جامعة بيرزيت
BIRZEIT UNIVERSITY

Mathematics Department
MATH 2351

Semester 1201

Lecture Notes

Prepared by Mohammad Madiah

(Reference: Mathematical Applications for the Management, Life, and Social Sciences by Hershberger and Reynolds. International Edition)

Lecture # 1

Section 1.6 page 106

Linear Models: Cost, Revenue, Profit, and Break Even

Review

For more details see your text book (sections 122 to 1.5). Be sure know how to find a straight line equation, and how to solve a system of two linear equations simultaneously.

A function f is a **rule** from a set A to a set B that assigns to each point x in A a unique element **y = f(x)** in B**.**

- x is called the **independent** variable – input –
- y is called the **dependent** variable – output –
- The set of all values of inputs is called the **domain**.
- The set of all values of outputs is called the **range**.
- A function may be defined as a set of ordered pairs (x, f(x)).

For example, if **A** is a circle area and **r** is its radius, then the area A (the dependent variable) is a function of r (the independent variable).

$$A(r) = \pi r^2$$

Linear Functions

A function of the **form** $\boxed{y = f(x) = mx + b}$ is called a **linear function**, m and b are constants. Where

- m is called the slope $(m = \frac{\text{change in y}}{\text{change in x}})$
- b is called the y – **intercepts**.
- **Intercepts**: To graph a linear function find x and y – intercepts then join both points.
  - x – Intercept :(x, 0)
    Solve for x, f(x) = 0
  - y – Intercept :(0, f(x)) = (0, b).
    Find f (0).
- The **graph** of a linear function is called a **straight line.**
- A straight line can be graphed by joining any **two points** belong to the this line

Forms of Linear Equations

- General form

$$ax + by + c = 0$$

a, b, and c are constants (not both a and b are zero)

$$2x + 3y = 10$$

- Point – slope form

Given a point $P(x_1, y_1)$ and a slope m, the equation is

$$y - y_1 = m(x - x_1).$$

- y – intercept–slope equation

Given y intercept b and slope m, the equation is

$$y = f(x) = mx + b.$$

- Vertical line: x = a, a line parallel to the y – axis, has x – intercept (a, 0), and undefined slope.
- Horizontal line: y = b, a line parallel to the x – axis, has y – intercept (0, b), and a zero slope.

Applications of Linear Functions in d Business and Economics (LINEAR MODELS)

1. TOTAL COST

A linear cost function expresses the total costs of producing a product C(x) as a linear function of the number of items produced x. The cost function is composed of two parts, **fixed costs (FC)** which are those costs **remain constant** regardless of the number of units produced (Included in the fixed cost is, for instance, mortgage payments, salaries, insurance, rent, utilities and so on,) and the **variable costs (VC)** which are the costs those are **directly related** to the number of units produced. In other words, the total cost is the sum of the variable costs and fixed costs.

Total Costs (TC) = Variable costs + Fixed costs

The variable cost is calculated by multiplying the cost per unit **m** by the total number of units produced **x**.

Variable Costs (VC) = (Cost per unit)*(Total number of units)

That is, the linear cost model is given by;

$$C(x) = mx + b$$

Remarks

- The slope m is also called the marginal cost $\overline{MC}$.
  The **marginal cost** is defined as the cost of producing one ***additional unit*** at any level of production.
- The fixed cost b is **C(0)** (the cost of producing no units)

Example 1

The daily total cost to produce x units of a product is given by

C(x) = 10 x + 200 (dollars).

(Note that C(x) is measured in dollars, and x is the number of units)

The marginal cost is m = $10, and the fixed cost is b = $200.

To understand what these quantities mean, let us compute some costs:

- The cost of producing no units is C (0) = b = $200. This allows us to interpret the fixed cost; $200, as that part of the cost that is not affected by the number of units produced.
- The daily cost to produce 15 units is C (15) = $350
- The daily cost to produce 16 units is C (16) = $360
- C (16) – C (15) = 360 – 350 = $10. (The cost of producing the unit # 16)
- In general, the daily cost to produce one additional unit at any level of production is called the **marginal cost (the slope) = $10.**
- **Note:** In linear models the marginal cost is always fixed.

Example 2

The cost of producing 50 units of a product is $1000, and the cost of producing 100 units of the same product is $1100. Write the cost equation.

To find the equation for the cost, we have two points (50, 1000) and (100, $1100). [(x, C(x))]

The slope is $\mathbf{m} = \frac{\Delta \mathbf{C}}{\Delta \mathbf{x}} = \frac{1100 - 1000}{100 - 50} = 2$.

Use the point slope equation to find this equation is :
$C(x) - 1000 = 2(x - 50) \Rightarrow \boxed{\mathbf{C(x) = 2x + 900}}$
Therefore, the fixed cost is $900, and the marginal cost is $2 per unit.

2. Total Revenue

Revenue results from the sale of items produced. If **R(x)** is the revenue from selling **x** items at a price of **p** each, then **R(x)** will be the linear function;

Total Revenue = (selling price per unit) (total number of units sold)

$$\mathbf{R(x) = p*x}$$

The selling price p also called the **marginal revenue (**the revenue of producing and selling one ***additional unit*** at any level of production**).**

Example 3:

Each unit produced (in example 1) sells for $10 each. The revenue function is then

$$R(x) = 10x \text{ dollars.}$$

The marginal revenue is p = $10 per unit. (Producing and selling one extra unit at any production level will increase the revenue by $10)

3. Total Profit

The profit is the net proceeds, or what remains of the revenue when costs are subtracted. If the profit depends linearly on the number of items, its slope is called the **marginal profit.**
Profit, revenue, and cost are related by the following formula.

Profit = Revenue – Cost

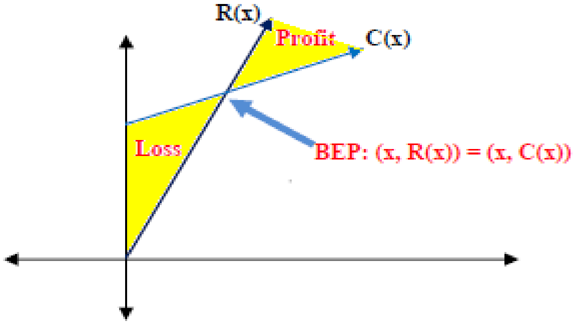
$$\mathbf{P(x) = R(x) - C(x).}$$

Both the revenue and profit depend on the number of items, x, we buy and sell, and so, like the cost function, they too are functions of x.

If the profit is negative, say -$1000, we refer to a **loss** (of $1000) in this case).

To **break even** means to make **neither a profit nor a loss**. Thus, break-even occurs when the profit is **ZERO** (That is; P(x) = 0, or R(x) = C (x)).

The break-even point (BEP) is the number of items x at which break-even occurs.
If **P(x) = 0** at x = a units, then the BEP is (a, C (a)) = (a, R (a)).



Example 4

The cost and revenue functions are given, respectively:

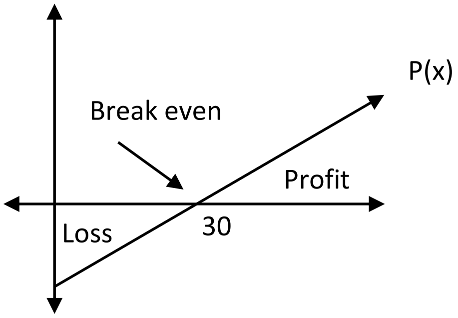
$$C(x) = 5x + 150, \text{ and } R(x) = 10x$$

To find the break-even level:

$$\mathbf{P(x) = R(x) - C(x)}$$
$$\mathbf{= 5x - 150}$$

$$P(x) = 0 \rightarrow 5x - 150 = 0 \rightarrow x = 30$$

Thus, 30 units per day should be produced and sold to break even



- There is a $50 **loss** when 20 units are produced and sold (P (20) = 100 – 150 = - 50).
- There is a $100 **profit** when 50 units are produced and sold (P (50) = 250 – 150 = 100).
- The **BEP** is (30, C (30)) = (30, R (30)) = (30, 300).

Example 5

The variable cost per unit for a product is $24, fixed costs are $8000 and the selling price is $32 per unit.

1. What is the loss or profit when 500 units are produced and sold?
2. Find the number of units that give a profit of $1600.
3. Find the break - even point.

Solution:

The cost function is **C(x) = 24x + 8000**, the revenue function is **R(x) = 32x**. So, the profit function is **P(x) = 8x – 8000**

1. $\mathrm{P}(500) = 8(500) - 8000 = -\$4000 \ \ (\mathrm{loss})$
2. $\mathrm{P}(x) = 1600 \Rightarrow 8x - 8000 = 1600 \Rightarrow x = 1200$
3. $\mathrm{R}(x) = \mathrm{C}(x)[\mathrm{P}(x) = 0] \Rightarrow 8x - 8000 = 0 \Rightarrow x = 1000$

   The **BEP** is (1000, 32000)

Extra Exercises

1. A company sells its products at $25 per unit. The fixed cost is $18000 and the cost per unit is $15. Find the level of production to break even. Find the break-even point.
2. A manufacturer sells 15 units of a product for $**225**. The fixed costs related to this product are $750 per month and the total costs of producing 100 units are $1750. How many units must be sold to guarantee no loss?
3. The total cost of producing 20 units of a product is $1880 and the cost of producing 25 units is $1950. The total revenue of selling 120 units of the same product is $2160. Assume linear cost and revenue models.
   a. Find the cost and revenue functions. Write the profit function.
   b. Find the profit (or loss) when 200 units are produced and sold.
   c. Find the profit (or loss) when 400 units are produced and sold.
   d. Find the break-even point.

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Lecture # 2

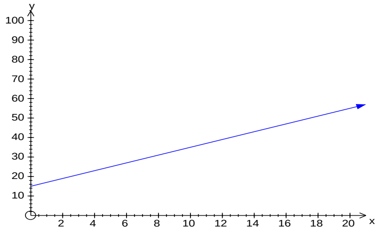
Supply, Demand, and Equilibrium point

Reference: Section 1.6 page 106 from the text book.

Linear Supply and Demand Functions
(Linear Models)

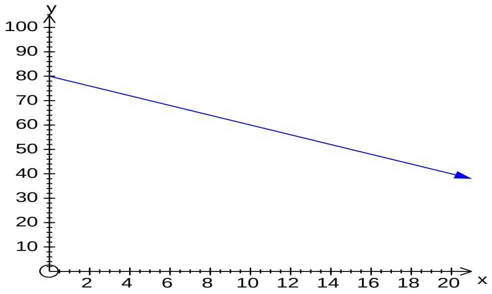
A supply curve describes the relationship between the quantity supplied and the selling price*. The amount of a good or service that producers plan to sell at a given price during a given period is called the* ***quantity supplied*****.** The quantity supplied is the maximum amount that producers are willing to supply at a given price. Quantity supplied is expressed as an amount per unit of time. For example, if a producer plans to sell 750 units per day at $15 per unit we say that the quantity supplied is 750 units per day at price $15.

The **law of supply** states that: **as price increases, the corresponding quantity supplied for sale will also increase.** A linear supply curve is a line with positive slope. The relationship between price and quantity is positive (direct).



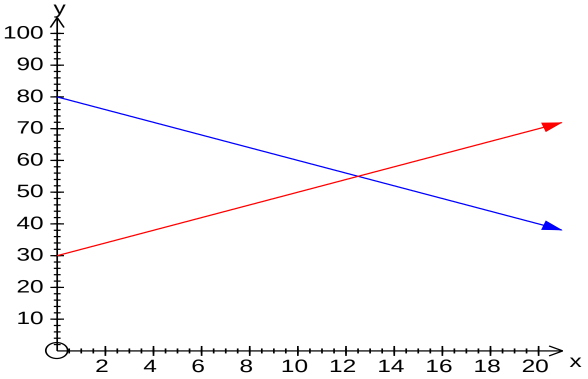
Similarly, *the amount of a good or service that consumers plan to buy at a given price during a given period is called the* ***quantity demanded*****.** The quantity demanded is the maximum amount that consumers can be expected to buy at a given price, and it also is expressed as amount per unit of time.

The **law of demand** states: that **as price increases, the corresponding quantity demanded will decrease**. A linear demand curve is a line with negative slope. The relationship between price and quantity is negative (indirect).



The equilibrium price is the price at which the quantity demanded equals the quantity supplied. The **equilibrium quantity** is the quantity bought and sold at the equilibrium price.

- If the curves are graphed on the same coordinate system, the point of intersection is the **equilibrium point,** and is where supply equals demand.
- If the price is below equilibrium there will be a **shortage** and the price will rise,
- While if the price is above equilibrium there will be a **surplus** and the price will fall.
- If the price is at equilibrium it will stay there unless other, factors enter to cause changes.



Example 1

Consider the following linear supply and demand relationships:

Producers will supply 1000 units when the selling price is $20 per unit, and 1500 units when the price is $25 per unit.

Consumers will demand 500 units when the selling price is $15 per unit but that the demand will decrease by 100 units if the price increases by $5.

Both supply and demand functions are linear. Determine the supply function, the demand function and the equilibrium point (using point – slope equations).

Solution

To determine the supply function, we use the two points (1000, 20) and (1500, 25), and write the equation of the line through the points

$$m = \frac{25 - 20}{1500 - 1000} = 0.01$$

Equation: $p - 20 = 0.01(q - 1000) \Rightarrow$ **$p = 0.01q + 10$**

For the demand function, one point is (500, 15). If the price increases to $20, the demand will decrease to 400. Thus the second point is (400, 20) and we can now determine the demand;

$$m = \frac{20 - 15}{400 - 500} = -0.05$$

Equation: $p - 15 = -0.05(q - 500) \Rightarrow$ **$p = -0.05q + 40$**

To find the equilibrium point let Supply = Demand

$0.01q + 10 = -0.05q + 40$

$\Rightarrow q = 500, p = 15$

So, the equilibrium point is (500, 15)

Example 2

Consider the following linear supply and demand

Demand: $p = 75 - 0.5q$
Supply: $P = 15 + 0.1q$

Determine whether there is a shortage or surpluses at the prices of $20, $30, and $25?

Solution

1. At price p =20

Demand: $20 = 75 - 0.5q$

$q=110$

Supply: $20 = 15+0.1q$

$q = 50$

Quantity demanded is greater than quantity supplied $\Rightarrow$ **Shortage**

Shortage occurs when quantity supplied is less than quantity demanded

2. At price $p = 30$: Quantity demanded = 90, Quantity supplied = 150

Quantity supplied is greater than quantity demanded $\Rightarrow$ **Surplus**

Surplus occurs when quantity supplied is more than quantity demanded

3. At price $p = 25$: Quantity demanded = Quantity supplied = 100 $\Rightarrow$ Market equilibrium

As you can see in this example the quantity supplied and the quantity demanded are not always equal. The difference in quantities will cause either a shortage or surplus

In our example: at $p = 20$ there is a shortage of $110 - 50 = 60$
at $p = 30$ there is a surplus of $150 - 90 = 60$
at $p = 25$ there is a market equilibrium

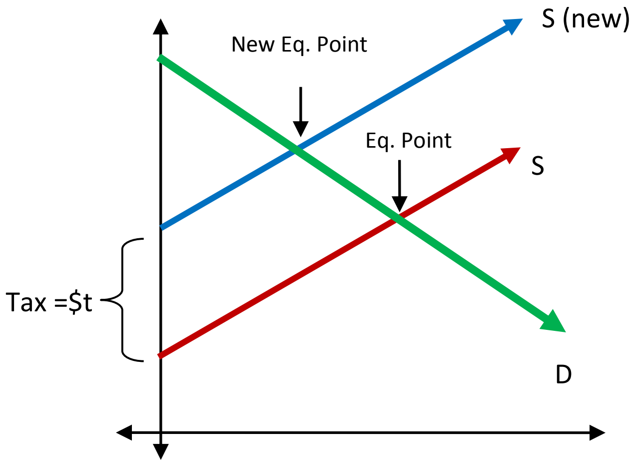
The equilibrium point is the point (q, p), q is called the equilibrium quantity and p is the equilibrium price**.**

Additive Tax and Market Equilibrium

Often government imposes taxes on certain commodities in order to raise more revenue. Suppose a supplier is taxed \$t per unit sold, and the tax is passed on to the consumers by adding \$t to the selling price of the product (we should assume that the quantity demanded by consumers depends only on the price alone, that is, the demand equation does not change).

If the original supply function is given by: $p = f(q)$, then the new supply function (supply after passing the tax on) is given by**:**

$p = f(q) + t$ (shifting the original supply t units above)



Example 3

The demand and supply curves for a certain product are given in terms of price, p, by:

$$D: p = 2500 - 20q$$

$$S: 2p = 20q - 1000$$

1. Find the equilibrium price and quantity. Solve the two equations simultaneously,

$$2500 - 20q = 10q - 500,$$

Solving for q, we get q = 100 units.

Demand and supply are equal at q =100 so the price, D (100) = 2500 – 20(100) = $500.

The equilibrium point is **(100, 500)**

2. If a tax of $60 is placed on each unit of the product, what are the new equilibrium price and quantity?

The new supply function is S (new): $p = (10q - 500) + 60 = 10\,q - 440$.

We have $2500 - 20q = 10q - 440$, solving for q, we get q = 98 units.

Demand and supply are equal at q= 98so the price, D (98) =2500-20(98) = $540.

The new equilibrium point is **(98, 540)**

Extra Exercises

1. Suppliers are willing to produce 56 items if the price is $440/ item and 136 items if the price is $530/ item.
   a. Write the supply function (linear).
   b. You are also told that the demand function is 2p + 6q =1414. Find equilibrium point.

2. At a price of \$100 per TV, the quantity demanded, x, is 150. At a price of \$250, the quantity demanded drops to 50. Given that the demand equation is linear, what is the highest price anyone would pay for a TV?
3. Suppose you are given the supply and demand curves, respectively, 12p – x = 24 and 8p + 10x = 80(x = # of units, p = price in dollars). What is the equilibrium point?
4. The demand for a certain commodity is **5p + 2x = 200** and the supply is $5p = 4x + 50$
    a. Find the equilibrium price and quantity.
    b. Find the equilibrium price and quantity after a tax of 6 per unit is imposed.

Solution for exercise 2 (Extra Exercises| Lecture # 2

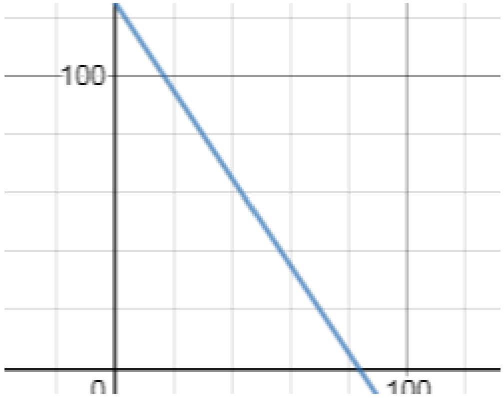
At a price of $100 per TV, the quantity demanded, x, is 150. At a price of $250, the quantity demanded drops to 50. Given that the demand equation is linear, what is the highest price anyone would pay for a TV?

- Two points (x, p): (150, 100) and (50, 250).
- The slope: $(250 – 100) / (50 – 150) = – 1.5$ $\boxed{\text{negative slope (DEMAND)}}$
- Demand equation: $p – 100 = – 1.5(x – 150)$

  $$\boxed{p = – 1.5x + 125}$$

- The highest price anyone **would** pay for a TV is p = 125 (the demand at this level of price is zero)

The point (0. 125): highest price
No demand



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Lecture # 3

Business Applications of Quadratic functions.
(Quadratic Models)

Reference: Section 2.3 page 165 from the text book.

In lectures 1 and 2 we dealt with the concepts of linear models (Cost, revenue, profit, supply, and demand). We'll now deal with the situation in which Revenue, Cost, Profit, Supply, Demand functions may be **quadratic.**

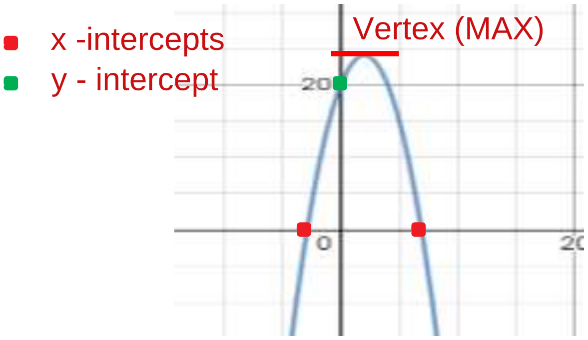
Quadratic Functions

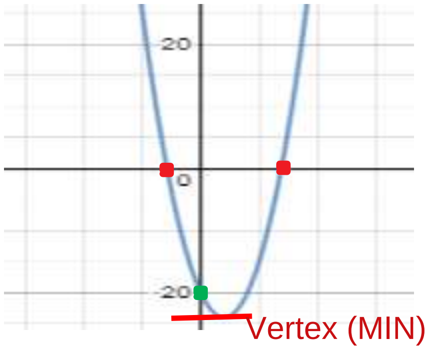
A function of the **form** $y = f(x) = ax^2 + bx + c$ is called a quadratic function, a, b and c are constants ($a \neq 0$).

To graph a quadratic function

- Find the x – Intercept :(x, 0)
  Solve for x, f(x) = 0 by using the **quadratic formula**;
  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
  Or solve f(x) = 0 by **factoring**.
- Find the y – Intercept :(0, f (0)) = (0, c).
- Find the **vertex**: $(\frac{-b}{2a}, f(\frac{-b}{2a}))$

- The graph of a quadratic function is called a **parabola**
- If a > 0, the parabola opens up **(a minimum point)**
- If a < 0, the parabola opens down **(a maximum point)**





Example 1

The profit of selling x units of a product is given by the function
$P(x) = 12x - 0.1x^2$

What is the **maximum profit** and how many **units** should be sold in order to earn this maximum profit.

Solution:
First, we look at the function $f(x) = -0.1x^2 + 12x$. We know that the graph of this function is a parabola opening downwards, in other words, a parabola with a maximum y-value.

Clearly, if we want to find the maximum y-value of this parabola we have to find the coordinates of the vertex: $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.
In this case we have a = – 0.1, b = 12 and c = 0. S

$\Rightarrow -\frac{b}{2a} = -\frac{12}{2\cdot(-0.1)} = 60$. We have found that the x – coordinate of the vertex is 60. The y – coordinate of the vertex is then
$f(60) = -0.1\cdot 60^2 + 12\cdot 60 = -360 + 720 = 360$.

We now see that the maximum y – value is y = 360 when 60 units are produced and sold.

Example 2

The total cost and the total revenue functions are given respectively,
$C(x) = 1600 + 1500x$, $R(x) = 1600x - x^2$

1. Find the break-even point(s).

The break-even point occurs when either (a) Profit = 0; or (b) Total Costs = Total Revenue.
We need to determine our profit Function, P(x). Recall that
Profit = Total Revenue – Total Cost.

$$Profit = P(x) = R(x) - C(x) = 1600x - x^2 - (1600 + 1500x)$$
$$= 1600x - x^2 - 1600 - 1500x$$
$$P(x) = -x^2 + 100x - 1600$$

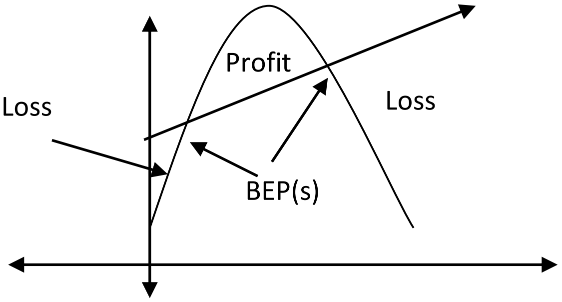
To determine the Break Even Point, we set P(x) = 0
That is,

$$-x^2 + 100x - 1600 = 0.$$

This is a quadratic equation, which we are going to solve (by factor method or using the quadratic formula).

$$-x^2 + 100x - 1600 = 0 \rightarrow x^2 - 100x + 1600 - 0$$
$$\rightarrow (x - 80)(x - 20) = 0$$
$$\rightarrow x = 80 \text{ or } x = 20$$

Therefore, there are two break even points: (20, 31600) and (80. 121600



2. Find maximum profit.

We have already determined the profit function: $P(x) = -x^2 + 100x - 1600$

The graph of P(x) will be a parabola. Since the value of "a" is negative, the parabola will point down. The vertex of this parabola will be at the top. The x-coordinates of this parabola represent the number of units manufactured/sold, and the y-coordinates are the profits for the given number of units. Since the vertex is at the top of the parabola, the y-coordinate of the vertex gives the maximum profit.
Therefore, we will determine the coordinates of the vertex.

