

12) If  $A_{n \times n}$  such that  $A^2 = A$ , then  $(A+I)$  is nonsingular and  $(A+I)^{-1} = I - \frac{1}{2}A$ . (True).

Proof:

$$\begin{aligned} \text{Consider } (A+I)(I - \frac{1}{2}A) &= A - \frac{1}{2}A^2 + I - \frac{1}{2}A \\ &= A - \frac{1}{2}A - \frac{1}{2}A + I \\ &= A - A + I = 0 + I = \underline{I} \end{aligned}$$

Also  $(I - \frac{1}{2}A)(A+I) = \dots = I$ .

So  $(A+I)$  is nonsingular and  $(A+I)^{-1} = I - \frac{1}{2}A$ .

13) If  $A_{n \times n}$ ,  $A^2 = 0$ , then  $(I-A)$  is nonsingular and  $(I-A)^{-1} = I+A$ . (True)

Proof:

$$\begin{aligned} \text{consider } (I-A)(I+A) &= I + A - A - A^2 \\ &= I + 0 - 0 = \underline{I} \end{aligned}$$

Also  $(I+A)(I-A) = I$ .

So  $I-A$  is nonsingular and  $(I-A)^{-1} = I+A$ .

$A = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \neq 0$   
 $A^2 = 0$

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$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$

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$A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

So  $I-A$  is nonsingular and  $(I-A)^{-1} = I+A$ .

(so  $I+A$  is nonsingular and  $(I+A)^{-1} = I-A$ )

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

\* If  $A$  is nonsingular, then  $\bar{A}^{-1}$  is nonsingular and  $(\bar{A}^{-1})^{-1} = A$ .

Proof: Assume  $A$  is nonsingular.

Consider  $(\bar{A}^{-1})(A) = \boxed{I} = A \bar{A}^{-1}$

so  $\bar{A}^{-1}$  is nonsingular and  $(\bar{A}^{-1})^{-1} = A$ .

* $A_{n \times n}$	Is $A$ nonsingular? If yes, find $\bar{A}^{-1}$	$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\Delta = ad - bc$ <p>If <math>\Delta \neq 0 \Rightarrow</math>  <math>A</math> is nonsingular          and <math>\bar{A}^{-1} = \frac{1}{\Delta} \begin{pmatrix} d &amp; -b \\ -c &amp; a \end{pmatrix}</math></p> <p>If <math>\Delta = 0 \Rightarrow A</math> is singular.</p>
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### 1.5 Elementary matrices:

Def: An  $n \times n$ -matrix  $E$  is called elementary if

$E$  can be obtained from  $I_n$  by applying one row operation on  $I$  once.

Ex: ①  $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$   $\xleftarrow{R_2 \leftrightarrow R_3}$   $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  is elementary of type I.

1 0 0 / 0 1 0 / 0 0 1 is elementary of type I.

3 Types of elementary matrices.

②  $E = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$-2R_3 + R_1$

row operation II

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

is elementary of type II.

③  $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -3 & 0 & 1 \end{pmatrix}$

not elementary

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$E = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

not elementary.

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

row operations:

① row operation I:  $R_i \leftrightarrow R_j$

type I

② row operation II:  $\alpha R_i, \alpha \neq 0$

type II

③ row operation III:  $\alpha R_i + R_j$

type III

Ex:  $E = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$2R_1 + R_2$

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

is elementary of type III.

row operation related to E is

$2R_1 + R_2$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Ex:  $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$  ← elementary of type III  $(-3R_2 + R_1)$   
 $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$  (column operation of  $E: -3C_2 + C_3$ )

①  $EA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} - 3a_{31} & a_{22} - 3a_{32} & a_{23} - 3a_{33} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

②  $AE = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & -3a_{12} + a_{13} \\ a_{21} & a_{22} & -3a_{22} + a_{23} \\ a_{31} & a_{32} & -3a_{32} + a_{33} \end{pmatrix}$   
 column operations  $-3C_2 + C_3$

Remark: 1) left multiplication of a matrix  $A$  by an elementary matrix  $E$   $\{EA\}$  has the same effect as performing the row operation of  $E$  on  $A$ .

2) right multiplication of a matrix  $A$  by an elementary matrix  $E$   $\{AE\}$  has the same effect as performing the column operation of  $E$  on  $A$ .

Ex: 
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 7 & 8 & 9 \\ 2 & 4 & 3 \\ -1 & 5 & -2 \end{pmatrix} = \begin{pmatrix} 7 & 8 & 9 \\ 10 & 20 & 15 \\ -1 & 5 & -2 \end{pmatrix}$$

elementary of type II.  $5R_2$

\*  $I_n$  nonsingular. ( $I^{-1} = I$ )

$E$  elementary Is  $E$  nonsingular?  
If yes,  $E^{-1}$ ??

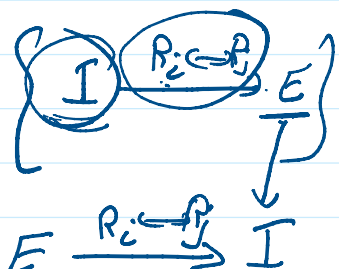
Theorem: If  $E$  is an elementary matrix, then  $E$  is nonsingular and  $E^{-1}$  is elementary of the same type as  $E$ .

Proof: Let  $E$  be elementary.

