Ex: 4

$R(x) = 50x \quad , \quad C(x) = 0.01x^2 + 30x + 1900$

$P = R - C$

$= 50x - (0.01x^2 + 30x + 1900)$

$= -0.01x^2 + 20x - 1900$

$P(500) = 5600\$$ producing and selling 500 units yeilds a profit of 5600 \$

$P' = 0.02x + 20$

$P'(500) = 10\$$ the approximated profit of unit number 501 is 10\$

$P(501) - P(500) = 9.99\$$

$P(501) - P(500) \approx P'(500)$

Find $\overline{MR}$ at $x = 2000$

$\Rightarrow P'(2000) = -20\$$ unit 2001 will decrease the profit by approximately 20 \$

$P'(1000) = 0$

يعني 1001 مش راح تزيد شي ع P

Ex: ② $c(x) = 0.001x^3 - 0.3x^2 + 32x + 2500$

① The $\overline{MC}$ function:

$C'(x) = 0.003x^2 - 0.6x + 32$

② find $C'(80)$

$C'(80) = 3.2\$$

producing one more unit at level of production 80 will increase the cost by opproximately 3.2$

The cost of unit 81 is opproximately 3.2$

$c(81) - c(80) = 3.14\$$ Exact $\overline{MC}$

Ex: ③ $P(x) = 20\sqrt{x+1} - 2x - 22$

profit function.

① $P'(x) = \frac{10}{\sqrt{x+1}} - 2$

② $\overline{MP}$ at $x = 3 = P'(3) = 3\$$ per unit

the profit from salling the 4^{th} unit is appraximately 3$

$P(4) - P(3) \approx P'(3)$

Exact approximated

9.9 Applications: Marginal and Derivatives

Recall: $R' = \overline{MR} \equiv$ marginal revenue

$C' = \overline{MC} \equiv$ " cost

$P' = \overline{MP} \equiv$ " profit

$P = R - C \Rightarrow P' = R' - C'$

Ex: ① The demand for product is given by

$p = 1000 - 20x$

① find the total revenue function

$R(x) = p(x) = (100 - 20x)x$

$= 100x - 20x^2$

② find the $\overline{MR}$ function

$R'(x) = 100 - 40x$

③ find and explain: $R(20)$, $R(21)$, $R'(20)$

$R(21) - R(20)$

$R(20) = 12000\ \$$

$R(21) = 12180$

$R(21) - R(20) = 180$ Exact $\overline{MR}$

$R'(20) = 200$ Approximated $\overline{MR}$

$R(21) - R(20) \simeq R'(20)$

If the demand for product is $p = 28 - 0.1x$

(a) Find the revenue function

$R(x) = px = (28 - 0.1x)x = 28x - 0.1x^2$

(b) Find $R(16) - R(15)$

$R(16) - R(15) = 24.9\ \$$

the exact revenue obtaned from selling unit #16 is 24.9 \$

(c) Find $R'(15)$

$R'(x) = 28 - 0.2x$

$R'(15) = 25\ \$$

the approximate revenue obtanned from selling unit #16 is 25 \$

كان مثال رقم ٩.٧ :

If $f(x) = \dfrac{x^3+1}{x^2}$ find $f'(2)$

$$F(x) = \frac{x^3}{x^2} + \frac{1}{x^2} = x + x^{-2}$$

$$F'(x) = 1 + -2x^{-3} = 1 - \frac{2}{x^3}$$

$$F'(2) = 1 - \frac{2}{8} =$$

If $y = \dfrac{1+x^2-x^4}{1+x^4}$ find y'

* هون ما كلل البسط
مشتقة سطى مقام كاري

③ At what rate is the marginal revenue changing when $x = 30$

$\Rightarrow$ This mean find $(R') = R''\Big|_{x=30}$

$$R = 20 - 9000(3x+10)^{-2}$$

$$R'' = (-2)(-9000)(3x+10)^{-3}(3)$$

$$R''(30) = \frac{54000}{(3x+10)^3} = 0.054$$

$$R''(31) - R''(30) = 0.054$$

when 1 more unit sold at $x = 30$, the m.r will increase by approximately ~~0.054~~ 0.054 (1000) dollars