Recall that the formula for the x-coordinate of the vertex is $x = \frac{-b}{2a}$.

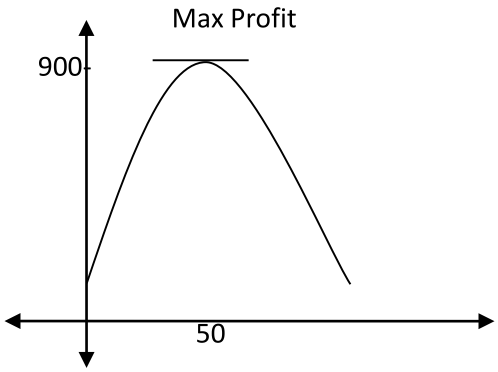
In our profit equation, a = -1 and b = 100. So the x –coordinate of the vertex is:

$$x = \frac{-100}{2 * -1} = \frac{-100}{-2} = 50$$

We maximize profit when we make/sell 50 units. To determine the amount of profit, we substitute x = 50 in the profit equation.

$P(50) = -50^2 + 100(50) – 1600 = -2500 + 5000 – 1600 = \$900$

The maximum profit is $900.



3. **3.** Compare the level of production to maximize profit with the level to maximize revenue. Do they agree?

We've already determine that a production level of 50 will maximize profit. Now to determine the quantity that will maximize revenue:
Recall that the revenue function is ***$R(x) = 1600x - x^2$***
The graph of this function is also a downward-pointing parabola. The x-coordinate of the vertex gives the number of units that should be manufactured and sold to maximize revenue. Again, we'll use the formula

$$x = \frac{-b}{2a}$$

For the revenue function, a = -1 and b = 1600. So the x –coordinate of the vertex is:

$$x = \frac{-1600}{2*-1} = \frac{-1600}{-2} = 800$$

We need to sell 800 to maximize revenue, but only 50 to maximize profit. These numbers are not equal.

4. **4.** How do the break even points compare with the zeros of P(x)?

The break even points are the zeros of P(x).

Example 3

If the demand and supply functions for a product are $p^2 + 2q = 1600$ and $200 - p^2 + 2q = 0$, respectively. Find the equilibrium price and quantity

Solution:

$$D : p^2 = 1600 - 2q$$
$$S : p^2 = 200 + 2q$$
$$S = D \rightarrow 200 + 2q = 1600 - 2q$$
$$\rightarrow q = 350, \; p = 30$$

The equilibrium price is $30 and the equilibrium quantity is 350 units. The equilibrium quantity is (350, 30).

Example 4

If the demand and supply functions for a product are
$pq = 100 + 20q$ and $2p - q = 50$, respectively.

1. Find the market equilibrium point.
2. If a $12.5 tax is placed on production and passed through the supplier , find the new equilibrium point

Solution:

1. We will use substitution to find the equilibrium point.

$$D : pq = 100 + 20q$$

$$S : p = 25 + \frac{q}{2}$$

$$\to (25 + \frac{q}{2})q = 100 + 20q$$

$$\to q^2 + 10q - 200 = 0$$

$$\to (q + 20)(q - 10) = 0$$

$$\to q = -20 \text{ or } q = 10$$

Thus the market equilibrium occurs when 10 items are sold at a price p = 25 + 5 = $30

2. The new supply function is

$$S_{\text{New}} : p = (25 + \frac{q}{2}) + 12.5$$

$$= 37.5 + \frac{q}{2}$$

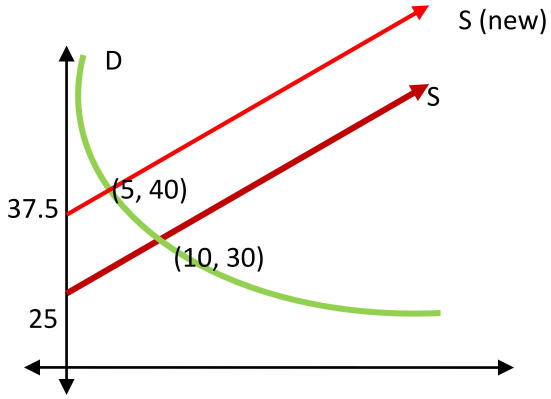
$$(37.5 + \frac{q}{2})q = 100 + 20q$$

$$(\to q^2 + 35q - 200 = 0 \quad)$$

$$\to (q + 40)(q - 5) = 0$$

$$\to q = 5$$

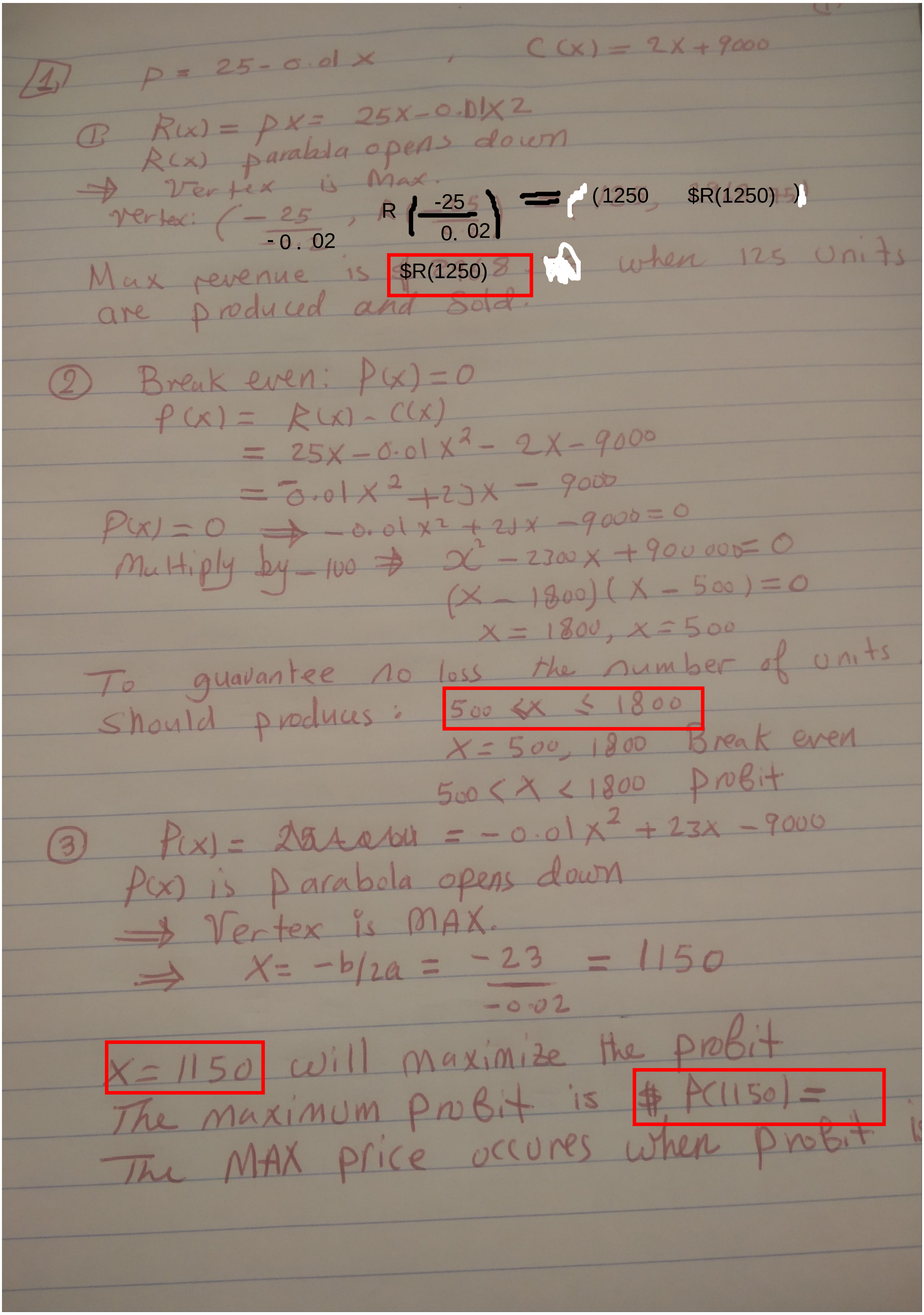
Thus the new market equilibrium occurs when 5 items are sold at a price p = 37.5 + 2.5 = $40



Extra Exercises

1. Let $p = 25 - 0.01x$ and $C(x) = 2x + 9000$ be the price-demand equation and cost function, respectively, for the manufacturer of umbrellas.
    a. Find the maximum revenue.
    b. Find the number of units that must be sold to guarantee no loss.
    c. What is the price per umbrella that produces the maximum profit?
    d. If the price – supply equation for the umbrellas is given by $p = 10 + 0.02x$, find the equilibrium point.
2. Suppose a company has fixed cost of $360 and variable cost of 10 + 0.2x dollars per unit, where x is the total number of units produced. Suppose further the selling price of the product is given by p = 50 – 0.2x dollars per unit
    a. Write the cost function.
    b. Write the revenue function.
    c. Write the profit function,
    d. Find the maximum profit.
    e. What price maximizes the profit?
3. The demand for x units of a product is given by p = 60 – 0.5x, if no more than 75 units can be sold, find the number of units that must be sold in order that the sales revenue be $1000.
4. A certain product has supply and demand functions **2p – q = 40** and **pq = 100 + 2q**, respectively.

a. If the price is $50, how many units are supplied and how many units are demanded. Is the price likely to increase from $50 or decrease from it. Explain
b. If a tax of $5 per item is levied on the supplier, who passes it on to the consumer as a price increase, find the market equilibrium point after the tax.



\boxed{1} $p = 25 - 0.01x$, $C(x) = 2x + 9000$

① $R(x) = px = 25x - 0.01x2$

$R(x)$ parabola opens down

$\Rightarrow$ Vertex is Max.

vertex: $\left(-\frac{25}{-0.02}\right.$, $R\left(\frac{-25}{0.02}\right) = (1250 \quad \$R(1250))$

Max revenue is $\boxed{\$R(1250)}$ [redacted] when 125 units are produced and sold.

② Break even: $P(x) = 0$

$P(x) = R(x) - C(x)$

$= 25x - 0.01x^2 - 2x - 9000$

$= -0.01x^2 + 23x - 9000$

$P(x) = 0 \Rightarrow -0.01x^2 + 23x - 9000 = 0$

Multiply by $-100 \Rightarrow x^2 - 2300x + 900000 = 0$

$(x - 1800)(x - 500) = 0$

$x = 1800, x = 500$

To guarantee no loss the number of units should produces: $\boxed{500 \leq x \leq 1800}$

$x = 500, 1800$ Break even

$500 < x < 1800$ probit

③ $P(x) =$ [illegible] $= -0.01x^2 + 23x - 9000$

$P(x)$ is parabola opens down

$\Rightarrow$ Vertex is MAX.

$\Rightarrow x = -b/2a = \frac{-23}{-0.02} = 1150$

$\boxed{x = 1150}$ will maximize the probit

The maximum probit is $\boxed{\$\ P(1150) =}$

The MAX price occures when probit

price $p = 25 - 0.01x$

MAX. price when $x = 1150$

$\Rightarrow$ MAX price $p\Big|_{x=1150} = 25 - (0.01)(1150)$

$= 25 - 11.5$

$= \boxed{\$\ 13.5}$

④ $p = 10 + 0.02x$ Supply

Equilibrium: $S = D$

$\Longrightarrow 10 + 0.02x = 25 - 0.01x$

$\Longrightarrow 0.03x = 15$

$x = \frac{15}{0.03} = 500$

$x = 500 \Rightarrow p = (25) - (0.01)(500)$

$= \$20$

Equilibrium point: $\boxed{(500, 20)}$

2. V. cost per unit $= 10 + 0.2x$

F. Cost = \$ 360

Price per unit $50 - 0.2x$

a), b), c)

$$C(x) = V.C + F.C$$
$$= (10 + 0.2x)\,x + 360$$
$$= \boxed{0.2x^2 + 10x + 360}$$

$$R(x) = px$$
$$= (50 - 0.2x)x$$
$$= \boxed{50x - 0.2x^2}$$

$$P(x) = R(x) - C(x)$$
$$= 50x - 0.2x^2 - (0.2x^2 + 10x + 360)$$
$$= \boxed{-0.4x^2 + 40x - 360}$$

d) Max. profit

P(x) is parabola ⇒ Vertex is MAX

Vertex is (50, P(50))

The max profit is $\boxed{\$ P(50) = \ldots\ldots}$ when 50 u

are produced and Sold.

e) Maximum Price happens when profit is M

$$p\Big|_{x=50} = 50 - (0.2)(50)$$

$$= \boxed{\$40}$$ is the Maximum price

Mathematics Department

MATH 2351

Semester 1201

Lecture Notes

Prepared by Mohammad Madiah

(Reference: Mathematical Applications for the Management, Life, and Social Sciences by Hershberger and Reynolds. International Edition)

Lecture # 4

Exponential and logarithmic functions

Reference: Sections 5.1, and 5.2 page 325 from the text book.

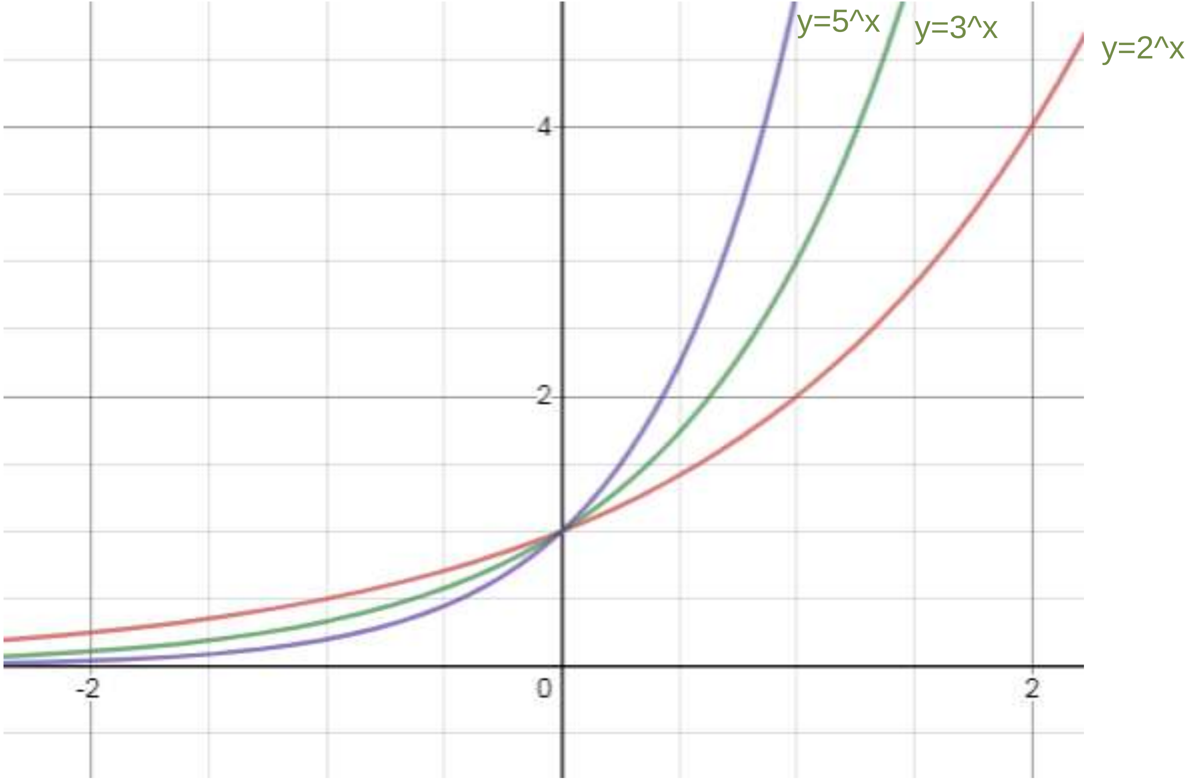
Review 1: Some Properties of exponential

Remember that $\mathrm{a^n = a \times a \times a.... \times a \ (n\ times)}$

For any real numbers a and b and positive integers m and n

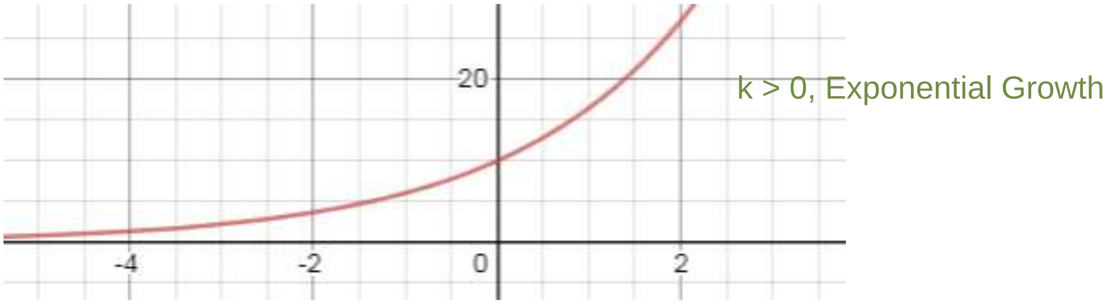
1) $a^m a^n = a^{m+n}$
2) For $a \neq 0$, $\frac{a^m}{a^n} = \begin{cases} a^{m-n} & m > n \\ 1 & m = n \\ \frac{1}{a^{n-m}} & m < n \end{cases}$
3) $(ab)^m = a^m b^m$
4) $(\frac{a}{b})^m = \frac{a^m}{b^m}$ $(b \neq 0)$
5) $(a^m)^n = a^{mn}$
6) $a^0 = 1(\ a \neq 0)$
7) $a^{-n} = (a^{-1})^n = \frac{1}{a^n}$ $(a \neq 0)$
8) $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$ (if n even, $a \geq 0$

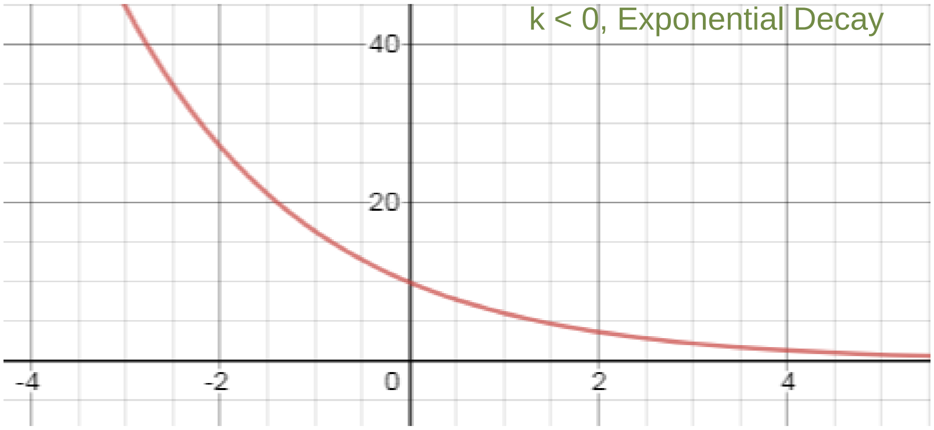
Exponential functions are functions written in the form $y = a^x$, where a is the **base**. a is positive, $a \neq 1$, and x is a real number.



The domain of the exponential function, the values for which x can equal, are all real number. The range however, is all positive numbers.

- For a >1, the function $y = y_0 a^{kx}$ is called the general exponential function
  - ✓ k > 0 means exponential **growth**.
  - ✓ k < 0 means exponential **decay.**
- **Special function:** $f(x) = y_0 e^{kx}$
- The shape of the curve will always be:

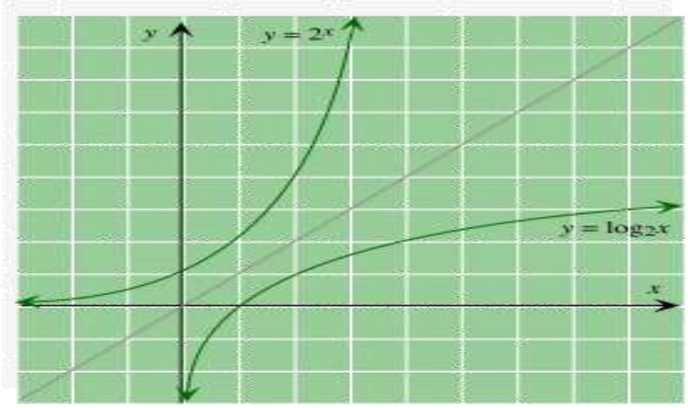




Logarithmic functions: For **a > 0, x > 0**, the function $y=\log_a x$ is called the Logarithmic function**.**

- $y=\log_a x \Leftrightarrow a^y = x$
- **Special function:** $f(x)=\ln x$.

  $\ln x = \log_e x$



Review 2: Some Properties logarithms

1. $\log_a x = y$ (logarithmic form) $\Leftrightarrow a^y = x$ (exponential form)
2. $\log_a 1 = 0$
3. $\log_a a = 1$
4. $\log_a a^x = x$
5. $a^{\log_a x} = x$
6. $\log_a MN = \log_a M + \log_a N$
7. $\log_a \dfrac{M}{N} = \log_a M - \log_a N$
8. $\log_a x^n = n\log_a x$

9. $\log_a x = \log_a y \Rightarrow x = y$

10. $\log_{10} x = \log x$ **(common logarithm)**

11. $\log_e x = \ln x$ **(natural logarithm)**

12. $\log_b a = \dfrac{\log_m a}{\log_m b} = \dfrac{\log a}{\log b} = \dfrac{\ln a}{\ln b}$ **(change of basis)**

Using Calculator for exponentials and logarithms

- **Power Key ^**

$3^{1.45}$ **: 3 ^ 1.45 = 4.92**

- **Logarithms Keys**

1. $\boxed{\textbf{log}}$ **key :** Use base **10**

   log 15 : $\boxed{\textbf{log}}$ **15 = 1.18**

2. $\boxed{\textbf{ln}}$ **key :** Use base **e**

   ln 15 : $\boxed{\textbf{ln}}$ **15 = 2.71**

- **Exponential Keys**

1. $\boxed{10^x}$ **key :** Shift + $\boxed{\textbf{log}}$

   $10^{1.5}$ **:** Shift + $\boxed{\textbf{log}}$ **1.5 = 31.62**

2. $\boxed{e^x}$ **key :** shift + $\boxed{\textbf{ln}}$

   $e^{1.5}$ **:** Shift + $\boxed{\textbf{ln}}$ **1.5 = 4.48**

Extra Problems and Excersises

* Solve for x

① $2^x = 3$

$\Rightarrow \ln 2^x = \ln 3$

$x \ln 2 = \ln 3$

$x = \dfrac{\ln 3}{\ln 2}$

$= 1.59^{*}$

* Calculator

② $e^{2x+1} = 10$

$\ln e^{2x+1} = \ln 10$

$(2x+1) \ln e = \ln 10$

$(2x+1) = \ln 10$

$2x = (\ln 10) - 1$

$x = \dfrac{\ln 10 - 1}{2} = 0.65$

③ $\log_2 x(x+2) = 3$

$2^3 = x(x+2)$

$\Rightarrow x^2 + 2x = 8$

$\Rightarrow x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$

$x = 2$ Ln x is defined

~~$x = -4$~~ for $x > 0$

Method 2 $a^{\log_a x} =$

$2^{\log_{2} x(x+2)} = 2^3$

$x(x+2) = 8$

$\Rightarrow x^2 + 2x - 8 = 0$

④ $\log_2 128 = x + 1$

⑤ $\log_2 x + \log_4 x^2 = 1$

** If $\ln 5 = a$, $\ln 9 = b$, $\ln 10 = 3$

Find in terms of a, b, c

1) $\ln 2$, $\ln 3$, $\ln 6$

2) $\ln \sqrt{45}$, $\ln 270$

*** Simplify

1) $\log_2 16 - \log_4 16 + \log_4 5$

2) $\ln e^2 + \ln(1/e) - e^{\ln 10} + e^{2 \ln \frac{1}{5}}$

Lecture # 5

Simple and Compound Interest

Reference: Section 6.1 and 6.2 page 369 from the textbook.

