E is an elementary matrix, then E is nonsingular and E^{-1} is elementary of the same type as E.

Proof: Let E be elementary.

① If E is of type I : $(R_i \leftrightarrow R_j)$.

$I \xrightarrow{R_i \leftrightarrow R_j} E$

$E \xrightarrow{R_i \leftrightarrow R_j} I$

consider $\boxed{EE = I}$

$\therefore$ E is nonsingular and $\boxed{E^{-1} = E}$ (of the same type I)

Ex: $E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\therefore$ $\boxed{\text{elementary of type I}}$ $(R_1 \leftrightarrow R_2)$

$EE = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \boxed{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$

elementary of type I $\Rightarrow$ E is nonsingular.

$\to$ elementary of type I $\Rightarrow$ E is nonsingular.

and $E^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

② If E is of type II (αR_i : $\alpha \neq 0$). $\left\{ I \xrightarrow{\alpha R_i} E \right\}$

let F be the matrix obtained from I by the row operation $\frac{1}{\alpha} R_i$.

F is elementary of type II.

$$F = \begin{array}{c} \\ \text{row } i \\ \\ \end{array} \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ 0 \cdots 0 & & \frac{1}{\alpha} & & 0 \cdots 0 \\ & & & 1 & \\ 0 & & & & \ddots \end{pmatrix}$$

consider $\underbrace{EF}_{\alpha R_i} = I$

also $\underbrace{FE}_{\frac{1}{\alpha}R_i} = I$.

elem.

$$E = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & \alpha & \\ 0 \cdots 0 & & & 1 \end{pmatrix}$$

so $EF = I = FE$.

$\therefore$ E is nonsingular and $E^{-1} = F$ (elementary of same type)

Ex: $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ elementary of type II ($-3R_2$)

$\therefore$ E is nonsingular.

and $E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

③ If E is elementary of type III. $\left[I \xrightarrow{cR_i + R_j} E \right]$.

② If E is elementary of type III. $\left[\ I \xrightarrow{\ cR_i+R_j\ } E\ \right]$.

$$E = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 \cdots & 1 & 0 & \cdots & 0 \\ 0 & c & 0 & 1 & \\ 0 & \cdots & & 0 & 1 \end{pmatrix}\begin{array}{l} \\ \text{row } i \\ \text{row } j \\ \\ \end{array} \longleftarrow I = \begin{pmatrix} 1 & 0 & & 0 \\ 0 & \boxed{1} & 0 & \\ & & \ddots & 0 \\ & 0 & \boxed{1} & \\ 0 & \cdots & 0 & 1 \end{pmatrix}\begin{array}{l} \\ \text{row } i \\ \\ \text{row } j \\ \\ \end{array}$$

Let F be the matrix obtained from I by $\boxed{-cR_i+R_j}$.

$$F = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ & \boxed{-c} & & \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$ elementary of type III.

Consider $\underbrace{E\,F}_{} = I$ $\quad F \xrightarrow{cR_i+R_j} I$ and $F\,E = I$ $\quad E \xrightarrow{-cR_i+R_j} I$

$\therefore$ E is nonsingular and $E^{-1} = F$ (elementary of same type)

Ex. $E = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ elem. of type III $\{2R_3+R_1\}$

$\therefore$ E is nonsingular and $E^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Ex. $E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xleftarrow{-5R_2+R_3} I$

elem. of type III

$$\begin{pmatrix} 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ elem. of type III

$\therefore$ E is nonsingular and $E^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

elem. of type III