Ex: ① $f(x) = x^4 + 10x^3 + 100$

$$f'(x) = 4x^3 + 30x^2$$

$$f''(x) = 12x^2 + 60x$$

$$f'''(x) = 24x + 60$$

$$f^{(4)}(x) = 24$$

$$f^{(5)}(x) = 0$$

② $f(x) = (x+1)^{-10}$

$$F'(x) = -10(x+1)^{-11}$$

$$f''(x) = 1\cancel{4}0(x+10)^{-12}$$

$$\vdots$$

* * * * *

Ex: If $R(x) = 20x - 3000(3x+10)^{-1} - 30$

$R(x)$ in \$ 1000

① find $R(30)$ $R(30) = 540\$$ (1000) total revenue

② find the marginal revenue at $x = 30$

$$R'\Big|_{x=30} = 20 - \frac{9000}{(3x+10)^2}\Bigg|_{x=30}$$

$$= 20 - \frac{9000}{(100)^2} = 19.1\$ \ (1000)$$

$$R(31) - R(30) = 19.1$$

③ $R(x) = 600x + \frac{4000}{x+10}$ Find and interpret

the M.r at $x=10$

$R(x) = 600x + 4000(x+10)^{-1}$

$R'(x) = 600 - 4000(x+10)^{-2}$

$R'(10) = 560$ ≈

الوحدة رقم ١١ ربح Producing and selling one extra unit

تزيد تقريباً الريفينيو at $x = 10$ will increase the revenue by

٥٦٠ دولار approximately 560$

* * * *

9.8 Higher order Derivatives:

If $y = f(x)$, $y' = f'(x) \equiv$ first derivative of $f(x)$ with respect to x

$$(y')' = (f')' = f'' = \frac{d^2y}{dx^2}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right)$$

$\equiv$ The second derivative of $f(x)$ with respect to x

* $y, y', y'', y''', y^{(4)}$

$\downarrow$

$(y''')'$

Ex: $y=\sqrt{h(x)}$ $h(1)=1$, $h'(1)=2$

Find $y'(1)$

$$y'=\frac{h'(x)}{2\sqrt{h(x)}}=\frac{\cancel{2}}{\cancel{2}\times 1}=1$$

$\star\star\star\star$

9.7 using Derivative formolas:

Ex①: $F(x)=\left(\frac{x^2}{2x+1}\right)^{10}$ find $f'(x)$

$$F(x)=\frac{(x^2)^{10}}{(2x+1)^{10}}=(x^2)^{10}(2x+1)^{-10}$$

$$f'(x)=(x^2)^{10}\times\left(-10(2x+1)^{-11}(2)\right)+20x^{19}\cdot(2x+1)^{1}$$

$=$

② $y=x^3\sqrt[3]{x^2+8}$ find $y'(0)$

$$y'=x^3\cdot\frac{2x}{2\sqrt[3]{(x^2+8)^2}}+\sqrt[3]{x^2+8}\cdot 3x^2$$

$y'(0)=0$

$$\left.\frac{dp}{dq}\right|_{q=4} = \left.(540)\left(-\frac{1}{2}\right)(2q+1)^{-\frac{3}{2}} \cdot 2\right|_{q=4}$$

$$= \left.\frac{-540}{(2q+1)^{\frac{3}{2}}}\right|_{q=4} = -20\ \$ \text{ Per pric}$$

If the demand increased by 1 unit at the level of production $q=4$, then the price will increase by 20 $

****

Ex: if $h(x) = (Fog)(x)$, $f(2)=3$, $g(3)=1$
$f'(1) = -2$, $g'(1)=3$, $f'(3)=2$
$g'(3)=4$ $f'(2)=-1$, $g'(2)=2$
find $(Fog)'(3)$

① $(Fog)'(3) = f'(g(3)) \cdot g'(3)$

$f'(1) \cdot g'(3)$

$= -2 \times 4 \quad = -8$

② $(gof)'(2) = g'(f(2)) \cdot f'(2)$

$= g'(3) \cdot f'(2)$

$= 4 \times -1 \quad = -4$

③ $g(x) = \frac{4}{3x^2+10x}$

$g(x) = 4(3x^2+10x)^{-1}$

$g'(x) = 4(-1)(3x^2+10x)^{-2} \cdot (6x+10)$

$$= \frac{-4(6x+10)}{(3x^2+10x)^2}$$

④ Find the equation of the tangent to $f(x) = (x^3+1)^{\frac{5}{2}}$ at $x=2$

Point: $f(2) = 243$ $\quad (2, 243)$

Slope $= f'(2) = 810$

Equation: $y - 243 = 810(x-2)$

* ...