- If **$P** is invested at an interest rate of **r** per year, then the simple interest, and the future value **S** after **t** years are

  S = P + I, where I = Prt

- If **$P** is invested at an interest rate of **r** per year, compound annually, the future value **S** after **t** years is

  $$\boxed{S=P(1+r)^t}$$

- If **$P** is invested for t years at a nominal interest rate **r**, compound **m** times per year, the future value **S** is

  $$\boxed{S=P(1+\frac{r}{m})^{mt}}$$

- If **$P** is invested for t years at a nominal interest rate of **r** compound **continuously**, the future value **S** is

  $$\boxed{S=Pe^{rt}}$$

Exercises

1. If $3000 is invested for 30 months at a simple interest rate of 5%
   - **a.** How much interest will be earned?
   - **b.** What is the future value of the investment after 30 months?
   - **c.** How long does it take the investment to be worth $7500?
2. If $1000 is invested at an annual rate of interest of 8%. What is the amount after 5 years?
   - **a.** simple interest
   - **b.** Compounding annually
   - **c.** Compounded semiannually
   - **d.** Compounded quarterly
   - **e.** Compounded continuously.
3. A bank is paying 5.5% (simple interest) on an account with $500. How much money is in the account after 30 months?
4. Find the future amount for $P invested at 2.5% simple interest for 72 months.

5. You have \$128500 for investment.
  a. What is your future value if you invest this money for 5 years at an annual rate of 4.5% compounded quarterly?
  b. How long will it take for your money to grow to \$150000 in account paying 6.5% compounded continuously?

6. How long would it take an investment to double if it is invested at
  a. 4.8% simple interest?
  b. 4.8% compounded annually?
  c. 4.8% compounded quarterly?
  d. 4.8% compounded continuously?
  e. Compounded continuously
  f. Compounded monthly.

7. If \$20000 has been invested on January 15, 2016, it would have been worth \$93300 on January 15, 2021. What interest rate compounded monthly is used?

Exercises

1. $P = \$3000$, $t = \frac{30}{12} = 2.5$ years, $r = 5\% = 0.05$

a. $I = Prt = (3000)(0.05)(2.5) = \boxed{\$375}$

b. $S = P + I = 3000 + 375 = \boxed{\$3375}$

c. $S = 7500$, $t = ?$

$$S = P(1 + rt)$$

$$7500 = 3000(1 + 0.05t)$$

$$2.5 = 1 + 0.05t$$

$$0.05t = 1.5$$

$$t = \frac{1.5}{0.05} = \boxed{30 \text{ years}}$$

2. $P = \$1000$, $r = 0.08$, $S = ?$, $t = 5$

a. Simple : $S = P(1 + rt)$

$$\Rightarrow S = 1000(1 + (0.08)(5))$$

$$= \boxed{\$1400}$$

b. Compound annually : $S = P(1 + r)^t$

$$\Rightarrow S = 1000(1 + 0.08)^5$$

$$= \boxed{\$1469.33}$$

c. Semiannually: $S = P(1 + r/2)^{2t}$

$$\Rightarrow S = 1000(1 + 0.04)^{10} = 1000(1.04)^{10}$$

$$= \boxed{\$1480.24}$$

d. Quarterly : $S = P(1 + r/4)^{4t}$

$$\Rightarrow S = 1000(1 + 0.02)^{20} = 1000(1.02)^{20}$$

$$= \boxed{\$1485.95}$$

e. Continuously: $S = Pe^{rt}$

$$\Rightarrow S = 1000e^{0.08t} = \boxed{\$1491.82}$$

6. $t = ?$, $S = 2P$ $r = 4.8\% = 0.048$

a. Simple : $S = P(1+rt)$

$$2P = P(1+0.048t) \Rightarrow 2 = 1+0.048t$$

$$\Rightarrow t = 1/0.048 = \boxed{20.83 \text{ yr}}$$

b. Compounded annually : $S = P(1+r)^t$

$$2P = P(1.048)^t \Rightarrow 2 = (1.048)^t$$

$$\Rightarrow \ln 2 = \ln(1.048)^t$$

$$\Rightarrow \ln 2 = t \ln(1.048)$$

$$\Rightarrow t = \frac{\ln 2}{\ln(1.048)} = \boxed{14.78 \text{ yr}}$$

d. Continuously: $S = Pe^{rt}$

$$\Rightarrow 2P = Pe^{0.048t} \Rightarrow 2 = e^{0.048t}$$

$$\Rightarrow \ln 2 = \ln e^{0.048t}$$

$$\Rightarrow \ln 2 = 0.048t \quad (\ln e = 1$$

$$\Rightarrow t = \frac{\ln 2}{0.048} = \boxed{14.44 \text{ yr}}$$

7. $P = \$20000$ $S = \$93300$ $t = 5$ $r = ?$

$m = 4$

$$S = P\left(1+\frac{r}{12}\right)^{12t}$$

$$93300 = 20000\left(1+\frac{r}{12}\right)^{60}$$

$$4.665 = \left(1+\frac{r}{12}\right)^{60}$$

$$\Rightarrow \left(\left(1+\frac{r}{12}\right)^{60}\right)^{1/60} = 1+\frac{r}{12} = (4.665)^{1/60}$$

⇑ use calculator 4.665 ^ (1/60)

$$\Rightarrow 1+\frac{r}{12} = 1.026$$

$$\Rightarrow r/12 = 0.026 \Rightarrow r = 0.312$$

$$= \boxed{31.2\%}$$

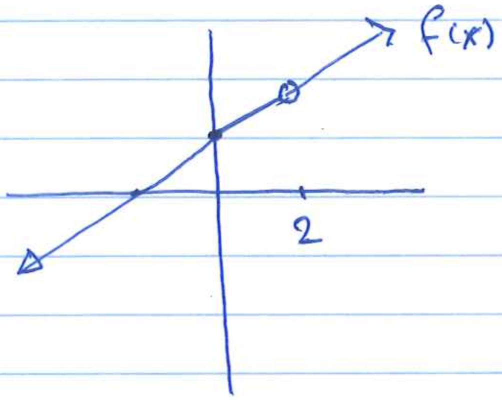
chapter 9

Derivatives

9.1 Limits

Given the function $f(x) = \frac{x^2+2x-8}{x-2}$ $\quad x \neq 2$

$= x+4 \quad x \neq 2$



$f(2)$ is not exist

$\equiv x=2$ is not on the domain of $f(x)$

what will happen near $x=2$

x	f(x)
1.9	5.9
1.95	5.95
1.99	5.99
2.01	6.01
2.05	6.05
2.1	6.1

x approaches 2

f(x) approaches 6

As x approaches 2

f(x) approaches 6

$x \to 2 \Rightarrow f(x) \to 6$

We say that the limit of $f(x)$ as x approaches 2 equals 6 and we write

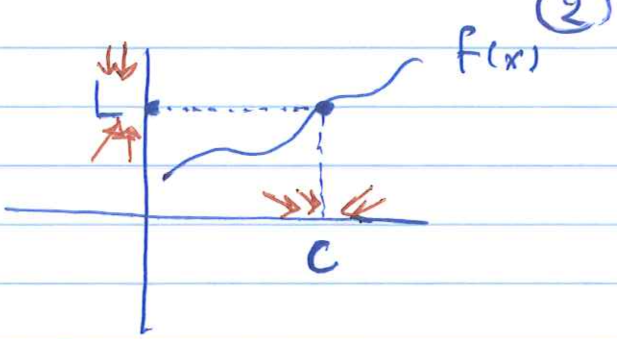
$$\lim_{x \to 2} f(x) = 6$$

The limit exists

If $f(x) = \frac{x^2+2x-8}{x-2}$, $x \neq 2$, then

$$\lim_{x \to 2} \frac{x^2+2x-8}{x-2} = 6,$$ the limit exists

f(x) is defined in an open interval containing c, except perhaps at $x = c$

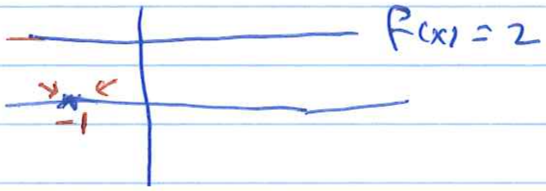


$$\lim_{x\to} f(x) = L$$

if as x gets close to c (on either side), the value of x approaches L

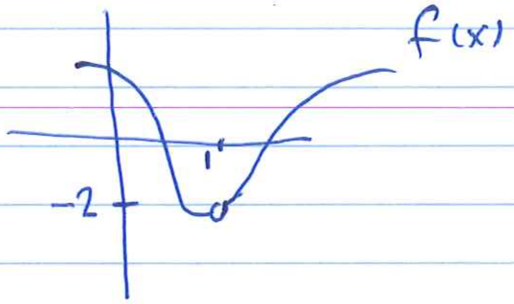
Example:

1) $\lim_{x\to -1} 2 = 2$



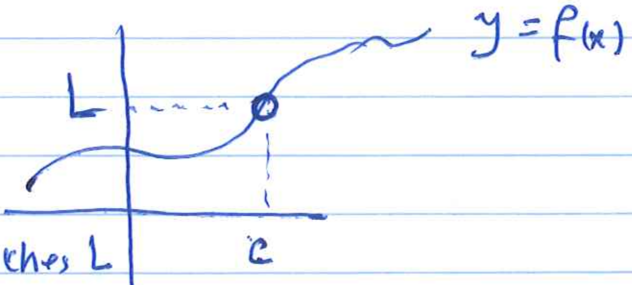
2) $\lim_{x\to 5} x = 5$

3) f(x)



$$\lim_{x\to 1} f(x) = -2$$

ONE SIDE LIMITS



limit from right

$\lim_{x\to c^+} f(x) = L$ f(x) approaches L as $x \to c$, $x > c$ (to the right of c)

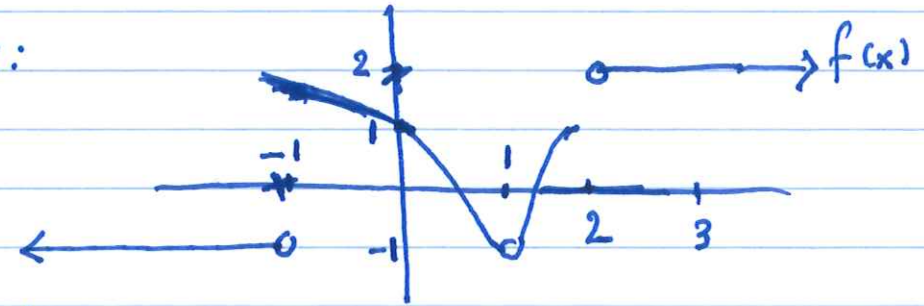
limit from left: $\lim_{x\to c^-} f(x) = M$

$f(x) \longrightarrow M$ as $x \longrightarrow c$ $x < c$

$$\lim_{x\to c} f(x) = L \iff \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$$

Both limits exist and equal

Ex:



* $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 1$ limit exists

* $\lim_{x\to -1^+} f(x) = 2$, $\lim_{x\to -1^-} f(x) = -1$

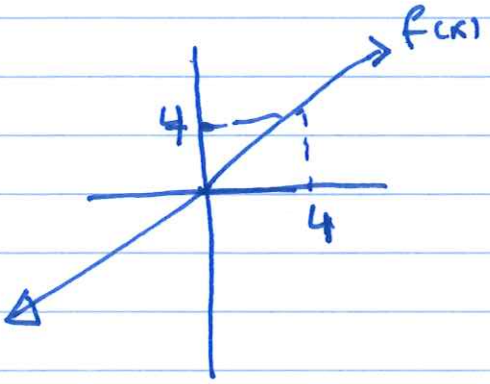
$\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x) \Longrightarrow \lim_{x\to -1} f(x)$ does not exists D.N.E

* $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = -1$

* $\lim_{x\to 2^+} f(x) = 2$, $\lim_{x\to 2^-} f(x) = 1 \Longrightarrow \lim_{x\to 1} f(x)$ D.N.E

Ex: $f(x) = x$ $c = 4$

$\lim_{x\to c} f(x) = \lim_{x\to 4} x = 4$



Properties of Limits

If $\lim\limits_{x\to c} f(x) = L$, $\lim\limits_{x\to c} g(x) = M$, $k =$ constant

1) $\lim\limits_{x\to c} k = k$ 2) $\lim\limits_{x\to c} x = c$

3) $\lim\limits_{x\to c} (f \pm g) = L \pm M$ 4) $\lim\limits_{x\to c} kf = k.L$

4) $\lim\limits_{x\to c} (f.g) = L.M$ 5) $\lim\limits_{x\to c} \frac{f}{g} = \frac{L}{M}$ $M \neq 0$

5) $\lim\limits_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to c} f(x)} = L^{1/n}$

6) If $f(x)$ is a polynomial of degree n

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

then $\lim\limits_{x\to c} f(x) = f(x)$

7) If $f(x) = \frac{g(x)}{h(x)}$ is a rational function

then $\lim\limits_{x\to c} f(x) = \frac{\lim\limits_{x\to c} g(x)}{\lim\limits_{x\to c} h(x)} = \frac{g(c)}{h(c)}$ $h(c) \neq 0$

* $g(x)$, $h(x)$ are both polynomials

Ex: 1) $\lim_{x \to -1} 10 = 10$

2) $\lim_{x \to 0} \dfrac{x^2 + 3x}{x + 5} = \dfrac{0 + 0}{0 + 5} = 0$

3) $\lim_{x \to 2} \sqrt{x^3 + 1} = \left(\lim_{x \to 2} x^3 + 1\right)^{1/2}$

$= (8 + 1)^{1/2} = 3$

$\lim_{x \to 2} \sqrt{x^3 + 1} = \sqrt{2^3 + 1} = \sqrt{9} = 3$

Rational Functions $\left(\dfrac{0}{0}\right)$ indeterminate form

* If $R(x) = \dfrac{f(x)}{g(x)}$ and $\lim_{x \to c} f(x) = 0$, $\lim_{x \to c} g(x) = 0$

then $\lim_{x \to c} R(x) = \lim_{x \to c} \dfrac{f(x)}{g(x)}$ has $\dfrac{0}{0}$ indeterminate form

(Reduce the fraction and find the limit of resulting expression)

* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{0}{a} = 0$

* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{a}{0}$ limit does not exists

Ex: $f(x) = \dfrac{x^2 - 2x - 15}{x - 5}$ $x \neq 5$

1) $\lim\limits_{x \to 0} f(x) = \dfrac{0 - 0 - 15}{0 - 5} = \boxed{3}$

2) $\lim\limits_{x \to -3} f(x) = \dfrac{9 + 6 - 15}{-3 - 5} = \dfrac{0}{-8} = 0$

3) $\lim\limits_{x \to 5} f(x) = \left(\dfrac{25 - 10 - 15}{5 - 5} = \dfrac{0}{0}\right)$ indeterminate form

$\Rightarrow \lim\limits_{x \to 5} \left(\dfrac{x^2 - 2x - 15}{x - 5}\right) = \lim\limits_{x \to 5} \dfrac{\cancel{(x-5)}(x+3)}{\cancel{(x-5)}}$

$= \lim\limits_{x \to 5} (x+3)$

$= 5 + 3 = 8$ exists

Ex: $f(x) = \dfrac{x^2 + x}{x^2 - 1}$

1) $\lim\limits_{x \to 0} f(x) = \dfrac{0 + 0}{0 - 1} = 0$ 2) $\lim\limits_{x \to 2} f(x) = \dfrac{4 + 2}{4 - 1} = \boxed{2}$

3) $\lim\limits_{x \to -1} f(x) = \dfrac{1 - 1}{1 - 1} = (0/0)$

$= \lim\limits_{x \to -1} \dfrac{x\cancel{(x+1)}}{\cancel{(x-1)}(x+1)} = \lim\limits_{x \to -1} \dfrac{x}{x-1} = \dfrac{-1}{-1-1} = \boxed{1/2}$

4) $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} \dfrac{x}{x-1} = \dfrac{1}{0}$, D.N.E

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(38) $\lim_{h\to 0} \frac{2(x+h)^2 - 2x^2}{h}$ $\left(\frac{0}{0}\right)$

$= 2\lim_{h\to 0} \frac{x^2+2xh+h^2 - \cancel{2}x^2}{h}$

$= 2\lim_{h\to 0} \frac{2xh+h^2}{h}$ $\left(\frac{0}{0}\right)$

$= 2\lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$

$= \boxed{4x}$

(52) $\lim_{x\to 5}[f(x)-g(x)] = 8$, $\lim_{x\to 5} g(x) = 2$

1) $\lim_{x\to 5} f(x)$, $\lim_{x\to 3} f(x) - \lim_{x\to 5} g(x) = 8$

$\lim_{x\to 5} f(x) = 10$

2) $\lim_{x\to 5}\left[(g(x))^2 - f(x)\right] = \left(\lim_{x\to 5} g(x)\right)^2 - \lim_{x\to 5} f(x)$

$= (2)^2 - 10 = \boxed{6}$

3) $\lim_{x\to 5}\left[\frac{2x\,g(x)}{4-f(x)}\right] = = \frac{\lim 2x\,g(x)}{\lim[4-f(x)]}$

$= \frac{\lim_{x\to 5} 2x \cdot \lim g(x)}{\lim 4 - \lim f(x)}$

$= \frac{10-2}{4-10} = \frac{8}{-6} = -4/3$

(45) $f(x) = \begin{cases} 12 - \frac{3}{4}x & x \le 4 \\ x^2 - 7 & x > 4 \end{cases}$, $\lim_{x\to 4} f(x)$

$\lim_{x\to 4^+} f(x) = \lim_{x\to 4} x^2-7 = 9$

$\lim_{x\to 4^-} f(x) = \lim_{x\to 4} 12 - \left(\frac{3}{4}\right)(4) = 9$

$\Rightarrow \lim_{x\to 4} f(x) = 9$

(55) $S(x) = \frac{4}{x} + 30 + \frac{x}{4}$ $4 \le x \le 100$

$S(x) \equiv$ Average monthly sales (\$ 1000)

$x =$ # of training of sales staff.

1) $\lim_{x\to 4^+} S(x) = \frac{4}{4} + 30 + \frac{4}{4} = 32$

2) $\lim_{x\to 100^-} S(x) = \frac{4}{100} + 30 + \frac{100}{4} = 55.04$

9.2 Continuous Functions

Def: The function $f(x)$ is Continuous at $x=c$

If 1) $\lim_{x \to c} f(x)$ exists

2) $f(c)$ exists (defined)

2) $\lim_{x \to c} f(x) = f(c)$

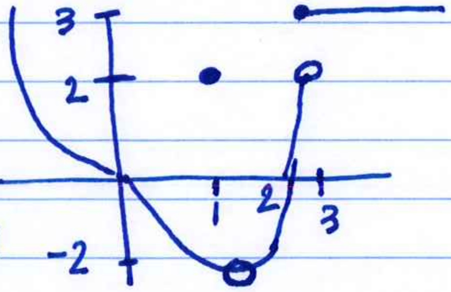
If one or more of the conditions above do not hold, we say $f(x)$ is discontinuous at $x=c$

Note: 1) If $p(x)$ is a polynomial, then $p(x)$ is continuous for all x

($\lim_{x \to c} p(x) = p(c)$ section 9.1)

2) If $R(x) = \frac{f(x)}{g(x)}$ is a rational function then $R(x)$ is continuous for x, such that $g(x) \neq 0$

Ex: Consider the graph of $f(x)$



at $x=1$: $\lim_{x \to 1} f(x) = -2$

$f(1) = 2$

$f(x)$ is discontinuous at $x=1$

at $x=2$: $f(2) = 3$

$\lim_{x \to 2^+} f(x) = 3 \neq \lim_{x \to 2^-} f(x) = 2 \Rightarrow \lim_{x \to 2} f(x)$ D.N.E

$\therefore f(x)$ is discontinuous at $x=2$

Ex① $f(x) = \begin{cases} x^2+2x-1 & x \geq 2 \\ \sqrt{2x+5} & x < 2 \end{cases}$

at $x=2$: $f(2) = (2)^2+(2)(2)-1 = 7$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2} (x^2+2x-1) = 7$

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2} \sqrt{2x+5} = \sqrt{9} = 3$

$\lim\limits_{x \to 2^+} f(x) \neq \lim\limits_{x \to 2} f(x) \Rightarrow \lim\limits_{x \to 2} f(x)$ D.N.E

$\therefore$ $f(x)$ is discontinuous at $x=2$

② $f(x) = \begin{cases} x^3 - 5 & x < 2 \\ 3 & x \geq 2 \end{cases}$

at $x=2$: $f(2) = \lim\limits_{x \to 2^+} f(x) = 3$

$\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 2} x^2 - 5 = 3$

$\therefore$ $f(x)$ is continuous for all x.

③ Given $f(x) = \begin{cases} x^2+3 & x \leq -1 \\ ax+2 & x > -1 \end{cases}$

Find the value of a such $f(x)$ is continuous for all x.