① If  $E$  is of type I: ( $R_i \leftrightarrow R_j$ ).

consider  $EE = I$

$\therefore E$  is nonsingular and  $E^{-1} = E$  (of the same type I)



Ex:  $E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

elementary of type I ( $R_1 \leftrightarrow R_2$ )

$$EE = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

elementary of type I  $\Rightarrow E$  is nonsingular.

elementary of type I  $\Rightarrow E$  is nonsingular.

$$\text{ad } E^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

② If  $E$  is of type II ( $\alpha R_i, \alpha \neq 0$ ).  $\{I \xrightarrow{\alpha R_i} E\}$

let  $F$  be the matrix obtained from  $I$  by the row operation  $\frac{1}{\alpha} R_i$ .

$F$  is elementary of type II.

consider  $EF = I$

also  $FE = I$

so  $EF = I = FE$

$\therefore E$  is nonsingular and  $E^{-1} = F$  (elementary of same type)

Ex:

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

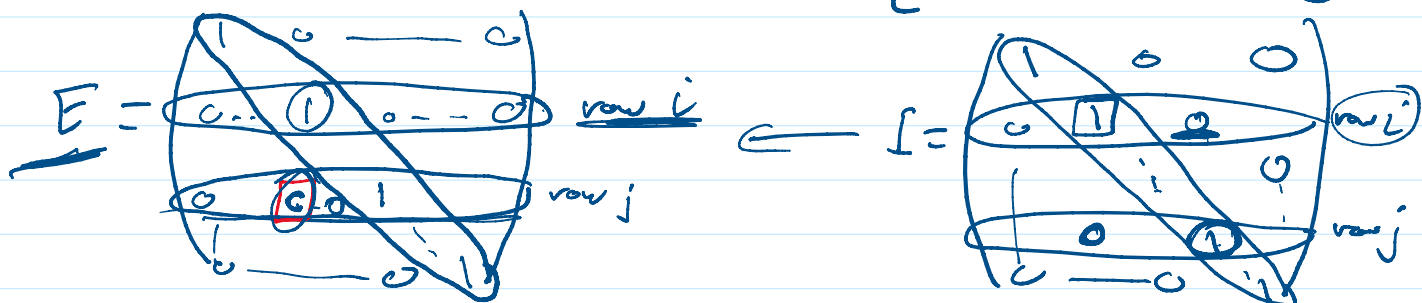
elementary of type II ( $-3R_2$ )

$\therefore E$  is nonsingular.

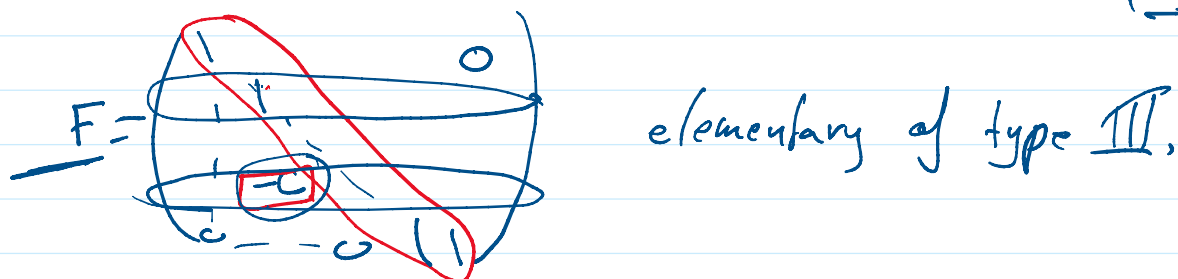
$$\text{ad } E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

② If  $E$  is elementary of type III.  $\{I \xrightarrow{R_i + R_j} E\}$ .

② If  $E$  is elementary of type III.  $[I \xrightarrow{cR_i}, E]$ .



Let  $F$  be the matrix obtained from  $I$  by  $\boxed{-cR_i + R_j}$ .

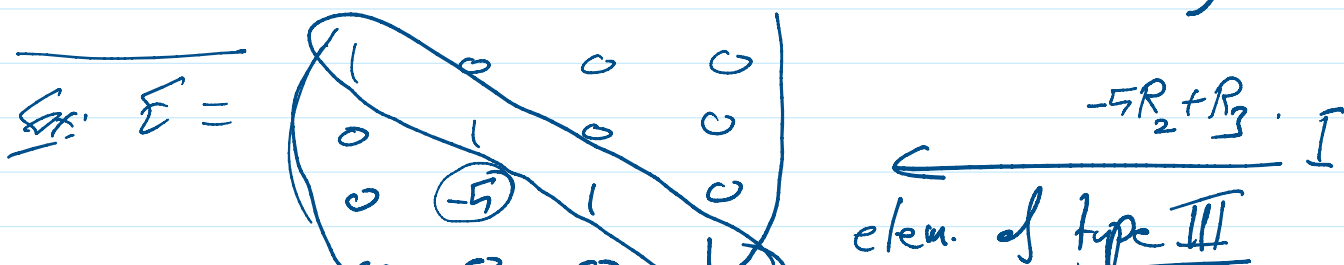


Consider  $\underbrace{EF}_c = I$  and  $\underbrace{FE}_{-cR_i + R_j} = I$ .

$\therefore E$  is nonsingular and  $E^{-1} = F$  (elementary of same type)

Ex.  $E = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  elem. of type III  $\left\{ \begin{matrix} 2R_3 + R_1 \end{matrix} \right\}$

$\therefore E$  is nonsingular and  $E^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ .



elem. of type III

$$\begin{pmatrix} 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

elem. of type III

$\therefore E$  is nonsingular and

$$E^{-1} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

elem. of type III