Ex: If $P = \frac{540}{\sqrt{2q+1}}$, P = price in dollars, q = quantity demanded.

Finde the rate of change of price P with respect to quantity demanded q at $q=4$ unit. Interpret your answer.

$$P = 540(2q+1)^{-\frac{1}{2}}$$

Chain Rule:

If $y = f(u)$, $u = g(x)$ $[y = f(g(x)) = f\circ g(x)]$

then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$f'(u) \cdot g'(x)$

$= f'(g(x)) \cdot g'(x)$

$y = (f\circ g)(x) \Rightarrow y' = [(f\circ g)(x)]' = f'(g(x)) \cdot g'(x)$

Power Rule:

If $y = (f(x))^n$

$y' = n(f(x))^{n-1} \cdot f'(x)$

Ex: ① $y = \sqrt{x^2+1} \Rightarrow y = (x^2+1)^{\frac{1}{2}}$

$y' = \left(\frac{1}{2}\right)(x^2+1)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$

② $y = \sqrt[3]{(x^3+5)^2}$

$y = (x^3+5)^{\frac{2}{3}}$

$y' = \frac{2}{3}(x^3+5)^{-\frac{1}{3}} \cdot 3x^2$

$= \frac{2x^2}{\sqrt{x^3+5}}$

9.6 The chain Rule and the Power Rule

Recall: Composite functions

If f and g are defined functions, then the Composite $g \circ f$ (g after f) is defined by:

$$(g \circ f)(x) = g(f(x))$$

Example: $f(x) = 2x + 5$, $g(x) = x^2 + 1$

$(g \circ f)(1) = g(f(1)) = g(7) = 50$

$(f \circ g)(1) = 9$

$(f \circ g)(x) = f(g(x)) = f(x^2+1)$

$= 2x^2 + 7$

$$\left.\frac{dq}{dp}\right|_{p=25} = -4 \text{ units per dollar}$$

يعني لو فكر التاجر يزيد السعر 1$ فيصبح 26$ سيكون تأثير
الزيادة هو أنه سيقل الطلب بمقدار 4 وحدات

Note: If $C(x) \equiv$ Cost function, $C'(x) = \overline{MC}$

$P(x) \equiv$ profit function, $P'(x) = \overline{MP}$

$$P = R - C \implies P' = R' - C'$$

$$\implies \overline{MP} = \overline{MR} - \overline{MC}$$

question 51 page 589:

$$q = \frac{1000}{\sqrt{p}} - 1 \qquad p > 0$$

لأن العلاقة بين q و p عكسية

Why this equation is a demand equation?

Find the rate of change of demand with respect to price at $p = 25\$$, $p = \$100$

P في ازدياد وعكس بيصير العكس.

Explain your answer.

Rate of change of demand with respect to price is $\frac{dq}{dp}$

$$q = 1000\, p^{-\frac{1}{2}} - 1$$

$$\frac{dq}{dp} = -(1000)\left(\frac{1}{2}\right) p^{-\frac{3}{2}}$$

لاننا معكي كن ديماند والعلاقة عكسية

فـ الريت أوف تشينج دائماً سالب

$$= \frac{-500}{p^{\frac{3}{2}}}$$

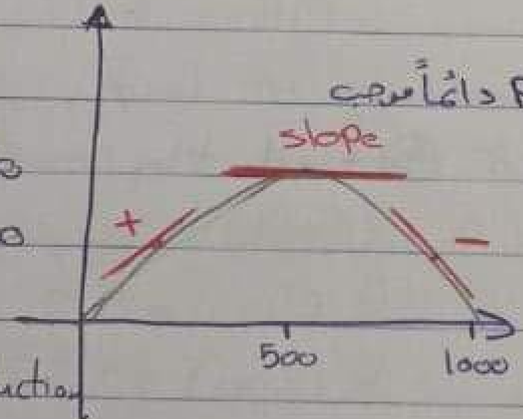
* $R(20) = 1960$, $R(21) = 2055.9$, $R'(20) = 96$
* $R(21) - R(20) = 95.9 \equiv$ The revenue of unit number 21 is \$95.9 <u>OR</u> The marginal revenue at $x = 20$ is \$95.9

* $R(21) - R(20)$ is the exact marginal revenue
  $R'(20) = 96$ is the approximated m.r