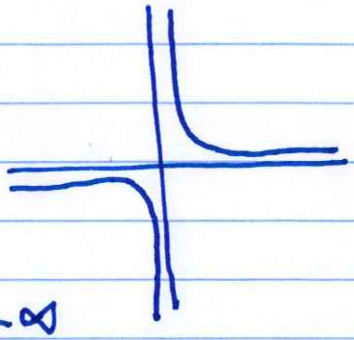
$\lim\limits_{x \to -1^-} f(x) = 4 = f(-1)$

$\lim\limits_{x \to -1^+} f(x) = (a)(-1)+2 = 2-a$

$2-a = 4 \Rightarrow \boxed{a = -2}$

limits at Infinity

Consider $y = f(x) = \frac{1}{x}$



$$\lim_{x \to \infty} \frac{1}{x} = \lim_{x \to -\infty} \frac{1}{x} = 0$$

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty \quad , \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

Note: 1) $\lim_{\substack{x \to \infty \\ -\infty}} c = c$

2) $\lim_{\substack{x \to \infty \\ x \to -\infty}} \frac{c}{x^n} = 0 \qquad n > 0$

3) $\lim_{n \to \infty} x^n = +\infty \qquad n > 0$

If $R(x) = \frac{p(x)}{q(x)}$, then

$$\lim_{x \to \infty} R(x) = \begin{cases} 0 & \text{deg. } p < \text{deg. } q \\ \text{Constant} & \text{deg } p = \text{deg } q \\ \pm\infty & \text{deg } p > \text{deg } q \end{cases}$$

Ex: 1) $\lim\limits_{x \to \infty} \dfrac{x^2+10x}{x^3-100} = 0$

2) $\lim \dfrac{\boxed{3}x^2+2x+4}{\boxed{5}x^2+10} = 5/3$

3) $\lim\limits_{x \to \infty} \dfrac{x^3+10}{x^2+1} = +\infty$

4) $\lim\limits_{x \to -\infty} \dfrac{x^3+10}{x^2+1} = -\infty$

Ex: $f(x) = \dfrac{x^2-1}{x^3+x^2}$

1) $\lim\limits_{x \to 1} f(x) = \dfrac{(1)^2-1}{1+1} = \dfrac{0}{2} = 0$

2) $\lim\limits_{x \to -1} f(x) = \left(\dfrac{(-1)^2-1}{(-1)^3+(-1)^2} = \dfrac{0}{0} \right)$

$\Rightarrow \lim\limits_{x \to -1} \dfrac{x^2-1}{x^3+x^2} = \lim\limits_{x \to -1} \dfrac{(x-1)\cancel{(x+1)}}{x^2\cancel{(x+1)}} = \dfrac{-1-1}{(-1)^2} = -2$

3) $\lim\limits_{x \to 0} f(x) = \dfrac{0-1}{0} = \dfrac{1}{0}$ limit DNE

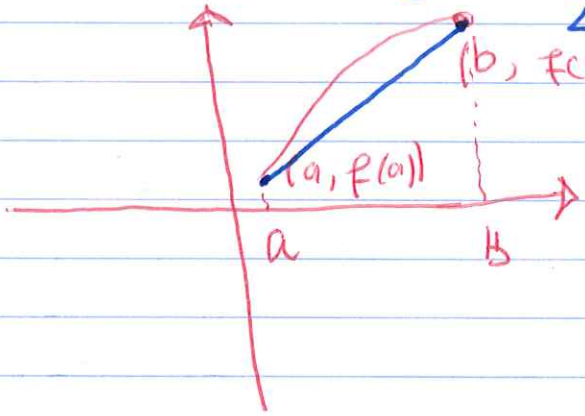
9.3 Rates of Changes and Derivatives

The slope m of a given line is defined by:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Definition: Consider a function $y = f(x)$ defined over an interval $[a, b]$, the average rate of change of $f(x)$ from $x = a$ to $x = b$ is given by:

$$\text{Average Rate of change} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$



$$\Delta x = b - a$$
$$b = a + \Delta x$$

$$\text{Average rate of change} = \frac{f(b) - f(a)}{b - a}$$
$$= \frac{f(a + \Delta x) - f(a)}{\Delta x}$$
$$= \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The average rate of change is the Slope of the secant Line (القاطع) joining $(a, f(a))$ and $(b, f(b))$

Ex ① $f(x) = \sqrt{2x+1}$ $\qquad x \in [0,4]$

$$\text{Average rate of change} = \frac{f(4) - f(0)}{4-0}$$

$$= \frac{\sqrt{9} - \sqrt{1}}{4-0} = \frac{2}{4} = \frac{1}{2}$$

② The Cost function for a product is given by $C(x) = x^2 + 5x + 100$

Find the average rate of change of $C(x)$ from $x = 10$ to $x = 20$ units

$$\text{Average rate} = \frac{C(20) - C(10)}{20-10}$$

$$= \frac{600 - 250}{10}$$

$= 35$ dollars per unit.

—o—

$$\text{Averag rate of change} = \frac{\Delta y}{\Delta x}$$

$$= \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{f(x+h) - f(x)}{h} \qquad \Delta x = h$$

As $\Delta x = h$ approaches 0, the slope of the secant approaches the slope of the tangent.

That is, $\displaystyle \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

if exists is the slope of the tangent line to $y = f(x)$.

- $\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}$ = Average Rate of change

- $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ if exists is called the instantaneous rate of change or simply the rate of change

- If the limit above exists, it is called the first <u>derivative</u> of the function $f(x)$ with respect to x. It is denoted by $f'(x)$

- $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ (limit exists)

- Notations: f', y', $\frac{dy}{dx}$, $\frac{df}{dx}$

- $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(x) \Big|_{x=a}$

- If the above limit exist, then $f(x)$ is said to be a differentiable function of x

④

Interpretations of the derivative

Given $y = f(x)$, find $y' = f'(x) = \frac{dy}{dx}$ means

1) The (rate) of $\overset{\text{change of}}{\uparrow}$ y with respect to x

2) The (velocity) (Instantaneous)

3) The (slope) of the tangent line to the graph of f(x)

4) (Marginal) (cost, revenue, profit)

$R'(x) = \overline{MR} \equiv$ marginal revenue

$C'(x) = \overline{MC} \equiv$ marginal Cost.

$P'(x) = \overline{MP} \equiv$ marginal profit

Derivative Formulas

Sections 9.4, 9.5 and 9.6 from the textbook.

- Recall that $f''(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

- Notations: $f', y', \frac{dy}{dx}, \frac{df}{dx}, \frac{d}{dx}(f) \equiv$ First derivative of $f'(x)$ with respect to x

- If $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists, the function

$f(x)$ is said to be a differentiable function of x.

Rules: If f, g are differentiable functions of x, c is a constant, then

Rule 1: $\frac{d}{dx}(x^n) = n x^{n-1}$ n is any real number

Example: ① $\frac{d}{dx}(x^{2020}) = 2020 x^{2020-1}$

$= 2020 x^{2019}$

② $\frac{d}{dx}\left(\frac{1}{x^{100}}\right) = \frac{d}{dx}\left(x^{-100}\right) = -100 x^{-101}$

③ $\frac{d}{dx}\left(\sqrt[5]{x^4}\right) = \frac{d}{dx}\left(x^{4/5}\right) = \frac{4}{5} x^{-1/5}$

$= \frac{4}{5\sqrt[5]{x}}$

Rule 2: $\frac{d}{dx}(c) = 0$ (c is a constant)

$y = c$ is a horizontal line $\Rightarrow$ slope $= 0$

Ex: $f(x) = 2020 \Rightarrow f'(x) = 0$

Rule 3: $\frac{d}{dx}[c\, f(x)] = c \cdot \frac{d}{dx}[f(x)] = c\, f'(x)$

Ex: 1) $\frac{d}{dx}(5x^8) = (5)(8x^7) = 40x^7$

2) $\frac{d}{dx}\left(\frac{5}{x^{10}}\right) = \frac{d}{dx}(5x^{-10}) = -50x^{-11}$

3) $\frac{d}{dx}\left(\frac{1}{4\sqrt[3]{x^5}}\right) =$

Rule 4: $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

(Term by Term Differentiation)

Ex: 1) $\frac{d}{dx}(3x^2 + 10x + 100) = 6x + 10 + 0$
$= 6x + 10$

2) $\frac{d}{dx}\left(\frac{4}{x^4} + 3\sqrt[4]{x} + 5x\right) =$

Rule 5: $\frac{d}{dx}(f\cdot g) = f\cdot g' + g\cdot f'$

Ex: $\frac{d}{dx}\underbrace{(x^3+10x)}_{f}\underbrace{(x^4+2x+5)}_{g}$

$= \underset{f}{(x^3+10x)}\cdot\underset{g'}{(4x^3+2)} + \underset{g}{(x^4+2x+5)}\cdot\underset{f'}{(3x^2+10)}$

Rule 6: $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\cdot f' - f\cdot g'}{g^2}$

Ex: $\frac{d}{dx}\left(\frac{\overset{f}{x^2+10}}{\underset{g}{x^3+3x^2}}\right) = \frac{\overset{g}{(x^3+3x^2)}\cdot\overset{f'}{(2x)} - \overset{f}{(x^2+10)}\overset{g'}{(3x^2+6x)}}{\underset{(g^2)}{(x^3+3x^2)^2}}$

Example: ① $f(x) = \frac{10}{x^4} + \frac{10}{\sqrt[4]{x}} + x^4 - 10$

$f(x) = 10x^{-4} + 10x^{-1/4} + x^4 - 10$

$f'(x) = -40x^{-5} - \frac{10}{4}x^{-5/4} + 4x^3$

$= \frac{-40}{x^5} - \frac{10}{4\sqrt[4]{x^5}} + 4x^3$

$f'(1) = f'(x)\Big|_{x=1} = \frac{-40}{(1)^5} - \frac{10}{\sqrt[4]{1}} + 4(1)^3$

$= -40 - 10 + 4$

$= -46$

2) If $f(x) = \dfrac{x^2+4x}{3x+2}$, find the equation of the tangent to the curve at $x=1$

Point: $(1, f(1)) = (1,1)$

$$\text{Slope} = f'(1) = \left.\frac{(3x+2)(2x+4)-(x^2+4x)(3)}{(3x+2)^2}\right|_{x=1}$$

$$= \frac{(5)(6)-(5)(3)}{(5)^2} = \frac{15}{25} = 3/5$$

Equation: $y - y_1 = m(x-x_1)$

$$y - 1 = 3/5\,(x-1)$$

$$y = 3/5\,x + 2/5$$

Example: (Application)

$R(x) = 100x - 0.1x^2$ (Revenue Function)

1) Find $R(400)$

$R(400) = (100)(400) - (0.1)(400)^2 = \$\,24000$

The revenue of producing and selling 400 Units is $\$\,24000$

2) Find the marginal revenue function

$\overline{MR} = R'(x) = 100 - 0.2x$

3) Find the marginal revenue at $x = 400, 500, 600$

$R'(400) = 100 - (0.2)(400) = \$\ 20$ per unit

$R'(500) = \$\ 0$

$R'(600) = -\$20$

Marginal revenue may be positive, negative or zero ($R(x) \geq 0$)

4) Find $R(20)$, $R(21)$, $R(21) - R(20)$, $R'(20)$
Explain your results

$R(20) = \$1960$, $R(21) = 2055.9$, $R'(20) = 96$

$R(21) - R(20) = 2055.9 - 1960 = 95.9$

$R(21) - R(20) = 95.9 \equiv$ The revenue of unit number 21 is \$95.9
<u>OR</u> The marginal revenue at $X=20$ is \$95.9

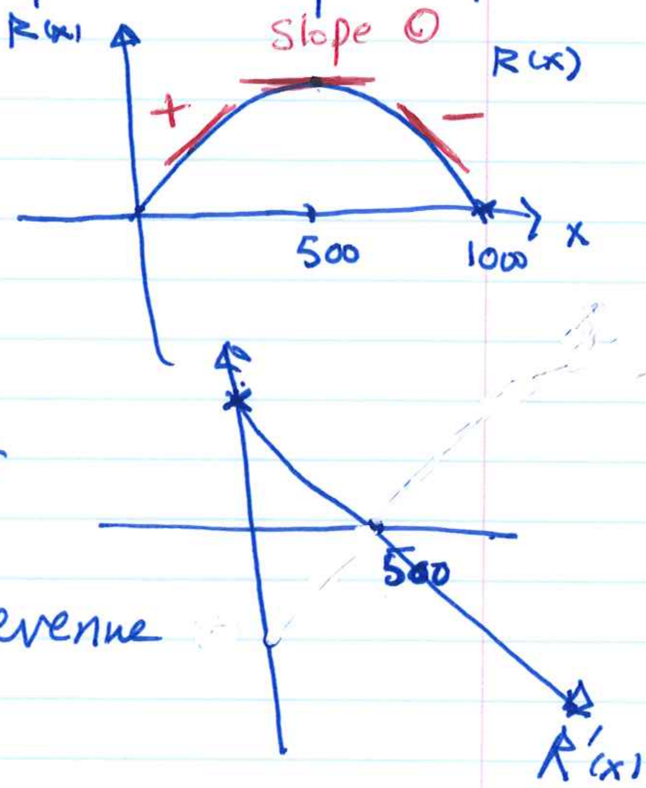
$R(21) - R(20)$ is the exact marginal revenue
$R'(20) = 96$ is the approximated m.r

$\Rightarrow\ R(21) - R(20) \approx R'(20)$

In general $\underbrace{R(a+1) - R(a)}_{\text{Exact}} \approx \underbrace{R'(a)}_{\text{Approximated}}$

$R(x) = 100x - 0.1x^2$ (parabola opens up)

$R'(x) = 100 - 0.2x$



(5) Explain : $R'(600) = -20$

$R(601) - R(600) \approx -20$

Producing one extra unit at level of production $x = 600$ will decrease the revenue by approximately $ 20

Note : If $C(x) \equiv$ Cost function, $C'(x) = \overline{MC}$
$P(x) \equiv$ profit function, $P'(x) = MP$

$$P = R - C \Rightarrow P' = R' - C$$

$$\overline{MP} = \overline{MR} - \overline{MC}$$

question # 51 page 589

$$q = \frac{1000}{\sqrt{p}} - 1 \qquad p > 0$$

(why this equation is a demand equation?)

Find the rate of change of demand with respect to price at p = \$25, p = \$100
Explain your answers.

Rate of change of demand with respect to price is $\frac{dq}{dp}$

$$q = 1000\, p^{-1/2} - 1$$

$$\frac{dq}{dp} = -(1000)\left(\tfrac{1}{2}\right) p^{-3/2}$$

$$= \frac{-500}{p^{3/2}}$$

$$\left.\frac{dq}{dp}\right|_{p=25} = -4$$ Units per dollar ??

It means: An increase of \$1 of price at p = \$25 will decrease the demand by four units

Ex: Find the equation of the tangent to $f(x) = \frac{x^3}{3} - \frac{3}{x^3}$ at $x = -1$

$\left(\frac{30}{589}\right)$

$f(x) = \frac{x^3}{3} - 3x^{-3}$

point: $x = -1 \Rightarrow f(-1) = \frac{(-1)^3}{3} - \frac{3}{(-1)^3}$

$= \frac{-1}{3} + 3 = \frac{8}{3}$

$f'(x) = x^2 + 9x^{-4}$

$f'(1) = (-1)^2 + + \frac{9}{(-1)^4} = 10$

Equation: $y - \frac{8}{3} = 10(x+1)$

$y = 10x + \frac{38}{3}$

Ex: Find the point(s) where f(x) has horizontal tangent(s).

$f(x) = x^3 - 6x^2 + 1$

Horizontal tangents $\Rightarrow f'(x) = 0$

$f'(x) = 3x^2 - 12x$

$f'(x) = 0 \Rightarrow 3x(x-4) = 0$

$x = 0, 4$

$f(0) = 1 \Rightarrow (0,1)$

$f(4) = 64 - 96 + 1 = -31 \Rightarrow (4, -31)$

at (0,1), (4,-31) f(x) has horizontal tangents

9.6 The chain Rule and the Power Rule

Recall: Composite Functions

If f and g are defined functions, then the composite $g \circ f$ (g after f) is defined by

$$(g \circ f)(x) = g(f(x))$$

Example: $f(x) = 2x+5$, $g(x) = x^2+1$

$$(g \circ f)(1) = g(f(1)) = g(7) = 49+1 = 50$$
$$(f \circ g)(1) = f(g(1)) = f(2) = 4+5 = 9$$

$$(f \circ g)(x) = f(g(x)) = f(x^2+1)$$
$$= 2(x^2+1)+5$$
$$= 2x^2+7$$

Chain Rule

If $y = f(u)$, $u = g(x)$ $[y = f(g(x)) = (f \circ g)(x)]$

then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$= f'(u) \cdot g'(x)$$
$$= f'(g(x)) \cdot g'(x)$$

$$y = (f \circ g)(x) \Longrightarrow y' = [(f \circ g)(x)]' = f'(g(x)) \cdot g'(x)$$

Power Rule: If $y = (f(x))^n$

$$y' = n(f(x))^{n-1} \cdot f'(x)$$

Ex: 1) $y = \sqrt{x^2+1} \Rightarrow y = (x^2+1)^{\frac{1}{2}}$

$$y' = \left(\frac{1}{2}\right)(x^2+1)^{-1/2} \cdot 2x$$

$$= \frac{x}{\sqrt{x^2+1}}$$

2) $y = \sqrt[3]{(x^3+5)^2}$

$$y = (x^3+5)^{2/3}$$

$$y' = \frac{2}{3}(x^3+5)^{-1/3} \cdot 3x^2$$

$$= \frac{2x^2}{\sqrt{x^3+5}}$$

3) $g(x) = \frac{4}{3x^2+10x}$

$$g(x) = 4(3x^2+10x)^{-1}$$

$$g'(x) = (4)(-1)(3x^2+10x)^{-2} \cdot (6x+10)$$

$$= \frac{(-4)(6x+10)}{(3x^2+10x)^2}$$

4) Find the equation of the tangent to $f(x) = (x^3+1)^{5/2}$ at $x=2$

Point: $f(2) = (9)^{5/2} = 243$

: $(2, 243)$

slope $= f'(2)$

$$= \left(\frac{5}{2}\right)(x^3+1)^{3/2}(3x^2)$$

$$= (5/2)(27)(12) = 810$$

Equation: $y - 243 = 810(x-2)$

Ex: If $p = \frac{540}{\sqrt{2q+1}}$, p = price in dollars
q = quantity demanded

Find the rate of change of price p with respect to quantity demanded q at $q=4$ unit. Interpret your answer.

$$p = 540(2q+1)^{-1/2}$$

$$\left.\frac{dp}{dq}\right|_{q=4} = \left.(540)\left(-\frac{1}{2}\right)(2q+1)^{-3/2}\cdot 2\right|_{q=-4}$$

$$= -\left.\frac{540}{(2q+1)^{3/2}}\right|_{q=4}$$

$$= \frac{-540}{27} = -20 \text{ dollar per price}$$

If the demand increased by 1 unit at the level of production $q=4$, then the price will increase by \$20

3^{*}

Ex: If $h(x) = (f \circ g)(x)$, $f(2) = 3$, $g(3) = 1$
$f'(1) = -2$, $g'(1) = 3$, $f'(3) = 2$, $g'(3) = 4$
$f'(2) = -1$, $g'(2) = 2$

Find 1) $(f \circ g)(3)$

$$(f \circ g)'(3) = f'(g(3)) \cdot g'(3)$$
$$= f'(1) \cdot g'(3)$$
$$= (-2)(4)$$
$$= -8$$

2) $(g \circ f)'(2)$

$$(g \circ f)'(2) = g'(f(2)) \cdot f'(2)$$
$$= g'(3) \cdot f'(2)$$
$$= (4)(-1)$$
$$= -4$$

Ex: $y = \sqrt{h(x)}$ $\quad h(1) = 1$, $h'(1) = -2$
Find $y'(1)$.

9.7 Using Derivative Formulas

Ex ①1) $f(x) = \left(\frac{x^2}{2x+1}\right)^{10}$, find $f'(x)$

$$f(x) = \frac{(x^2)^{10}}{(2x+1)^{10}}$$

$$= x^{20} \cdot (2x+1)^{-10}$$

$$f'(x) = (20)(x^{19}) \cdot (2x+1)^{-10} + x^{20}\left(-10(2x+1)^{-11}(2)\right)$$

$$=$$

② $y = x^3 \sqrt[3]{x^2+8}$. Find $y'(0)$

$$y' = (x^3)\left(\tfrac{1}{3}\right)(x^2+8)^{-2/3} \cdot 2x + \left(\sqrt[3]{x^2+8}\right)(3x^2)$$

$$y'(0) = 0$$

(3) $R(x) = 600x + \frac{4000}{x+10}$

Find and interpret the m.r at $x=10$

$$R = 600x + 4000(x+10)^{-1}$$

$$R' = 600 - 400(x+10)^{-2}$$

$$R'(10) = 600 - \frac{4000}{(10+10)^2} = \#560$$

Producing and selling one extra unit at $x=10$ will increase the revenue by approximately # 560

9.8 Higher Order Derivatives

If $y = f(x)$, $y' = f'(x) \equiv 1^{st}$ derivative of $f(x)$ with respect to x

$$(y')' = (f')' = f'' = \frac{d^2y}{dx^2} \quad \left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right)$$

$\equiv$ The second derivative of $f(x)$ with respect to x

$$y, \; y', \; y'', \; y''', \; y^{(4)}$$

$$\Downarrow$$

$$(y''')' = y''''$$

Ex: 1) $f(x) = x^4 + 10x^3 + 100$

$f' = 4x^3 + 30x^2$

$f'' = 12x^2 + 60x$

$f''' = 24x + 60$

$f^{(4)} = 24$

$f^{(5)} = 0$

2) $f(x) = (x+1)^{-10}$

$f' = (-10)(x+1)^{-11}$

$f'' = 121(x+1)^{-12}$

$\Downarrow$

Ex: If $R(x) = 20x - 3000(3x+10)^{-1} - 30$

$R(x)$ in \$1000

1) Find $R(30)$

$$R(30) = (20)(30) - \frac{3000}{100} - 30$$

$$= 600 - 30 - 30$$

$= \$540$ (1000) Total revenue

2) Find the marginal revenue at $x=30$

$$R'\Big|_{x=30} = 20 - \frac{9000}{(3x+10)^2}\Big|_{x=30}$$

$$= 20 - \frac{9000}{(100)^2}$$

$$= 20 - 0.9$$

$= \$19.1$ (1000)

$R(31) - R(30) \approx 19.1$

3) At what rate is the marginal revenue changing when $x=30$

⇒ This mean Find $(R')' = R''\Big|_{x=30}$

$$R = 20 - 9000(3x+10)^{-2}$$

$$R'' = (-2)(-9000)(3x+10)^{-3}(3)$$

$$R''(30) = \frac{54000}{(3x+10)^3} = 0.054$$

$R''(31) - R''(30 \approx 0.054$: When 1 more unit sold at $x=30$, the marginal revenue will increase by approximately (0.054)(1000) dollars.

9.9 Applications : Marginals and Derivatives

Recall: $R' = \overline{MR} \equiv$ Marginal Revenue

$C' = \overline{MC} \equiv$ Marginal Cost

$P' = \overline{MP} \equiv$ Marginal Profit

$P = R - C \Rightarrow P' = R' - C'$

Example(1): The demand for a product is given by $p = 1000 - 20x$

1) Find the total revenue function

$$R(x) = px = (1000 - 20x)x$$
$$= 1000x - 20x^2$$

2) Find the marginal revenue function

$$\overline{MR} = R' = 1000 - 40x$$

3) Find and Explain: $R(20)$, $R(21)$, $R'(20)$

$R(21 - R(20)$

$R(20) = (1000)(20) - (20)(20)^2$

$= \$ 12000$

$R(21) = \$ 12180$

$R(21) - R(20) = 180$ Exact $\overline{MR}$

$R'(20) = 1000 - (40)(20) = 200$ Approximated $\overline{MR}$

$R(21) - R(20) \approx R'(20)$

Example 2: $C(x) = 0.001x^3 - 0.3x^2 + 32x + 2500$

1) The marginal Cost function

$C' = \overline{MC} = 0.003x^2 + 0.6x + 32$

2) Find $C'(80)$

$C'(80) = (0.003)(80)^2 + (0.6)(80) + 32$

$= \$3.2$

3) Interpret your answer.

$C'(80) = \$3.2$ means that: producing one more unit (at level of production 80) will increase the cost by approximately $3.2

The Cost of unit #21 is approximately $3.2

Note: $C(81) - (80) = \$3.14$ Exact $\overline{MC}$

Example 3: $P(x) = 20\sqrt{x+1} - 2x - 22$

↑ Profit function

1) $\overline{MP} = P' = \frac{10}{\sqrt{x+1}} - 2$

2) $\overline{MP}$ at $x = 3 = P'(3) = \frac{10}{\sqrt{4}} - 2 = \$3$ per unit

3) Explain: The profit from saling the 4^{th} unit is approximately $3

$\underset{\text{Exact}}{P(4) - P(3)} \simeq \underset{\text{Approximated.}}{P'(3)}$

Example 4: (27/625)

Given $R(x) = 50x$, $C(x) = 0.01x^2 + 30x + 1900$

$\Rightarrow P = R - C$

$= -0.01x^2 + 20x - 1900$.

$P(500) = \$5600 \equiv$ producing and selling 500 units yeilds a probit of \$5600

$P' = -0.02x + 20$

$P'(500) = -0.02x + 20 \Big|_{x=50}$

$= \$10 \equiv$ The approximated probit of unit number 501 is \$10

$P(501) - P(500) = \$9.99$

$P(501) - P(500) \approx P'(500)$

$\$9.9 \approx \$10$

Exact Approximated

Find $\overline{MP}$ at $x = 2000$

$P'(2000) = (-0.02)(2000) + 20$

$= -\$20$

Unit # 2001 will decrease the probit by approximately \$20

$P'(1000) = \$0$. Explain

BIRZEIT UNIVERSITY
MATHEMATICS DEPARTMENT
MATH 2351
Chapter 9 Additional Exercises

1. The profit function for a company is given by

$$P(x) = -0.25x^2 + 100x - 1000$$

Find the **rate of change** in profit if 600 items are manufactured and sold.

2. Find the `point(s) for which the function $f(x) = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + 1$ has (have) horizontal tangent(s).

3. Find $\lim\limits_{h \to 0} \frac{(2+h)^4 - 16}{h}$

4. Suppose that a demand function for a product is given by $p = 20 - 0.8q$ (q = # of units, p = price in dollars). Find and **interpret** the marginal revenue when q = 10.

5. If $f(x) = \frac{8(9-3x)^5}{5}$, find $f(x)'$.

6. Determine the point(s) at which f(x) is not continuous

$$f(x) = \begin{cases} x^2 - 4 & x < -1 \\ 0 & -1 \le x \le 1 \\ x^2 + 4 & x > 1 \end{cases}$$

7. If $f(x) = x^2 g(x)$, $g(1) = 1$, and $g'(1) = -2$, find $f'(2)$.

8. Given $h(x) = \sqrt{f(x)}$, $f(3) = 16$ and $f'(3) = 9$. What is $h'(3)$?

9. Find the average rate of change for $f(x) = \sqrt{\frac{x+2}{2}}$ from x = 6 to 30.

10. Suppose the demand function is $p(x) = \frac{100}{\sqrt{x}}$ and the cost function is C(x) = x +50.

- **a.** Find the rate of change of price with respect to the number of units demanded at x = 2500. Explain your answer
- **b.** Find the **marginal profit** when x = 2500.

11. A firm has a monthly costs given by $C(x) = 45000 + 100x + x^3$, where x is the number of units produced per month. The firm sells its product in a competitive market for $4600.

- **a.** Find the **exact** profit obtained from selling the **21st unit**.
- **b.** Find the **approximated** profit obtained from selling the 21st unit.

12. Let $f(x)=\frac{2x^2+x-3}{x^2+4x-5}$, find

- **a.** $\lim\limits_{x\to 0} f(x)$
- **b.** $\lim\limits_{x\to 1} f(x)$
- **c.** $\lim\limits_{x\to -5} f(x)$
- **d.** $\lim\limits_{x\to \infty} f(x)$

13. If $f(x)=\frac{x+2}{x-2}$, find $f'(4)$

14. Write the equation of the **tangent** to the curve $f(x)=(3+x)^{\frac{2}{3}}$ at x = -10.

15. If $C(x)=2x^{\frac{5}{2}}+100$

- **a.** What is the marginal cost function?
- **b.** Use marginal cost to find the approximated cost to produce the 5th unit

16. Let f and g be two functions satisfy $f(4)=2$, $f'(4)=-1$ and $g'(4)=-3$. Find $h'(4)$ for $h(x)=3(f(x))^2-g(x)+2x$.

17. The demand for a product is given by

$$x=100+\frac{1000}{p^2+1}$$

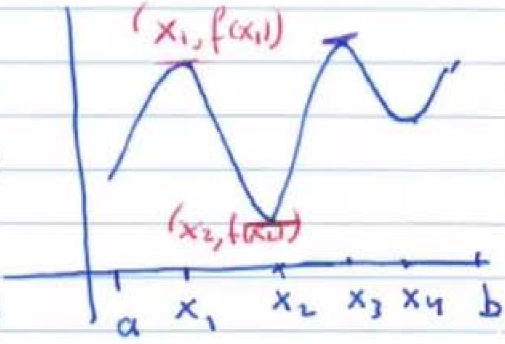
Find and **explain** the rate of change of demand with respect to price at p = $3.

GOOD LUCK

Chapter 10: Applications of Derivatives

Definitions:

① The point $(x_1, f(x_1))$ is a relative (local) maximum point of $f(x)$ if $f(x_1) \geq f(x)$ for all x in an interval around x_1



② $(x_2, f(x_2))$ is a relative minimom point of $f(x)$ if $f(x_2) \leq f(x)$ for all x in an interval around x_2

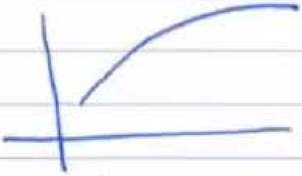
③ $(c, f(c))$ is an absolute (Global) maximum point of $f(x)$ if $f(c) \geq f(x)$ for all x in the domain of f

④ $(c, f(c))$ is an absolute maximom point for $f(x)$ if $f(x) \geq f(c)$ for all x in the domain of f.

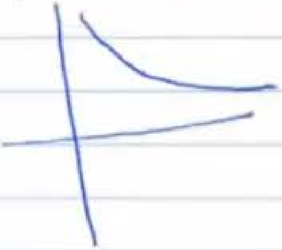
- $x = x_1$ } Max
  $x = x_3$ }
- $x = x_2$ } Min
  $x = x_4$ }
- $f(x) \geq f(x_2)$ for all x: x_2 is ABS Max
- $f(x) \leq f(x_4)$ for all x: x_4 is ABS Max.

Definitions:

① $f(x)$ is increasing on an interval I if: $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$



② $f(x)$ is decreasing on an interval I if: $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$



First Derivative sign

① If $f'(x) > 0$ for all x in (a, b), then $f(x)$ is increasing on (a, b)

② If $f'(x) < 0$ for all x in (a, b), then $f(x)$ is decreasing on (a, b)

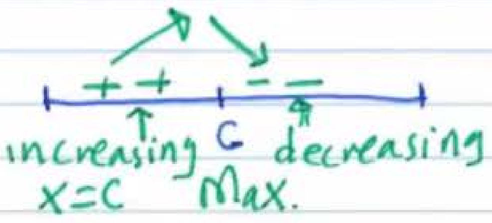
③ If $f(x)$ has a relative maximum (minimum) at $x = c$, then $f'(c) = 0$ or $f'(c)$ is undefined

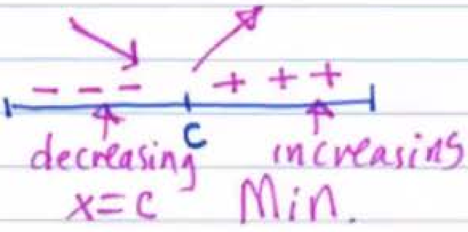
④ $x = c$ is a critical value for $f(x)$ if : $f'(c) = 0$ or $f'(c)$ is undefined
$(c, f(c))$ is a critical point.

⑤ If $x = c$ is a critical value for $f(x)$, then $f(x)$ may or may not have a relative maximum or a relative minimum at $x = c$

First Derivative Test : Increasing, Decreasing

1) Find $f'(x)$
2) Find critical value(s) for $f(x)$
   let $x = c$ be a critical value for $f(x)$
3) Create a sign diagram for $f'(x)$
4)





Neither Max, Nor Min.

Increasing always Decreasing always

Second Derivative Sign

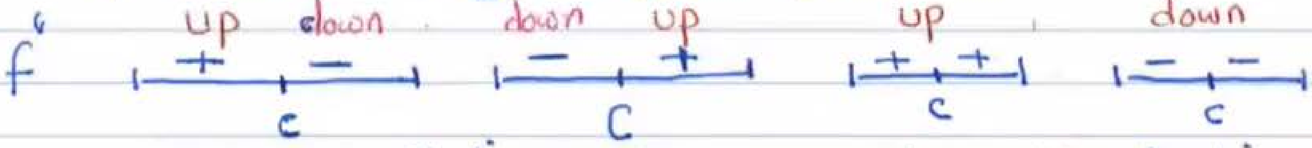
① A function $f(x)$ is concave up on intervals where $f''(x) > 0$ (f' is increasing)

② $f(x)$ is concave down on intervals where $f''(x) < 0$ (f' is decreasing)

③ The point where concavity changes is called an *inflection point*.

Second Derivative Test for Concavity

1) Find f', f''
2) Find $f''=0$ or f'' is undefined
   let $x=c$ be such a value.
3) Create a sign diagram for $f''(x)$



$x=c$ inflection point

$x=c$ Not inflection

Example: $f(x) = x^3 - \frac{3}{2}x^2 - 18x + 5$

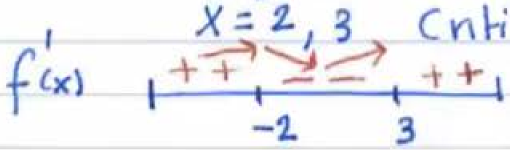
f'

$f'(x) = 3x^2 - 3x - 18$

$f'(x) = 0 \Rightarrow 3(x^2 - x - 6) = 0$

$\Rightarrow 3(x-3)(x+2) = 0$

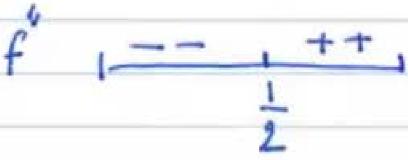
$x = -2, 3$ Critical values



f''

$f''(x) = 6x - 3$

$f''(x) = 0 \Rightarrow x = 1/2$



- f(x) is increasing on : $(-\infty, -2)$, $(3,\infty)$
  decreasing on: $(-2, 3)$
  $x=-2$ Maximom
  $x=3$ minimum
- f(x) is concave up on: $(\frac{1}{2}, \infty)$
  concave down on: $(-\infty, 1/2)$
  $(\frac{1}{2}, f(\frac{1}{2}))$ inflection point.

Example: $f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2$, $f''(x) = 12x^2 - 24x$

$f'(x) = 0$

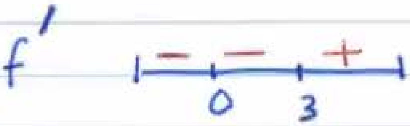
$4x^2(x-3) = 0$

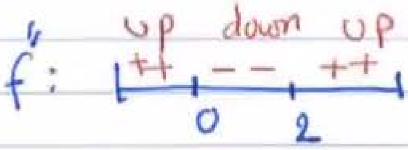
$x = 0, 3$

$f''(x) = 0$

$12x(x-2) = 0$

$x = 0, 2$





Increasing: $(3, \infty)$
Decreasing: $(-\infty, 3)$
$x=3$ Minimom (Absolute)
$x=0$ Horizontal point of inflection
Note that $f'(0)=0$ but is not Max. or Min

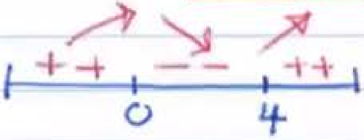
Inflection Points
$(0, f(0))$, $(2, f(2))$

Example: Given $f(x) = x^{4/3}(x-7)$
$f'(x) = 7x^{1/3}(x-4)$
$f''(x) = \frac{28(x-1)}{9x^{2/3}}$

- $f'(x)=0 \Rightarrow \frac{7x^{1/3}(x-4)}{3}=0$

$7x^{1/3}=0 \Rightarrow x=0 \qquad (x-4)=0 \Rightarrow x=4$

$x=0, 4$ Critical values

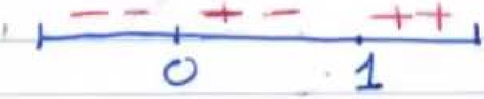


$x=0$ Maximom

$x=4$ Minimom

- $f''(x)=0 \Rightarrow 28(x-1)=0 \Rightarrow x=1$

$f''(x)$ undefined when $x^{2/3}=0 \Rightarrow x=0$



Concave up: $(1, \infty)$

concave down $(-\infty, 1)$

$(1, f(1))$ inflection point

Note that $f''(0)$ is not defined but it is not inflection point.

Second Derivative Test For Maxima and Minima

To find max. or min. for $f(x)$

1) Find f', f''

2) Find the critical values for $f(x)$

let $x=c$ be a critical value

3) $f''(c)>0 \Rightarrow x=c$ is minimom

$f''(c)<0 \Rightarrow x=c$ is maximom

$f''(c)=0 \Rightarrow$ test fails

Example: $f(x) = 4x^3 - 3x^4 + 1$

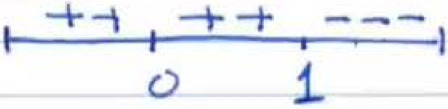
$f'(x) = 12x^2 - 12x^3$ $\quad$ $f''(x) = 24x - 36x^2$

$f'(x) = 0 \Rightarrow 12x^2(1-x) = 0$

$\Rightarrow x = 0, 1$

$f''(1) = 24 - 36 < 0 \Rightarrow x = 1$ Maximom

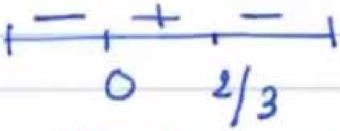
$f''(0) = 0 \Rightarrow$ Second derivative test fails



$x = 0$ Horizontal point of inflection

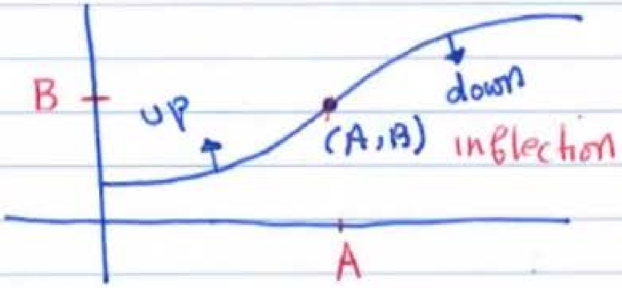
$f''(x) = 0 \Rightarrow 12x(2 - 3x) = 0$

$x = 0, 2/3$



$(0, f(0))$
$(2/3, f(2/3))$ inflection points

Point of Diminishing Returns



$f(x)$ is increasing

$\underbrace{f'(x) > 0 \quad x < A}_{\text{concave up}}$

$\underbrace{f'(x) < 0 \quad x > A}_{\text{concave down}}$

$x = A$ maximum of f'

The point (A, B) is called the point of diminishing returns (The point where $f'(x)$ has its Maximum)

Example: Given $p(x) = -0.2x^3 + 3x^2 + 6$

$x \geqslant 0$

1) Find the Maximum profit.

$$p'(x) = -0.6x^2 + 6x$$

$$p' = 0 \Rightarrow -0.6x^2 + 6x = 0$$

$$x(-0.6x + 6) = 0$$

$$x = 0, 10$$

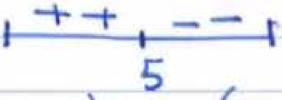
$x = 10$ is maximum

The maximum profit is \$ $P(10)$

2) Find the maximum of the marginal profit (the point of diminshing retournes)

$$P'' = -1.2x + 6$$

$$P'' = 0 \Rightarrow -1.2x + 6 = 0 \Rightarrow x = 5$$



$x = 5$ is Max.

$\Rightarrow (5, P(5)) = (5, 56)$ is the point of diminshing retournes.

Example: Given $C(x) = 5000x^2 + 125000, \quad x > 0$

The average cost is defined by

$$\bar{C}(x) = \frac{C(x)}{x}$$

$$= 5000x + \frac{125000}{x}$$

Find the minimum average cost per unit.

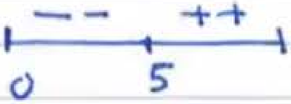
$$(\bar{C}(x))' = 5000 - \frac{125000}{x^2}$$

$$(\bar{C}(x))' = 0 \implies 5000 - \frac{125000}{x^2} = 0$$

$$\implies 5000x^2 = 125000$$

$$x^2 = 25$$

$$x = \pm 5$$



$x = C$ is minimum

The minimum average cost per unit is

$$\bar{C}(5) = (5000)(5) + \frac{125000}{5}$$

$$= \$\,50000$$

Optimization in Busines and Economics

To optimize ≡ to find the max. or min. (Absolute)

Recall: Definitions 1-4 page 1

Notes:

(1) If $f(x)$ is continuous on a closed interval $[a, b]$ then $f(x)$ takes both an absolute maximum and an absolute minimom on $[a, b]$.

(2) The abs. max. min. may occur at the endpoints of the interval or at the critical value(s).

(3) To find the absolute, compare the function values at the endpoints and the critical value(s).

(4) To test for max or min use 1st derivative test or 2nd derivative test.

Example: 8/671

Find the Maximum revenue for

$$R(x) = 2800x - 8x^2 - x^3.$$

$R' = 2800 - 16x - 3x^2$

$R' = 0 \Rightarrow 3x^2 + 16x - 2800 = 0 \Rightarrow (3x+100)(x-28) = 0$

$x = 28$

$R'' = -6x - 16$, $R''(28) < 0 \Rightarrow x = 28$ MAX.

The MAX revenue is $\$R(28) = \$\ 50167$

⑩

37/673 Demand: $p = 600 - \frac{1}{2}x$

Average Cost: $\bar{C}(x) = 300 + 2x$

1) Write the profit function. Find the quantity that will maximize the profit (optimal level of production) ≡ optimal quantity

$$R(x) = px = 600x - \frac{1}{2}x^2$$

$$C(x) = x\bar{C}(x) = 300x + 2x^2$$

$$P(x) = R(x) - C(x) = 300x - \frac{5}{2}x^2$$

$$P' = 300 - 5x$$
$$P'' = -5$$

$$P' = 0 \Longrightarrow x = 60$$

$P''(60) < 0 \Longrightarrow x = 60$ is Max.

$x = 60$ is the optimal level of production

Max. profit is \$ $P(60)$ when 60 units are produced and Sold

2) Find the selling price at the optimal level of production

$$p = 600 - \frac{1}{2}x \Longrightarrow p\Big|_{x=60} = 600 - \left(\frac{1}{2}\right)(60) = \$570$$

$\boxed{20/672}$ $C(x)=(x+5)^3$ $x=$ # of (100) of units producing

$C(x)$ in $

Find the minimum average Cost per unit

$$\bar{C}(x)=\frac{C(x)}{x}=\frac{(x+5)^3}{x} \qquad x>0$$

$$(\bar{C}(x))'=\frac{(x)(3)(x+5)^2-(x+5)^3}{x^2}$$

$$=\frac{(x+5)^2(3x-x-5)}{x^2}$$

$$=\frac{(x+5)^2}{x^2}(2x-5)$$

$$(\bar{C}(x))'=0 \Rightarrow 2x-5=0 \Rightarrow x=2.5$$

-- +++ $(\bar{C})'$

0 2.5

$x=(2.5)(1000)=2500$ units

minimizes the average Cost per unit

The minimum average cost per unit is

$$\bar{C}(2.5)=\frac{(2.5+5)^3}{2.5}=\frac{(7.5)^3}{2.5}$$

$\boxed{12/671}$

# of persons (x)	charge $ (P)
1000	5
900	6
800	7
⋮	⋮

Each additional increase of $1 will decrease the number of persons by 100 persons

(12)

- Find equation of price:

Points: (1000, 5), (900, 6), ...

Slope $= \frac{\Delta P}{\Delta X} = \frac{-1}{100} = -0.01$

Equation: $P - P_1 = m(X - x_1)$

$$P - 5 = \frac{-1}{100}(X - 1000)$$

$$P = -0.01X + 15$$

$R = PX = -0.01X^2 + 15X$

$R' = -0.02X + 15$

$R'' = -0.02$

$R' = 0 \Rightarrow X = 750$

$R''(750) < 0 \Rightarrow X = 750$ is MAX

Max revenue is $R(750) = \$\ 5625$

12. If club members charge \$5 admission to a classic car show, 1000 people will attend, and for each \$1 increase in price, 100 fewer people will attend. What price will give the maximum revenue for the show? Find the maximum revenue.

Chapter 11: Derivatives Continued

11.1, 11.2: Derivatives of Logarithmic and Exponential Functions

Recall from chapter 5

1) $y = \log_a x \Leftrightarrow a^y = x$

$\Updownarrow$ logarithmic Form

$\Updownarrow$ Exponential Form

2) $\log x \equiv \log_{10} x$, $\ln x = \log_e x$

3) $\log_a xy = \log_a x + \log_a y$

4) $\log_a \left(\frac{x}{y}\right) = \log_a x + \log_a y$

5) $\log_a x^n = n \log_a x$

6) $\log_a a^x = x$, $a^{\log_a x} = x$

Derivatives

$\boxed{1}$ If $y = \ln x$, then $\boxed{\frac{dy}{dx} = \frac{1}{x}}$

$y = \ln[u(x)] \Rightarrow \boxed{y' = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u}}$

Ex ① $y = x + 4\ln x$

$y' = 1 + (4)\left(\frac{1}{x}\right) = 1 + 4/x$

2) $y = x^3 \ln x$

$y' = x^3 \cdot \frac{1}{x} + (\ln x)(3x^2)$

$= x^2 + 3x^2 \ln x$

3) $$y = \ln(x^3+5x^2+10)$$

$$y' = \frac{3x^2+10x}{x3+5x^2+10}$$

4) $$y = \ln x^{10}$$

$$= 10 \ln x$$

$$y' = \frac{10}{x}$$

5) $$y = \ln\left(\frac{2x^4}{(x+5)^{10}}\right)$$

$$= \ln(2x^4) - 10\ln(x+5)$$

$$= \ln 2 + 4\ln x - 10\ln(x+5)$$

$$y' = \frac{4}{x} - \frac{10}{x+5}$$

6) $$y = \sqrt{\ln x}$$

$$= (\ln x)^{1/2}$$

$$y' = \left(\tfrac{1}{2}\right)(\ln x)^{-1/2} \cdot \frac{1}{x}$$

$$= \frac{1}{2x\sqrt{\ln x}}$$

7) $$y = \frac{\ln x}{x}$$

Example: If $C(x) = 10\ \ln(3x+1) + 100$. Find the maginal cost $x = 8$

$$\overline{MC} = C'(x) = \frac{30}{3x+1} \quad , \quad C'(8) = \frac{30}{25} = 1.2$$

The next unit (unit #9) will increase the cost by approximately \$1.2

$\boxed{2}$ Since $\log_a x = \frac{\ln x}{\ln a}$, then

$$\frac{d}{dx}\left(\log_a x\right) = \frac{d}{dx}\left(\left(\frac{1}{\ln a}\right)\ln x\right)$$

$$= \frac{1}{(\ln a)x}$$

$$\boxed{\begin{array}{l} y = \log_a x \Longrightarrow y' = \dfrac{1}{(\ln a)(x)} \\ y = \log_a u(x) \Longrightarrow y' = \dfrac{u'(x)}{(\ln a)u(x)} \end{array}}$$

Eample: 1) $y = \log_5 x \Longrightarrow y' = \frac{1}{(\ln 5)x}$

2) $y = \log x \longrightarrow y' = \frac{1}{(\ln 10)x}$

3) $y = \log_2(x^2+4) \Longrightarrow y' = \frac{2x}{(\ln 2)(x^2+4)}$

4) $y = \log_3 x^3(x^2+5)^4$

$$= 3\log_3 x + 4\log_3(x^2+5)$$

$$y' = \frac{3}{(\ln 3)x} + \frac{(4)(2x)}{(\ln 3)(x^2+5)}$$

③
$$\boxed{\begin{aligned} y = e^{x} &\Longrightarrow y' = e^{x} \\ y = e^{u(x)} &\longrightarrow y' = e^{u(x)} \cdot u'(x). \end{aligned}}$$

Ex: 1) $y = e^{3x^2+5}$

$y' = e^{3x^2+5} \cdot 6x$

$= 6x\, e^{3x^2+5}$

2) $y = (e^{3x}+1)^{10}$

$y' = (10)(e^{3x}+1)^{9} \cdot e^{3x} \cdot 3$

$= 30\, e^{3x}(e^{3x}+1)^{9}$

3) $y = \frac{x^2}{e^x}$

$= x^2 e^{-x}$

$y' = (x^2)(e^{-x})(-1) + (e^{-x})(2x)$

$= -x^2 e^{-x} + 2x e^{-x}$

④
$$\boxed{\begin{aligned} y &= a^{x} \\ y' &= a^{x} \cdot \ln a \end{aligned}} \quad a > 0,\ a \neq 1$$

$$\boxed{y = a^{u(x)} \Longrightarrow y' = a^{u(x)} \cdot \ln a \cdot u'}$$

Ex: 1) $\frac{d}{dx}(10^x) = 10^x \ln 10$

2) $\frac{d}{dx}\left(2^{3x+4}\right) = 2^{3x+4} \cdot \ln 2 \cdot 3$

$= (\ln 8)\, e^{3x+4}$

3) $y = e^x \Longrightarrow y' = (e^x)(\ln e) = e^x$

Example

27/714 $y = \frac{1+e^{5x}}{e^{2x}}$, find y'

$= \frac{1}{e^{2x}} + \frac{e^{5x}}{e^{2x}}$

$= e^{-2x} + e^{3x}$

y'

35/714 $y = xe^{-x}$. Find the tangent equation at $x = 1$

point: $(1, f(1)) = (1, \frac{1}{e})$

Slope $= f'(1) = -xe^{-x} + e^{-x} \Big|_{x=1} = -e^{-x} + e^{-x} = 0$

H.T

$\Rightarrow$ equation: $y = \frac{1}{e}$

48/715 $R(x) = 1000\,x\,e^{-x/50}$

Find the marginal revenue function

$R' = (1000x)\left(e^{-x/50} \cdot -\frac{1}{50}\right) + \left(e^{-x/50}\right)(1000)$

$= -20\,xe^{-x/50} + 1000\,e^{-x/50}$

11.3 Implicit Differentiation

$y = f(x)$ is an explicit function
y is written in terms of x explicitly

$*$ $xy^2 + x^3 + y = 10xy$: y can't be written as a function of x explicitly

If $y = f(x) \Rightarrow y^{10} = (f(x))^{10}$

$$(y^{10})' = ((f(x))^{10})'$$
$$= 10(f(x))^9 \cdot f'(x)$$
$$= 10 \cdot y^9 \cdot y'$$

To find $\frac{dy}{dx}$ for equation $*$ we will use implicit differentiation

Ex: 11 If $x^3 + y^2 = 10x + y$ Find y'

$$3x^2 + \underline{2yy'} = 10 + \underline{y'}$$
$$2yy' - y' = 10 - 3x^2$$
$$y'(2y-1) = 10 - 3x^2$$
$$y' = \frac{10 - 3x^2}{2y-1}$$

2) $x^4 + 5xy^4 = 2y^2 + x^2 + 10$ Find y'

$$4x^3 + \underline{(5x)(4y^3y')} + (y^4)(5) = \underline{(2)(2)yy'} + 2x$$

$$(20xy^3 - 4y)y' = 2x - 4x^3 - 5y^4$$

$$y' = \frac{2x - 4x^3 - 5y^4}{20xy^3 - 4y} \checkmark$$

Ex: (6|721) Given: $x^2 + 4y^2 - 2x + 4y - 2 = 0$
At what point(s) does the curve have horizontal tangent? Vertical tangent?

H.T: $y' = 0$ V.T: y' undefined

$$2x + 8yy' - 2 + 4y' = 0$$

$$\Longrightarrow y' = \frac{1 - x}{4y + 2}$$

$y' = 0 \Longrightarrow 1 - x = 0 \Longrightarrow x = 1$

$x = 1 \Longrightarrow 1 + 4y^2 - 2 + 4y - 2 = 0$

$4y^2 + 4y - 3 = 0$

$(2y - 1)(2y + 3) = 0 \Longrightarrow y = \frac{1}{2}, -3/2$

points: $(1, 1/2)$, $(1, -3/2)$ HT

y' undefined $\Longrightarrow 4y + 2 = 0 \Longrightarrow y = -1/2$

$y = -1/2 \Longrightarrow x^2 + 1 - 2x - 2 - 2 = 0$

$x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0 \Longrightarrow x = -1, 3$

Points: $(-1, -1/2)$, $(3, -1/2)$ V.T

Ex: 1) $\ln(xy) = x^2+y^2$ find y'

$\ln x + \ln y = x^2+y^2$

$\frac{1}{x} + \frac{y'}{y} = 2x + 2yy'$

2) $x^2 \ln y = 10$

$x^2 \cdot \frac{y'}{y} + 2x \ln y = 0$

$y' = (-2x \ln y)\left(\frac{y}{x^2}\right) = \frac{-2y \ln y}{x}$

$\frac{59}{725}$ 3) $p(q+1)^2 = 200\ 000$

Find the rate of change of quantity with respect to price at $p = \$80$.

(Find $\frac{dq}{dp}\Big|_{p=80}$)

$p \cdot (2)(q+1)^1 \cdot \frac{dq}{dP} + (q+1)^2 = 0$

$2p(q+1)\frac{dq}{dP} = -(q+1)^2$

$dq = - (q+1)^2 = -(q+1)$

1. If $\boxed{x^2 + y^2 = 1}$, find y'' when $x = 1$ and $y = 1$,
   - **a.** -2
   - **b.** 2
   - **c.** 0
   - **d.** $\sqrt{2}$.
   - **e.** None of the above.

2. The function $f(x) = e^x (x + 3)$ has an inflection point when x =
   - **a.** -1
   - **b.** -3
   - **c.** -5
   - **d.** -6
   - **e.** None of the above

3. If $4\ln x + 2x^2 y = y^3 + 10x$, then $\dfrac{dy}{dx} =$

2

6. The demand function for a q units of a product at \$p per unit is given by $\boldsymbol{p(q+1)^2 = 2000}$. Find the rate of change of quantity with respect to price at q = 19.

$$p \cdot 2(q+1)\frac{dq}{dp} + (q+1)^2 = 0$$

$$q = 19 \Rightarrow p = \frac{200}{40} = 5$$

$$\frac{dq}{dp} = -\left.\frac{q+1}{2p}\right|_{(5,19)} = -\frac{20}{10} = -2$$

MATH2351

Review Examples

Example 1: *Second derivative*

$$y = \ln e^{x^2}$$
$$\Rightarrow y = x^2 \ln e = x^2$$
$$\Rightarrow y' = 2x$$
$$\Rightarrow y'' = 2$$

Example 2: *First derivative*

$$g(x) = (2e^{3x+1} - 5)^3$$

Using the chain rule:

$$g'(x) = 3(2e^{3x+1} - 5)^2 (2e^{3x+1})(3)$$
$$= \mathbf{18(2e^{3x+1}} - 5)^2 (e^{3x+1})$$

Example 3: *First derivative*

$$f(x) = 1 + \log_8 10^x$$
$$f(x) = 1 + x \log_8 10$$
$$f'(x) = \log_8 10$$

Example 4: *Maxima, Minima*

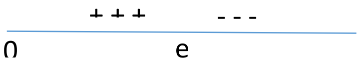
$$f(x) = \frac{\ln x}{x}, \quad x > 0$$

$$f'(x) = \frac{x.\frac{1}{x} - \ln x}{x^2}$$

$$f'(x) = \frac{1 - \ln x}{x^2}$$

$$f'(x) = 0 \Rightarrow 1 - \ln x = 0$$
$$\Rightarrow \ln x = 1 \Rightarrow e^{\ln x} = e^1 = e$$

x = e is a critical value



f(x) is increasing: $0 < x < e$

f(x) is decreasing: $x > e$

x = e is an absolute maximum, the maximum value is f (e) = 1/e

Example 5: *Equation of the tangent*

$y = x \ln x$, at x =1

Point: $y(1) = 0 \Rightarrow (1,0)$

Slope: $y'(1) = x\frac{1}{x} + \ln x = 1 + \ln x\Big|_{x=1} = 1$

Equation: y – 0 = 1(x – 1), y = x – 1

Example 6: *Inflection point*

The function $f(x) = e^{x}(x+3)$ has an inflection point when x =

- **a.** -1
- **b.** -3
- **c.** -5
- **d.** -6
- **e.** None of the above

Example 7: *Extreme values*

The function $f(x) = \frac{x^2}{e^x}$

- **a.** Has a local maximum at x = 2 and a local minimum at x= 0.
- **b.** Has neither maximum nor minimum
- **c.** Has a local maximum at x = 0 and a local minimum at x= 2.
- **d.** None.

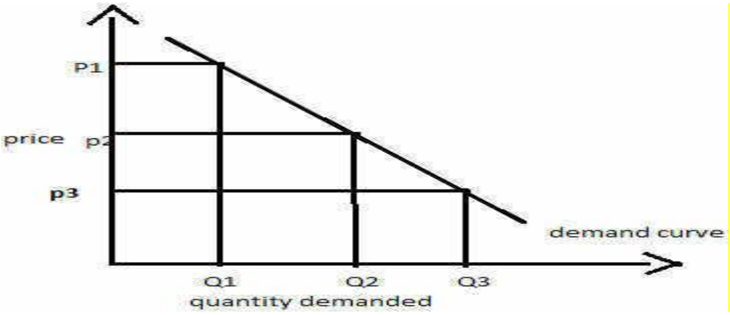
Example 8: *Find the derivative*

- **a.** $\frac{d}{dx}(\log(\sqrt{\frac{(x^2+1)^5}{5x^3}})$
- **b.** $\frac{d}{dx}(5^{2x+2} + 10^{x} + 3)$
- **c.** $\frac{d}{dx}(5^{2x+2})$ at x = 0

11.5 Elasticity of Demand

Recall from 1.6 the law of demand:

"Law of demand explains **consumer choice behavior** when the price changes. In the market, assuming other factors affecting demand being constant, when the price of a good rises, it leads to a fall in the demand of that good. This is the natural consumer choice behavior. This happens because a consumer hesitates to spend more for the good with the fear of going out of cash."



The elasticity of demand is a measure of the **responsiveness** of consumers (quantity demanded) to a change in a product's price. That is, demand elasticity measures the **impact** of a change in any of a variety of factors including the product's **price.**

Given **q = f (p);** where **q** is the quantity demanded at a price **p**, the formula for any calculation of demand elasticity is:

Elasticity = – [Change in quantity demanded / original quantity demanded] ÷ [Change in price/ original price]

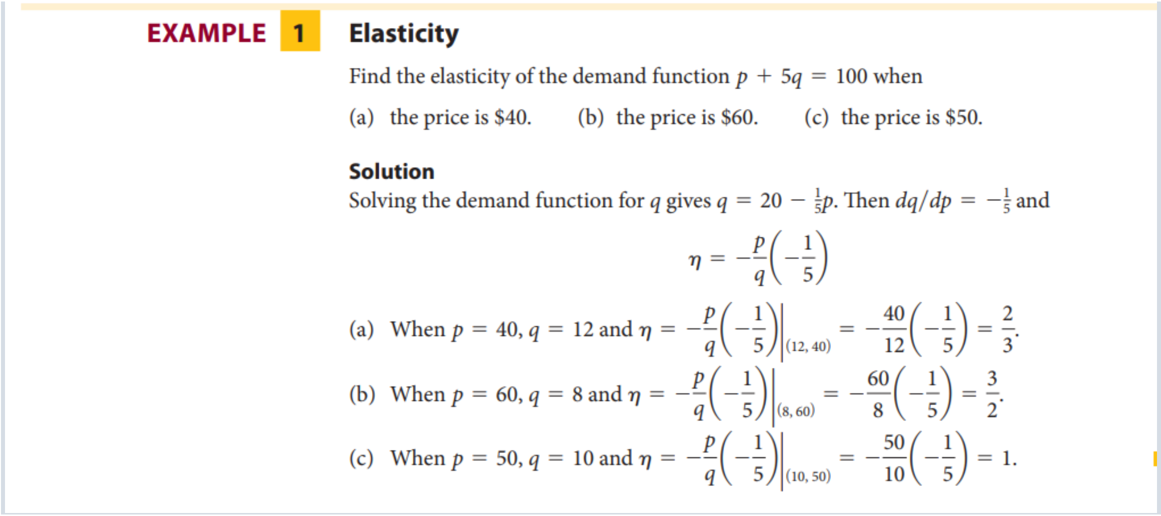
$$= -\frac{\Delta q}{q} \div \frac{\Delta p}{p} = -\frac{q}{p}\frac{\Delta q}{\Delta p}$$

Point elasticity of demand is given by the following formula

$$\eta = \lim_{\Delta q \leftarrow 0} -\frac{q}{p}\frac{\Delta q}{\Delta p} = -\frac{p}{q}\frac{dq}{dp}$$

Economists use η to measure how responsive demand is to price at different points on the demand curve for a product.

- If $\eta > 1$ the demand is elastic, and the percent decrease in demand is greater than the corresponding percent increase in price.
- If $\eta < 1$, the demand is inelastic, and the percent decrease in demand is less than the corresponding percent increase in price.
- If $\eta = 1$, the demand is unitary elastic, and the percent decrease in demand is approximately equal to the corresponding percent increase in price.



EXAMPLE 1 **Elasticity**

Find the elasticity of the demand function $p + 5q = 100$ when

(a) the price is \$40. (b) the price is \$60. (c) the price is \$50.

Solution

Solving the demand function for q gives $q = 20 - \frac{1}{5}p$. Then $dq/dp = -\frac{1}{5}$ and

$$\eta = -\frac{p}{q}\left(-\frac{1}{5}\right)$$

(a) When $p = 40$, $q = 12$ and $\eta = -\frac{p}{q}\left(-\frac{1}{5}\right)\bigg|_{(12,\,40)} = -\frac{40}{12}\left(-\frac{1}{5}\right) = \frac{2}{3}.$

(b) When $p = 60$, $q = 8$ and $\eta = -\frac{p}{q}\left(-\frac{1}{5}\right)\bigg|_{(8,\,60)} = -\frac{60}{8}\left(-\frac{1}{5}\right) = \frac{3}{2}.$

(c) When $p = 50$, $q = 10$ and $\eta = -\frac{p}{q}\left(-\frac{1}{5}\right)\bigg|_{(10,\,50)} = -\frac{50}{10}\left(-\frac{1}{5}\right) = 1.$

EXAMPLE 2 Elasticity | APPLICATION PREVIEW |

The demand for a certain product is given by

$$p = \frac{1000}{(q+1)^2}$$

where p is the price per unit in dollars and q is demand in units of the product. Find the elasticity of demand with respect to price when $q = 19$.

Solution

To find the elasticity, we need to find dq/dp. Using implicit differentiation, we get the following:

$$\frac{d}{dp}(p) = \frac{d}{dp}[1000(q+1)^{-2}]$$

$$1 = 1000\left[-2(q+1)^{-3}\frac{dq}{dp}\right]$$

$$1 = \frac{-2000}{(q+1)^3}\frac{dq}{dp}$$

$$\frac{(q+1)^3}{-2000} = \frac{dq}{dp}$$

When $q = 19$, we have $p = 1000/(19+1)^2 = 1000/400 = 5/2$ and

$$\left.\frac{dq}{dp}\right|_{(q=19)} = \frac{(19+1)^3}{-2000} = \frac{8000}{-2000} = -4$$

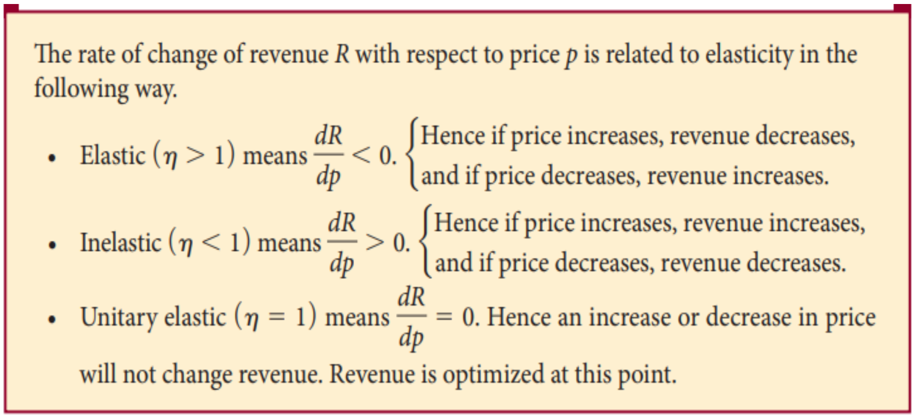
The elasticity of demand when $q = 19$ is

$$\eta = \frac{-p}{q}\cdot\frac{dq}{dp} = -\frac{(5/2)}{19}\cdot(-4) = \frac{10}{19} < 1$$

Thus the demand for this product is inelastic. ■

Elasticity of Demand and Revenue

$$\begin{aligned}
R &= pq \\
\frac{dR}{dp} &= p\frac{dq}{dp} + q.1 \\
&= \frac{q.}{q}p\frac{dq}{dp} + q \\
&= q(\frac{p}{q}\frac{dq}{dp} + 1) \\
&= q(1 - \eta)
\end{aligned}$$



The rate of change of revenue R with respect to price p is related to elasticity in the following way.

- Elastic $(\eta > 1)$ means $\frac{dR}{dp} < 0$. $\begin{cases}\text{Hence if price increases, revenue decreases,} \\ \text{and if price decreases, revenue increases.}\end{cases}$
- Inelastic $(\eta < 1)$ means $\frac{dR}{dp} > 0$. $\begin{cases}\text{Hence if price increases, revenue increases,} \\ \text{and if price decreases, revenue decreases.}\end{cases}$
- Unitary elastic $(\eta = 1)$ means $\frac{dR}{dp} = 0$. Hence an increase or decrease in price will not change revenue. Revenue is optimized at this point.

3. (a) Find the elasticity of the demand function $p^2 + 2p + q = 49$ at $p = 6$.

   (b) How will a price increase affect total revenue?

4. Find the elasticity of the demand function $pq = 81$ at $p = 3$.

Example

The demand function for a product is given by $q = \sqrt{900 - p}$, $\mathbf{0 \leq p \leq 900}$.

1. Find the **elasticity** of demand as a **function of p**.

$$q = (900 - p)^{\frac{1}{2}}$$

$$\frac{dq}{dp} = \frac{-1}{2(900 - p)^{\frac{1}{2}}}$$

$$\eta = -\frac{p}{q}\frac{dq}{dp}$$

$$= -\frac{p}{(900 - p)^{\frac{1}{2}}}\,\frac{-1}{2(900 - p)^{\frac{1}{2}}}$$

$$\boxed{= \frac{p}{1800 - 2p}}$$

2. Find the point at which the demand of **unitary** elasticity.

$$\eta = 1 \Rightarrow \frac{p}{1800 - 2p} = 1$$

$$p = 1800 - 2p$$

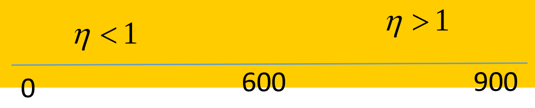
$$3p = 1800$$

$$p = 600$$

At price p = \$600, the demand is of unitary elasticity. That is, the demand is of unitary elasticity, when $q = \sqrt{900 - 600} = \sqrt{300}$ and p = \$600.

3. Find the intervals in which the demand is elastic, and in which the demand is inelastic?

The demand is elastic for 0 < p < 600
The demand is inelastic for 600 < p < 900



4. At p = $ 500, should price increased or decreased to increase revenue?
   At p =$500, the demand is inelastic, so the relation between revenue and price is positive, so to increase revenue, price should be increased.

5. At p =$ 800, how would a price increase affect the revenue?
   At p =$800, the demand is elastic, and the relation between revenue and price negative. Therefore, a price increase will decrease the revenue.

6. What is the maximum revenue?
   The maximum revenue is at the point where demand is of unitary elasticity ($\eta = 1$), that is at p = $600 and $q = \sqrt{300}$
   Maximum Revenue = (p)(q) = $\$600\sqrt{300}$

If the elasticity of the demand function for a product is $\boxed{E_d = \frac{2p}{600-p}}$, where p is unit price and $0 < p < 600$

1. Is the demand elastic or inelastic at p = \$ 200.

$$E(200) = \frac{(2)(200)}{600-200} = \frac{400}{400} = 1$$

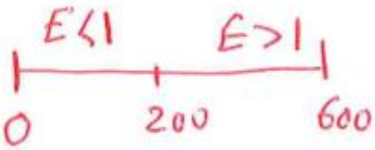
2. At p = \$200, should the price be increased or decreased to increase the revenue?

At p=\$200, the demand is unit elastic, so the revenue is Max. Price should not change

3. Find the price at which the demand is of unitary elasticity, and find the intervals in which the demand is inelastic and in which is elastic.

$E_d = 1 \Rightarrow p = 200$

Unit elastic at p=\$200



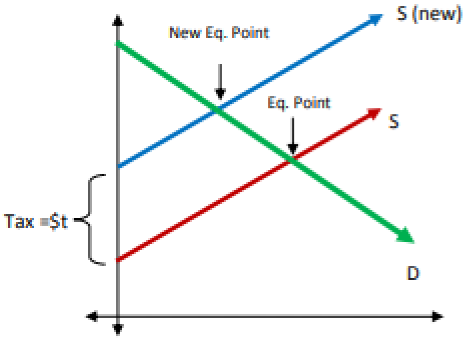
elastic $200 < p < 600$

inelastic $0 < p < 200$

Please watch the following video**: https://youtu.be/2g8zJzMViXU**

Maximum Tax Revenue

A demand function p = and a supply functions are given by **p = f (q) and p = g(q);** respectively. Suppose a tax **$t** is imposed by the government, then the new supply is **p = f (q) + t.**



The new equilibrium point occurred when **f (q) + t = g (q)**

So, **t = g (q) – f (q)**

Let **T** be the tax revenue, then

T (q) = t*q

To find the maximum of **T, use the procedures of chapter 10**

4. If the demand and supply function for a product are (respectively)

$$\boxed{D: p = 600 - q}, \qquad \boxed{S: p = 200 + \frac{1}{3}q}$$

Find the **maximum tax revenue.**

$$t = D - S = 600 - q - 200 - 1/3\, q$$

$$= 400 - \frac{4}{3}q$$

$$T = t \cdot q = 400q - \frac{4}{3}q^2$$

$$T' = 400 - \frac{8}{3}q, \quad T'' = -8/3$$

$$T' = 0 \Rightarrow 400 - 8/3\, q = 0 \Rightarrow \boxed{q = 150}$$

$$q = 150 \Rightarrow t = 400 - \frac{(4)(150)}{3} = 200$$

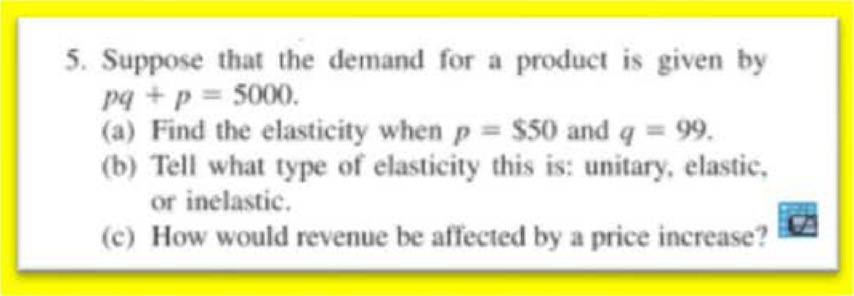
Max. tax. revenue $= (150)(200) = \$\ 30\,000$

If the price-demand equation is $D: p = 200 - 2x^2$ **and the price-supply equation is** $S: p = 20 + 2x$ **(x = # of units, p = price).**

3. Find the equilibrium point.

4. Find the elasticity of demand at p = $56. Is the demand elastic, inelastic or unitary elastic.

5. If the price is increased slightly from $56, will revenue increase or decrease? Why?



5. Suppose that the demand for a product is given by $pq + p = 5000$.
   - (a) Find the elasticity when $p = \$50$ and $q = 99$.
   - (b) Tell what type of elasticity this is: unitary, elastic, or inelastic.
   - (c) How would revenue be affected by a price increase?

HINT:

$$pq + p = 5000 \Rightarrow p(q+1) = 5000$$
$$\Rightarrow q + 1 = \frac{5000}{p} \Rightarrow q = \frac{5000}{p} - 1$$
$$\frac{dq}{dp} = \frac{-5000}{p^2}$$

Chapter 12
Indefinite Integrals

Derivatives: Given $f(x)$, find $f'(x)$ — 9, 10, 11

Integrals: Given $f'(x)$, find $f(x)$

Indefinite integrals: Antidifferentiation — 12

Definite integrals: Area — 13

12.1 Indefinite Integrals

Definition: A function $F(x)$ is called an antiderivative of $f(x)$ if $F'(x) = f(x)$ for all x in the domain of f.

Note: $F(x)$ is not unique. For example

$F(x) = x^3 + 1$, $G(x) = x^3 - 5$, $H(x) = x^3 + c$ (constant)

are all anti derivatives of $f(x) = 3x^2$

$F' = G' = H' = f(x)$

The set of all antiderivatives of $f(x)$ $\left(F(x)+c\right)$ is called the indefinite integral of $f(x)$.

In symbol:

$$\underset{\text{integral sign}}{\int} \underset{\text{integrand}}{f(x)}\,dx = F(x) + \underset{\text{Constant of integration}}{C}$$

$F(x) + C$ is called the indefinite integral of $f(x)$ with respect to the independent variable x.

Using previous example of $F(x) = x^3$ and $f(x) = 3x^2$, we can write

$$\int 3x^2\,dx = x^3 + C$$

Ex: Find the antiderivative of x^{100}.

$$\int x^{100}\,dx = \frac{x^{101}}{101} + C \qquad F(x) = \frac{x^{101}}{101}$$

Rule: 1) $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

Ex: 1) $\int x^5\,dx = \frac{x^6}{6} + C$

2) $\int \frac{1}{\sqrt{x}}\,dx = \int x^{-1/2}\,dx = \frac{x^{-1/2+1}}{-1/2+1} + C = 2x^{1/2} + C$

Rules: 2) $\int k\,f(x)\,dx = k\int f(x)\,dx$

3) $\int (f \pm g)\,dx = \int f\,dx \pm \int g\,dx$

Ex: 1) $\int (3x^2 + 4x + 10)\,dx = x^3 + 2x^2 + 10x + C$

2) $\int \left(x^4 + \frac{4}{x^4} + \sqrt[4]{x} + 4\right)dx$

$= \int (x^4 + 4x^{-4} + x^{1/4} + 4)\,dx$

$= \frac{x^5}{5} + 4\frac{x^{-3}}{-3} + \frac{x^{5/4}}{5/4} + 4x + C$

$=$

Ex: $\int (x^3+1)^2 dx$

$= \int (x^6 + 2x^3 + 1)\, dx$

$= \frac{x^7}{7} + \frac{x^4}{2} + x + c$

Ex: The Marginal revenue for a product is given by: $MR = 100 - 0.4X$. Find the total revenue from the sale of 100 units

$R(x) = \int \overline{MR}\, dx = \int (100 - 0.4x)\, dx$

$= 100x - 0.2x^2 + c$

$R(0) = 0 \Rightarrow (100)(0) - (0.2)(0)^2 + c = 0 \Rightarrow c = 0$

$\Rightarrow R(x) = 100x - 0.2x^2$

$R(100) = (100)(100) - (0.2)(100)^2$

$= \$\, 8000$

Example: If $\int f(x)\, dx = 11x^{10} - 4x^3 + c$. Find $f(x)$

$f(x) = (11x^{10} - 4x^3 + c)'$

$= 110x^9 - 12x^4$

Example: $\int 3x^8 + \frac{4}{x^8} - \frac{5}{\sqrt[5]{x}})\, dx$

$= \int (3x^8 + 4x^{-8} - 5x^{-1/5})\, dx$

$= \frac{x^9}{3} - \frac{4}{x^7} - \frac{25}{4} x^{4/5} + c$

12.2 The Power Rule

Recall from 9.6 that $(f\circ g)'(x) = f'(g(x))\cdot g'(x)$

Definition: If $y=f(x)$ is a differentiable function of $f(x)$, $y'=f'(x)=\frac{dy}{dx}$, then

dy is called the differential of y and dx is the differential of x

$$\frac{dy}{dx}=f'(x) \Rightarrow dy = f'(x)\,dx$$

Ex: $y=f(x)=(x^3+1)^{10} \Rightarrow dy = (10)(x^3+1)^9(3x^2)\,dx$
$= 30x^2(x^3+1)^9\,dx$

The Power Rule

* If $y=(u(x))^n$, then $\frac{dy}{dx}=n\,u(x)^{n-1}u'(x)$

$$\Rightarrow \int (u(x))^n\cdot u'(x)\,dx = \int u^n\,du$$
$$= \frac{u^{n+1}}{n+1}+C \qquad n\neq -1$$

* $$\int f'(g(x))\,g'(x)\,dx = (f\circ g)(x)+C$$

Ex: $\int 2\sqrt{2x+1}\,dx = \int (2x+1)^{1/2}\,2\,dx$
$= \frac{2}{3}(2x+1)^{3/2}+C$

Ex: $I = \int 4x^3 (x^4+1)^{15} dx$

$$= \frac{(x^4+1)^{16}}{16} + c$$

$u = x^4+1$

$du = 4x^3 dx$

$$I = \int u^{15} du$$

$$= \frac{u^{16}}{16} + c$$

$$= \frac{(x^4+1)^{16}}{16} + c$$

Integration by Substitution

Ex: $I = \int \frac{x^2}{\sqrt{x^3+1}} dx$

$$= \int x^2 (x^3+1)^{-1/2} dx$$

$$= \int (x^3+1)^{-1/2} \frac{3}{3} x^2 dx$$

$$= 1/3 \int (x^3+1)^{-1/2} 3x^2 dx$$

$$= \frac{1}{3} \frac{(x^3+1)^{1/2}}{1/2} + c$$

$$= \frac{2}{3} (x^3+1)^{1/2} + c$$

$u = x^3+1$

$du = 3x^2 dx$

$\frac{du}{3} = x^2 dx$

$$I = \int \frac{1}{\sqrt{u}} \frac{du}{3}$$

$$= \frac{1}{3} \int u^{-1/2} du$$

$=$

Example: $\int \frac{x^2-1}{(x^3-3x+1)^5} dx$

Example: If $\overline{MR} = \frac{1000}{\sqrt{5x+10}} + 10$, find $R(x)$

$$R(x) = \int R'(x)\,dx = \int \left(1000(5x+16)^{-1/2} + 10\right) dx$$

$$= \frac{1000}{5} \frac{(5x+16)^{1/2}}{1/2} + \frac{10x^2}{2} + c$$

$$= 400(5x+16)^{1/2} + 5x^2 + c$$

$$R(0)=0 \Rightarrow (400)(16)^{1/2} + 0 + C = 0$$

$$\Rightarrow C = -1600$$

$$R(x) = 400(5x+16)^{1/2} + 5x^2 - 1600$$

Example: page 763

41, 50

29. $\displaystyle\int \frac{x^3-1}{(x^4-4x)^3}\,dx$ 30. $\displaystyle\int \frac{3x^5-2x^3}{(x^6-x^4)^5}\,dx$

31. $\displaystyle\int \frac{x^2-4x}{\sqrt{x^3-6x^2+2}}\,dx$ 32. $\displaystyle\int \frac{x^2+1}{\sqrt{x^3+3x+10}}\,dx$

Solution 31

Method1: $I=\displaystyle\int \frac{x^2-4x}{\sqrt{x^3-6x^2+2}}dx=\frac{1}{3}\int \frac{3(x^2-4x)}{(x^3-6x^2+2)^{\frac{1}{2}}}dx$

$$=\frac{1}{3}\frac{(x^3-6x^2+2)^{\frac{1}{2}}}{\frac{1}{2}}+c$$

$$=\frac{2}{3}(x^3-6x^2+2)^{\frac{1}{2}}+c$$

$$=\frac{2}{3}\sqrt{x^3-6x^2+2}+c$$

$\left[(x^3-6x^2+2)'=3x^2-12x=3(x^2\text{-}4x)\right]$

****

Method 2: Let $u=x^3\ \text{-}6x^2\ +2$

$\rightarrow du=(3x^2\ \text{-}12x)dx\ \ =3(x^2\text{-}4x)dx$

$\rightarrow \dfrac{du}{3}\ \ =(x^2\ \text{-}4x)dx$

$\rightarrow I=\displaystyle\int \frac{1}{u^{\frac{1}{2}}}\,\frac{du}{3}\ \ =\frac{1}{3}\int u^{\frac{-1}{2}}du=\frac{2}{3}\sqrt{u}\ \ +c=\frac{2}{3}\sqrt{x^3-6x^2+2}+c$

12.3 Integrals Involving Exponential and Logarithmic Functions

Recall that (from chapter 11):

- ❖ $\frac{d}{dx}(e^{u(x)}) = e^{u(x)}.u'(x)$
- ❖ $\frac{d}{dx}(a^{u(x)}) = a^{u(x)}(\ln a.)(u'(x))$
- ❖ $\frac{d}{dx}(\ln u(x)) = \frac{1}{u(x)}.u'(x)$

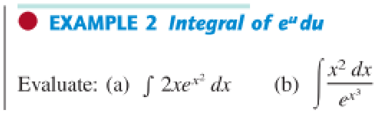
If u is a differentiable function of x, then

$**\int e^u u' du = \int e^u du = e^u + c$

**In particular $\int e^x dx = e^x + c$

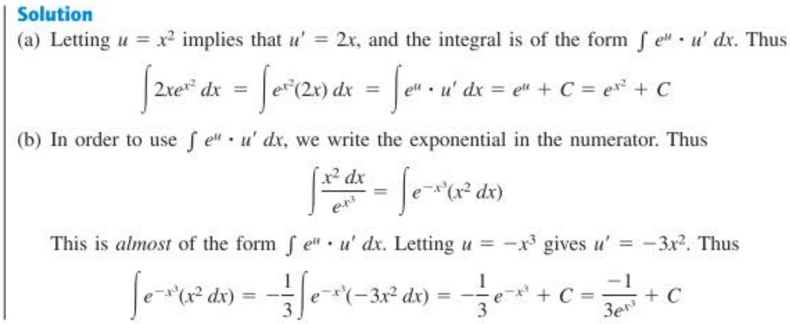
$**\int \frac{u'}{u} du = \int \frac{du}{u} = \ln|u| + c$

**In particular $\int \frac{dx}{x} = \ln|x| + c$



EXAMPLE 2 ***Integral of $e^u du$***

Evaluate: (a) $\int 2xe^{x^2}\,dx$ (b) $\displaystyle\int \frac{x^2\,dx}{e^{x^3}}$



Solution

(a) Letting $u = x^2$ implies that $u' = 2x$, and the integral is of the form $\int e^u \cdot u'\,dx$. Thus

$$\int 2xe^{x^2}\,dx = \int e^{x^2}(2x)\,dx = \int e^u \cdot u'\,dx = e^u + C = e^{x^2} + C$$

(b) In order to use $\int e^u \cdot u'\,dx$, we write the exponential in the numerator. Thus

$$\int \frac{x^2\,dx}{e^{x^3}} = \int e^{-x^3}(x^2\,dx)$$

This is *almost* of the form $\int e^u \cdot u'\,dx$. Letting $u = -x^3$ gives $u' = -3x^2$. Thus

$$\int e^{-x^3}(x^2\,dx) = -\frac{1}{3}\int e^{-x^3}(-3x^2\,dx) = -\frac{1}{3}e^{-x^3} + C = \frac{-1}{3e^{x^3}} + C$$

● EXAMPLE 5 *Integrals Resulting in Logarithmic Functions*

Evaluate $\int \frac{4}{4x + 8}\,dx$.

Solution

This integral is of the form

$$\int \frac{u'}{u}\,dx = \ln|u| + C$$

with $u = 4x + 8$ and $u' = 4$. Thus

$$\int \frac{4}{4x + 8}\,dx = \ln|4x + 8| + C$$

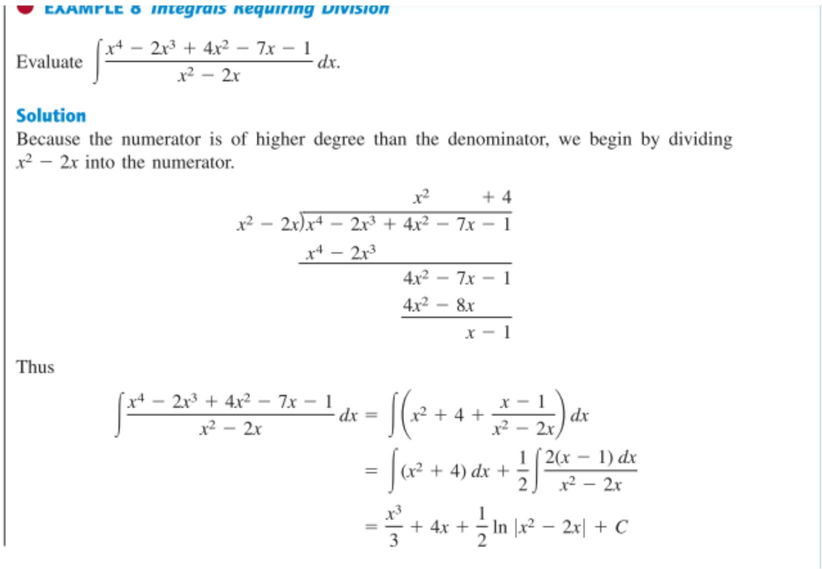
● EXAMPLE 6 *Integral of du/u*

Evaluate $\int \frac{x - 3}{x^2 - 6x + 1}\,dx$.

Solution

This integral is of the form $\int (u'/u)\,dx$, *almost.* If we let $u = x^2 - 6x + 1$, then $u' = 2x - 6$. If we multiply (and divide) the numerator by 2, we get

$$\begin{aligned}\int \frac{x - 3}{x^2 - 6x + 1}\,dx &= \frac{1}{2}\int \frac{2(x - 3)}{x^2 - 6x + 1}\,dx \\ &= \frac{1}{2}\int \frac{2x - 6}{x^2 - 6x + 1}\,dx \\ &= \frac{1}{2}\int \frac{u'}{u}\,dx = \frac{1}{2}\ln|u| + C \\ &= \frac{1}{2}\ln|x^2 - 6x + 1| + C\end{aligned}$$



EXAMPLE 8 ***Integrals Requiring Division***

Evaluate $\int \frac{x^4 - 2x^3 + 4x^2 - 7x - 1}{x^2 - 2x}\,dx$.

Solution

Because the numerator is of higher degree than the denominator, we begin by dividing $x^2 - 2x$ into the numerator.

$$\begin{array}{r|l} & x^2 \qquad\qquad + 4 \\ \hline x^2 - 2x & x^4 - 2x^3 + 4x^2 - 7x - 1 \\ & \underline{x^4 - 2x^3} \\ & \qquad\qquad 4x^2 - 7x - 1 \\ & \qquad\qquad \underline{4x^2 - 8x} \\ & \qquad\qquad\qquad x - 1 \end{array}$$

Thus

$$\begin{aligned} \int \frac{x^4 - 2x^3 + 4x^2 - 7x - 1}{x^2 - 2x}\,dx &= \int \left(x^2 + 4 + \frac{x - 1}{x^2 - 2x}\right) dx \\ &= \int (x^2 + 4)\,dx + \frac{1}{2}\int \frac{2(x - 1)\,dx}{x^2 - 2x} \\ &= \frac{x^3}{3} + 4x + \frac{1}{2}\ln|x^2 - 2x| + C \end{aligned}$$

13. $\int \frac{x^5}{e^{2-3x^6}}\,dx$

15. $\int \left(e^{4x} - \frac{3}{e^{x/2}}\right) dx$

Solution 13

$$\int \frac{x^5}{e^{2-3x^6}}dx = \frac{1}{\mathbf{18}}\int e^{3x^6-2}(\mathbf{18}x^5)dx \Rightarrow \left[\frac{1}{18}\int e^u du,\ u = 3x^6 - 2\right]$$

$$= \frac{1}{18}e^{3x^6-2} + c$$

25. $\int \frac{3x^2 - 2}{x^3 - 2x}\,dx$

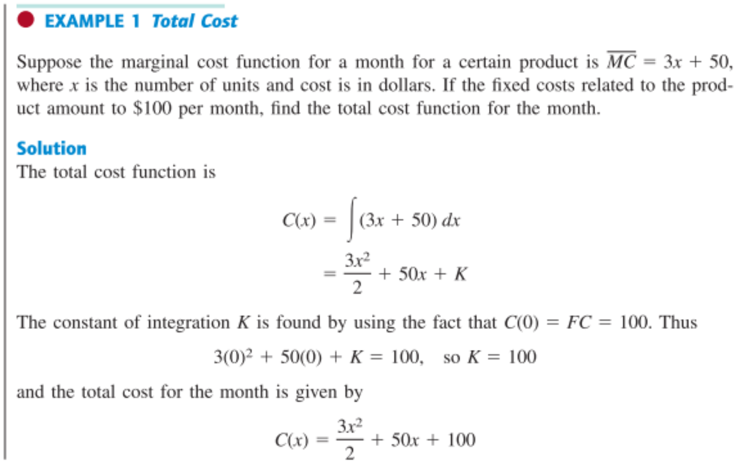
27. $\int \frac{z^2 + 1}{z^3 + 3z + 17}\,dz$

Solution 27

$$I = \int \frac{z^2+1}{z^3+3z+17}dz = \frac{1}{\mathbf{3}}\int \frac{\mathbf{3}(z^2+1)}{z^3+3z+17}\,dz \qquad \left[\ u = z^3+3z+17, du = 3z^2+3 = 3(z^2+1)\right]$$

$$= \frac{1}{3}\ln\left|z^3+3z+17\right| + c \qquad I = \frac{1}{3}\int\frac{du}{u} = \ln|u| + c =$$

12.4 Applications of the Indefinite Integrals in Business and Economics



● **EXAMPLE 1** ***Total Cost***

Suppose the marginal cost function for a month for a certain product is $\overline{MC} = 3x + 50$, where x is the number of units and cost is in dollars. If the fixed costs related to the product amount to \$100 per month, find the total cost function for the month.

Solution

The total cost function is

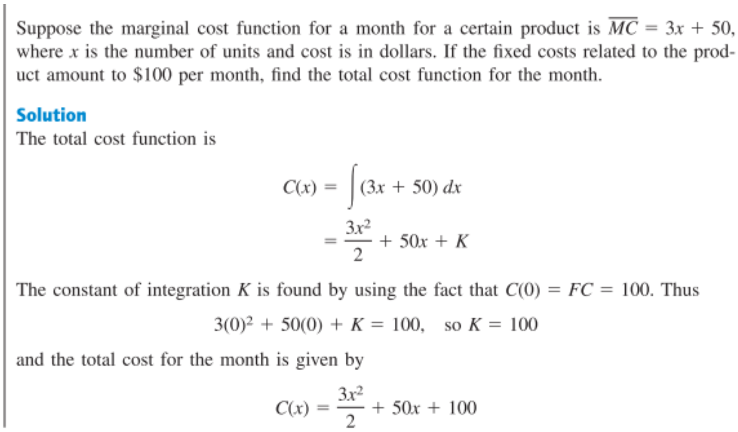
$$C(x) = \int (3x + 50)\, dx$$
$$= \frac{3x^2}{2} + 50x + K$$

The constant of integration K is found by using the fact that $C(0) = FC = 100$. Thus

$$3(0)^2 + 50(0) + K = 100, \quad \text{so } K = 100$$

and the total cost for the month is given by

$$C(x) = \frac{3x^2}{2} + 50x + 100$$



Suppose the marginal cost function for a month for a certain product is $\overline{MC} = 3x + 50$, where x is the number of units and cost is in dollars. If the fixed costs related to the product amount to \$100 per month, find the total cost function for the month.

Solution

The total cost function is

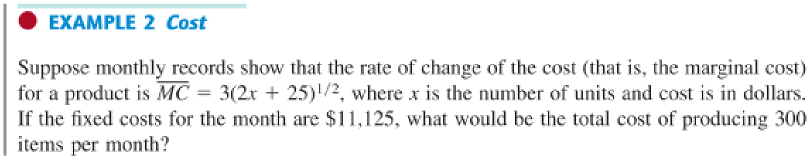
$$C(x) = \int (3x + 50)\, dx$$
$$= \frac{3x^2}{2} + 50x + K$$

The constant of integration K is found by using the fact that $C(0) = FC = 100$. Thus

$$3(0)^2 + 50(0) + K = 100, \quad \text{so } K = 100$$

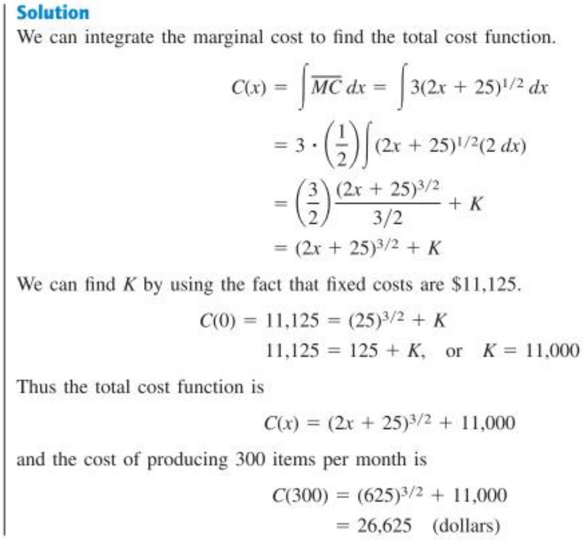
and the total cost for the month is given by

$$C(x) = \frac{3x^2}{2} + 50x + 100$$



EXAMPLE 2 ***Cost***

Suppose monthly records show that the rate of change of the cost (that is, the marginal cost) for a product is $\overline{MC} = 3(2x + 25)^{1/2}$, where x is the number of units and cost is in dollars. If the fixed costs for the month are \$11,125, what would be the total cost of producing 300 items per month?



Solution

We can integrate the marginal cost to find the total cost function.

$$\begin{aligned} C(x) &= \int \overline{MC}\,dx = \int 3(2x + 25)^{1/2}\,dx \\ &= 3 \cdot \left(\frac{1}{2}\right)\int (2x + 25)^{1/2}(2\,dx) \\ &= \left(\frac{3}{2}\right)\frac{(2x + 25)^{3/2}}{3/2} + K \\ &= (2x + 25)^{3/2} + K \end{aligned}$$

We can find K by using the fact that fixed costs are \$11,125.

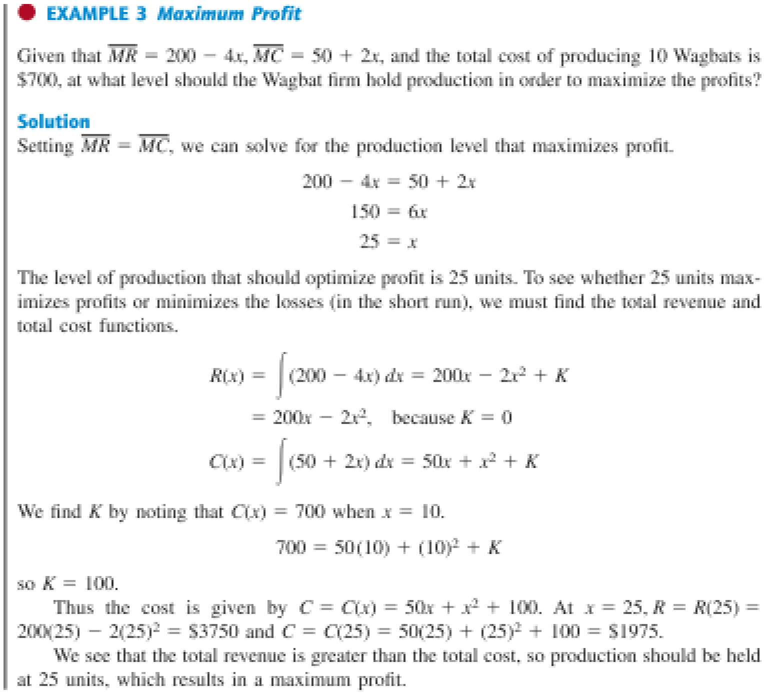
$$\begin{aligned} C(0) = 11{,}125 &= (25)^{3/2} + K \\ 11{,}125 &= 125 + K, \quad \text{or} \quad K = 11{,}000 \end{aligned}$$

Thus the total cost function is

$$C(x) = (2x + 25)^{3/2} + 11{,}000$$

and the cost of producing 300 items per month is

$$\begin{aligned} C(300) &= (625)^{3/2} + 11{,}000 \\ &= 26{,}625 \quad \text{(dollars)} \end{aligned}$$



● **EXAMPLE 3** ***Maximum Profit***

Given that $\overline{MR} = 200 - 4x$, $\overline{MC} = 50 + 2x$, and the total cost of producing 10 Wagbats is \$700, at what level should the Wagbat firm hold production in order to maximize the profits?

Solution

Setting $\overline{MR} = \overline{MC}$, we can solve for the production level that maximizes profit.

$$200 - 4x = 50 + 2x$$
$$150 = 6x$$
$$25 = x$$

The level of production that should optimize profit is 25 units. To see whether 25 units maximizes profits or minimizes the losses (in the short run), we must find the total revenue and total cost functions.

$$R(x) = \int (200 - 4x)\,dx = 200x - 2x^2 + K$$
$$= 200x - 2x^2, \quad \text{because } K = 0$$
$$C(x) = \int (50 + 2x)\,dx = 50x + x^2 + K$$

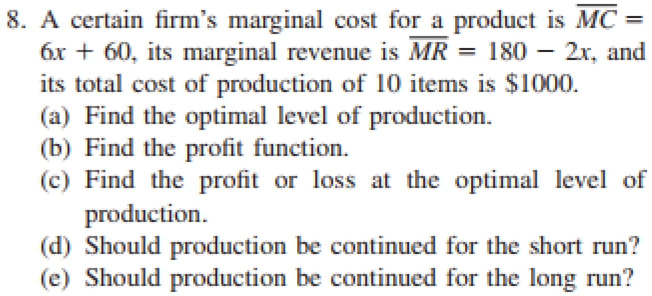
We find K by noting that $C(x) = 700$ when $x = 10$.

$$700 = 50(10) + (10)^2 + K$$

so $K = 100$.

Thus the cost is given by $C = C(x) = 50x + x^2 + 100$. At $x = 25$, $R = R(25) = 200(25) - 2(25)^2 = \$3750$ and $C = C(25) = 50(25) + (25)^2 + 100 = \$1975$.

We see that the total revenue is greater than the total cost, so production should be held at 25 units, which results in a maximum profit.



8. A certain firm's marginal cost for a product is $\overline{MC} = 6x + 60$, its marginal revenue is $\overline{MR} = 180 - 2x$, and its total cost of production of 10 items is $1000.
   (a) Find the optimal level of production.
   (b) Find the profit function.
   (c) Find the profit or loss at the optimal level of production.
   (d) Should production be continued for the short run?
   (e) Should production be continued for the long run?

Solution

(a) Optimal level production: $\overline{MC} = \overline{MR}$

$$\to 6x + 60 = 180 - 2x \to 8x = 120 \to x = 15.$$

(b)

$$\int \overline{MC} dx = \int (6x + 60) dx = 3x^2 + 60x + K$$

$$C(10) = 3(100) + 60(10) + k = 1000 \to K = 100$$

$$C(x) = 3x^2 + 60x + 100$$

$$\int \overline{MR} dx = \int (180 - 2x) dx = 180x - x^2 + K$$

$$R(0) = 0 + k = 0 \to K = 0$$

$$R(x) = 180x - x^2$$

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= (180x - x^2) - (3x^2 + 60x + 100) \\ &= -4x^2 + 120x - 100 \end{aligned}$$

(c) $P(15) = -4(15)^2 + 120(15) - 100 = \$800$

(d) Yes

(e) No

CHAPTER 13

DEFINITE INTEGRALS

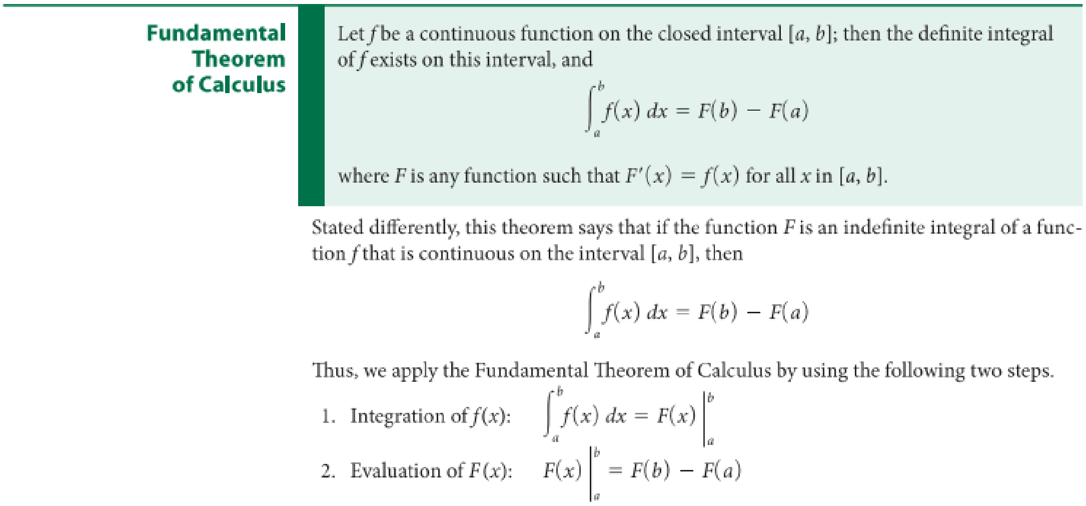
13.2 DEFINITE INTEGRALS: The Fundamental Theorem of Calculus

Recall from Chapter 12

F(x) is called an antiderivative of f(x) if $F'(x) = f(x)$ and we defined the indefinite integral of f(x) to be the set of all antiderivatives F(x) + c and it is denoted by $\int f(x)dx = F(x) + c$

The definite integral of the function f(x) over the interval [a, b] is denoted by

$\int_a^b f(x)dx$, a and b are the limits of integration



Fundamental Theorem of Calculus

Let f be a continuous function on the closed interval $[a, b]$; then the definite integral of f exists on this interval, and

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any function such that $F'(x) = f(x)$ for all x in $[a, b]$.

Stated differently, this theorem says that if the function F is an indefinite integral of a function f that is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Thus, we apply the Fundamental Theorem of Calculus by using the following two steps.

1. Integration of $f(x)$: $\int_a^b f(x)\,dx = F(x)\Big|_a^b$
2. Evaluation of $F(x)$: $F(x)\Big|_a^b = F(b) - F(a)$

Examples

1. Evaluate $\int_{1}^{3} (3x^2 + 6x)\, dx.$

2. Evaluate $\int_{3}^{5} (\sqrt{x^2 - 9} + 2)x\, dx.$

Examples

17. $\int_0^4 \sqrt{4x+9}xdx = \frac{1}{4}\int_0^4 (4x+9)^{\frac{1}{2}}(4)dx = \left\langle \frac{1}{4}\frac{(4x+9)^{\frac{3}{2}}}{\frac{3}{2}} \right|_0^4 = \frac{1}{6}(125-27) = \frac{98}{6}$

19. $\int_1^3 \frac{3}{y^2}dy = \int_1^3 3y^{-2}dy = -3y^{-1} = \left\langle \frac{-3}{y} \right|_1^3 = -1++3 = 2$

22. $\int_0^2 e^{4x-3}dx = \frac{1}{4}\int_0^2 e^{4x-3}4dx = \frac{1}{4}\left\langle e^{4x-3} \right|_0^2 = \frac{1}{4}(e^5 - e^{-3})$

23. $\int_1^e \frac{4}{z}dz = \left\langle 4\ln|z| \right|_1^e = 4\ln e - 4\ln 1 = 4$

Properties of definite Integrals

1. $\int_a^b [f \pm g]dx = \int_a^b fdx \pm \int_a^b gdx$

2. $\int_a^b kfd = k\int_a^b fdx$

3. $\int_a^b fdx = -\int_b^a fdx$

4. $\int_a^a fdx = 0$

5. $\int_a^b fdx = \int_a^c fdx + \int_c^b fdx$

6. If $f(x) \geq 0$, then $\int_a^b f dx \geq 0$

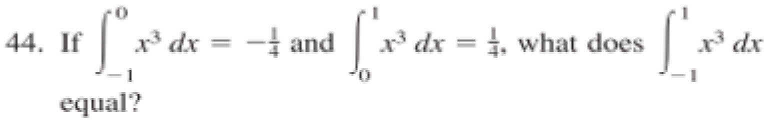
7. If $f(x)$ is continuous on $[a, b]$ and $f(x) \geq 0$, then $\int_a^b f dx$ is the area beween $f(x)$ and the the x - axis from $x = a$ to $x = b$ is

$$A = \int_a^b f dx$$

8. If $f(x)$ is continuous on $[a, b]$, then the average value of $f(x)$ on $[a, b]$ is

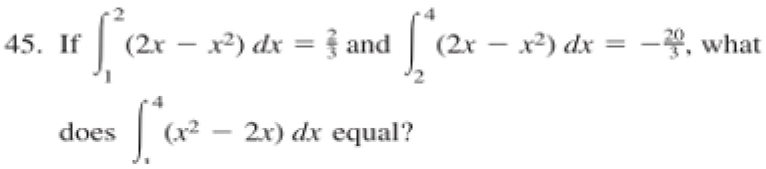
$$\frac{1}{b-a} \int_a^b f dx$$

Examples



44. If $\int_{-1}^{0} x^3\,dx = -\frac{1}{4}$ and $\int_{0}^{1} x^3\,dx = \frac{1}{4}$, what does $\int_{-1}^{1} x^3\,dx$ equal?

$$\int_{-1}^{1} x^3 dx = \int_{-1}^{0} x^3 dx + \int_{0}^{1} x^3 dx = 0$$



45. If $\int_{1}^{2} (2x - x^2)\,dx = \frac{2}{3}$ and $\int_{2}^{4} (2x - x^2)\,dx = -\frac{20}{3}$, what does $\int_{1}^{4} (x^2 - 2x)\,dx$ equal?

$$\int_{1}^{4} (2x - x^2) dx = \int_{1}^{2} (2x - x^2) dx + \int_{2}^{4} (2x - x^2) dx = \frac{2}{3} + \frac{-20}{3} = -6$$

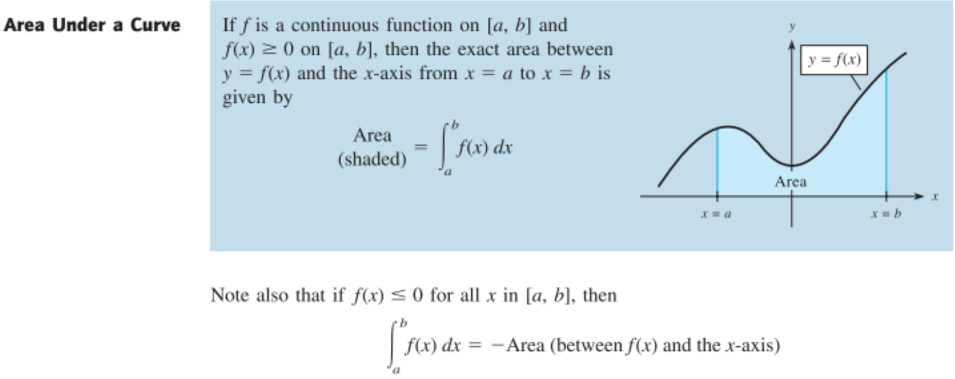
Examples

1. If $\int_1^5 f(x)dx = 28$, find the average value of f(x) over the interval [1, 5]

2. If $\int_0^6 f(x)dx = 9$ and $\int_4^6 f(x)dx = 7$, what is $\int_0^4 4f(x)dx$?

3. Given $\int_3^5 f(x)dx = 7,\ \int_3^5 g(x)dx = 1$, find $\int_3^5 [4f(x) + 2g(x) - 2]dx$

Area under a Curve



Area Under a Curve

If f is a continuous function on $[a, b]$ and $f(x) \ge 0$ on $[a, b]$, then the exact area between $y = f(x)$ and the x-axis from $x = a$ to $x = b$ is given by

$$\begin{matrix}\text{Area}\\ \text{(shaded)}\end{matrix} = \int_a^b f(x)\,dx$$

Note also that if $f(x) \le 0$ for all x in $[a, b]$, then

$$\int_a^b f(x)\,dx = -\text{Area (between } f(x) \text{ and the } x\text{-axis)}$$

Example

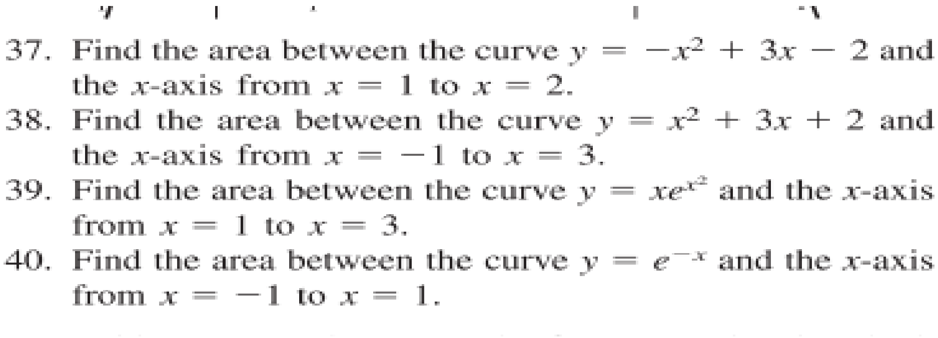
Find the area between $y = -x^2 + 4x - 3$ and the x-axis.

First solve: y = 0

$$\Rightarrow -x^2 + 4x - 3 = 0$$
$$\Rightarrow x^2 - 4x + 3 = 0$$
$$\Rightarrow (x-3)(x-1) = 0$$
$$x = 1, x = 3$$

$$A = \int_1^3 (-x^2 + 4x - 3)dx$$
$$= -\frac{x^3}{3} + 2x^2 - 3x \,\Big|_1^3$$
$$= (-9 + 18 - 9) - (\frac{-1}{3} + 2 - 3)$$
$$= \frac{4}{3}$$

Examples



37. Find the area between the curve $y = -x^2 + 3x - 2$ and the x-axis from $x = 1$ to $x = 2$.
38. Find the area between the curve $y = x^2 + 3x + 2$ and the x-axis from $x = -1$ to $x = 3$.
39. Find the area between the curve $y = xe^{x^2}$ and the x-axis from $x = 1$ to $x = 3$.
40. Find the area between the curve $y = e^{-x}$ and the x-axis from $x = -1$ to $x = 1$.

BIRZEIT UNIVERSITY
MATHEMATICS DEPARTMENT
MATH 2351

Instructor: Mohammad Mdiah
FINAL EXAM_SAMPLE
Time: 120 minutes

Part I: True/ False

1. In profit – loss analysis, point where revenue equals cost is equilibrium point.
   a. True
   b. False
2. If P(x) is the profit function, then $P(10) - P(9) \approx P'(10)$.
   a. True.
   b. False
3. If R(x) is the revenue function, then the marginal revenue $R'(x)$ is always nonnegative.
   a. True.
   b. False
4. If P(x) is the profit function, then the approximated profit of producing and selling unit number 11 is $P'(11)$.
   a. True
   b. False
5. If $C(x) = \int C'(x)dx$, then the constant of integration equals the fixed cost.
   a. True.
   b. False
6. $\int_a^b f(x)dx + \int_b^a f(x)dx = 0$.
   a. True.
   b. False
7. A market shortage courses when quantity demanded is greater than quantity supplied
   a. True
   b. False

Part II: Multiple choices questions: Circle the correct answer

1. The point where supply equals demand.
   - a. Profit – loss point
   - b. Inflection point
   - c. Break even point
   - d. Equilibrium point
2. If $x^2+y^2=5$, find y'' when $x=1$ and $y=2$,
   - a. 0
   - b. -5
   - c. $-\frac{5}{8}$
   - d. $\frac{-7}{4}$.
   - e. None of the above

(3 & 4) If the total cost function for a product is $C(x)=100+x^2$ dollars (x = total number of units produced)

3. Find the **average value** of the cost function over the interval from 3 units to 9 units produced.
   - a. 127
   - b. 834
   - c. 363
   - d. None of the above
4. Producing how many units, x, will result in a **minimum average cost per unit**?
   - a. 100.
   - b. 10.
   - c. 20.
   - d. There is no minimum average cost per unit.
5. A bank is paying 5.5% compounded continuously on an account with $500. How much money is in the account after 6 months?
   - a. $1056.54
   - b. $542.99
   - c. $695.48
   - d. $513.94
   - e. None of the above.

6. If $f(x)=\dfrac{x+2}{x-2}$, find $f'(x)$

- **a.** $f'(x)=\dfrac{4}{(x+2)^2}$
- **b.** $f'(x)=\dfrac{4}{(x-2)^2}$
- **c.** $f'(x)=\dfrac{-4}{(x+2)^2}$
- **d.** $f'(x)=\dfrac{-4}{(x-2)^2}$

7. Find the slope of the tangent to the curve $f(x)=(3+x)^{\frac{2}{3}}$ at x = -2.

- **a.** $\dfrac{2}{3}$
- **b.** $\dfrac{1}{2}$
- **c.** 1
- **d.** $\dfrac{3}{2}$

(8 & 9) If $C(x)=2x^{\frac{5}{2}}+100$

8. What is the marginal cost function

- **a.** $C'(x)=5x^{\frac{3}{2}}$
- **b.** $C'(x)=2x^{\frac{5}{2}}+100$
- **c.** $C'(x)=2x^{\frac{5}{2}}$
- **d.** $C'(x)=5x^{\frac{3}{2}}+100$

9. Use marginal cost to find the approximated cost to produce the 5th unit.

- **a.** 25.98
- **b.** 40
- **c.** 32.82
- **d.** 140
- **e.** None of the above.

10. $\frac{d}{dx}(10^x) =$

- **a.** 10^{x-1}
- **b.** 10^{x+1}
- **c.** $10^x \ln 10$
- **d.** $\frac{10^x}{\ln 10}$

11. Given the function $f(x) = x^4 - 18x^2 + 15$, where are the inflection values?

- **a.** $0, \pm 3$
- **b.** $\pm\sqrt{3}$
- **c.** $0, \pm\sqrt{3}$
- **d.** ± 3

12. Find the value at which $f(x) = x^4 - 8x^2 + 3$ takes on its absolute maximum on [0, 3]

- **a.** x = 0.
- **b.** x = 2
- **c.** x =3
- **d.** x =2.76

13. $\int \frac{1}{3x+1} dx =$

- **a.** $\frac{1}{3}\ln|3x+1|| + c$
- **b.** $3\ln|3x+1|| + c$
- **c.** $\ln|3x+1|| + c$
- **d.** $\frac{-1}{6}(3x+1)^{-2} + c$

14. How long will it take for $5500 to grow to $40300 at an interest rate of 4.8% compounded continuously

- **a.** 4.15 years
- **b.** 0.41 years
- **c.** 4149.17 years
- **d.** 41.49 years

15. The marginal revenue for a product is given by $R'(x)=5+e^{x}$. Find the revenue function.

- **a.** $R(x)=5x+e^{x}-1$
- **b.** $R(x)=5x+e^{x}$
- **c.** $R(x)=5x+e^{x}+1$
- **d.** None of the above.

16. What is the average cost per unit for the first 100 units produced if the cost function is given by $C(x)=3+\sqrt{x}$

- **a.** 18
- **b.** 9.67
- **c.** 0.13
- **d.** 13

17. Find the area between $y=x^{2}$ and $y=4$ on [0, 2]

- **a.** 8
- **b.** 0
- **c.** $\frac{8}{3}$
- **d.** $\frac{16}{3}$

18. If f is continuous, $f'(3)=0$, $f'(x)<0$ for $2<x<3$ and $f'(x)>0$ for $3<x<4$, then which of the following must hold?

- **a.** f has a relative maximum at x = 3;
- **b.** f has a relative minimum at x = 3;
- **c.** f has an inflection point at x = 3;
- **d.** f has neither a relative maximum nor a relative minimum at x = 3;
- **e.** None of the above.

19. If $\ln x+\ln(x-2)=\ln 8$, then x =

- **a.** -2,- 4
- **b.** 2, 4
- **c.** 2
- **d.** 4

20. Suppose you invest $500in each of 2 bank accounts. The first compounds quarterly at rate of 5% and the second compounds monthly at a rate of 4%. At the end of the year, which has more money on it?

- **a.** The first;
- **b.** The second;
- **c.** Both have the same
- **d.** Can't tell.

21. If $f'(3) = 0$ and $f''(3) > 0$, then which of the following must hold?

- **a.** f has a relative maximum at x = 3;
- **b.** f has a relative minimum at x = 3;
- **c.** f has an inflection point at x = 3;
- **d.** Cannot conclude.

(22 – 25) The demand and the supply for a product are given, respectively, by

$$D(x): p = 10 - 0.5x$$
$$S(x): p = 0.5x$$

where x is the number of units and p in dollars.

22. The point elasticity of demand when **x = 4** is

- **a.** 0.76
- **b.** 4
- **c.** 0.25
- **d.** 6

23. The equilibrium point is

- **a.** (5, 10)
- **b.** (5, 5)
- **c.** (10, 5)
- **d.** None of the above

24. The consumer surplus at the equilibrium point is

- **a.** $25
- **b.** $37.5
- **c.** $6.25
- **d.** None of the above

25. The revenue is maximum when p =

- **a.** $2.5
- **b.** $5
- **c.** $10
- **d.** $50

26. Given $f'(x) = 16x^4 - 9x^2 - 8x$, which of the following could be f(x)?

- **a.** $f(x) = 32x^3 - 18x - 8$
- **b.** $f(x) = 4x^5 - 3x^3 - 4x^2 + 10$
- **c.** $f(x) = 4x^5 - 3x^3 - 4x^2 + 15$
- **d.** a, b and c
- **e.** b and c
- **f.** a and b

27. If $\int_1^4 f(x)dx = 5$, then $\int_4^1 2f(x)\,dx =$

- **a.** – 10
- **b.** 5
- **c.** 0
- **d.** 10

28. If $\int_1^2 f(x)dx = 6, \int_3^2 f(x)dx = -6,$, the average value of f(x) over the interval [1, 3]

- **a.** 12
- **b.** 3
- **c.** 0
- **d.** 6
- **e.** None of the above

29. If the total cost function for a product is $\boxed{C(x) = 0.01x^2 + 20x + 2500}$ dollars (x = total number of units produced). Find the **minimum average cost per unit**?

- **a.** 500.
- **b.** 30.
- **c.** 50.
- **d.** There is no minimum average cost per unit.

30. If $f(x) = 3x^4 - 6x^3 + x - 8$, determine the x coordinates of all inflection points

- **a.** x = 0.
- **b.** x = 0 and x = 1.
- **c.** x = 1.
- **d.** x = - 1, x = 0 and x = 1.
- **e.** x = -1 and x = 1.

31. The demand for a product is given by $\boxed{p+0.05q=6}$, where q = the number of units, and p is the price in dollars; determine the elasticity of demand when the price is equal to $5?

- **a.** 1
- **b.** 5
- **c.** – 5
- **d.** $\frac{5}{11}$
- **e.** $\frac{11}{5}$

32. $\int_1^3 (x+\frac{6}{x^2}-2)dx=$

- **a.** $\frac{-7}{2}$
- **b.** $\frac{26}{3}$
- **c.** 5
- **d.** 4
- **e.** None of the above.

33. Find the area between the two curves: $y=x^3-1, y=x-1$

- **a.** 1
- **b.** 3
- **c.** 0.5
- **d.** 0.25
- **e.** None of the above.

34. $\int_0^2 \frac{2}{1+2x}dx=$

- **a.** $\frac{1}{2}\ln 5$
- **b.** $\ln 5-\ln 2$
- **c.** $\ln 5$
- **d.** $-\ln 5$