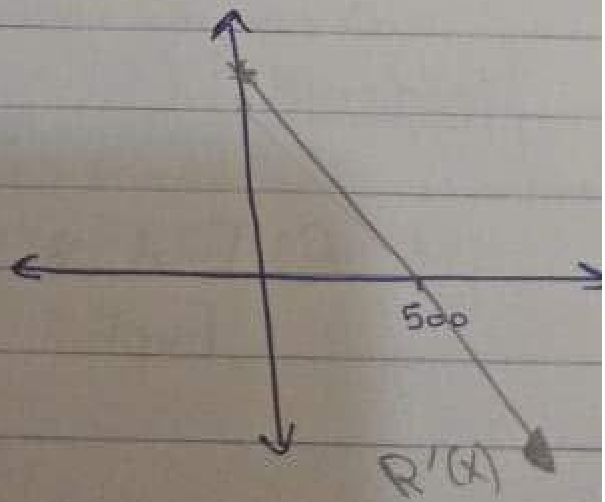
$$\Rightarrow R(21) - R(20) \simeq R'(20)$$

In general $\underbrace{R(a+1) - R(a)}_{\text{Exact}} \simeq \underbrace{R'(a)}_{\text{approximated}}$

$R(X) = 100X - 0.1X^2$ (Parabola opens up)
$R'(X) = 100 - 0.2X$



Explain : $R'(600) = -20$
$R(601) - R(600) \simeq -20$
Producing one extra unit at level of production by approximately \$ 20



Equation: $y - y_1 = m(x - x_1)$

$$y - 1 = \frac{3}{5}(x - 1)$$

اقتران الريفنيو دائماً موجب

$R(x) \geq 0$

$$y = \frac{3}{5}x + \frac{2}{5}$$

Example: (Applkation)

$R(x) = 100x - 0.1x^2$ (Revenue function)

① find $R(400)$

$R(400) = 100 \cdot 400 - 0.1 \cdot 400^2 = \$\,24{,}000$

The revenue of producing and selling 400 units is \$ 24,000

ممكن تطلع بالسا ② find The marginal revenue function

$MR = R'(x) = 100 - 0.2x$

③ find the marginal revenue at $x = 400, 500, 600$

$R'(400) = 100 - (0.2)(400) = \$20$ per unit

$R'(500) = \$0$

$R'(600) = -\$20$

$\Rightarrow$ marginal revenue maybe positive, negative or zero $(R(x) \geq 0)$

④ Find $R(20)$, $R(21)$, $R(21) - R(20)$, $R'(20)$

Explain your results

Ex:

① $f(x) = \dfrac{10}{x^4} + \dfrac{10}{\sqrt[4]{x}} + x^4 - 10$

$= 10x^{-4} + 10x^{-\frac{1}{4}} + x^4 - 10$

$F'(x) = -40x^{-5} + -\dfrac{10}{4}x^{-\frac{5}{4}} + 4x^3$

$= \dfrac{-40}{x^{+5}} - \dfrac{10}{4\sqrt[4]{x^5}} + 4x^3$

$f'(x)\Big|_{x=1} = f'(1) = \dfrac{-40}{(1)^{+5}} - \dfrac{10}{4\sqrt[4]{1^5}} + 4(1)^3$

$= -40 - \dfrac{10}{4} + 4 = -36 - \dfrac{10}{4}$

② if $f(x) = \dfrac{x^2+4x}{3x+2}$, find th equation of the tangent to the curve at $x=1$

حتى اكون معادلة بحتاج نقطة وميل :

Point: $(1, f(1)) = (1, 1)$

Slope $= f'(1) = \dfrac{(3x+2)(2x+4) - (x^2+4x)(3)}{(3x+2)^2}\Bigg|_{x=1}$

$= \dfrac{(5)(6) - (5)(3)}{5^2} = \dfrac{3}{5}$

Rule 4: $\frac{d}{dx}[f(x) \mp g(x)] = f'(x) \pm g'(x)$

حد حد نشتق (Term by Term Differentiation)

Ex: ① $\frac{d}{dx}(3x^2 + 10x + 100) = 6x + 10 + 0$

$= 6x + 10$

$4x^{-4}$ اكتر

② $\frac{d}{dx}\left(\frac{4}{x^4} + 3\sqrt[4]{x} + 5x\right) =$

$= \frac{\cancel{x^4 \times 0} - 4 \times 4x^3}{x^8} + 3 \times \frac{1}{4} x^{\frac{-3}{4}} + 5 = \frac{-16}{x^5} + \frac{3}{4} x^{\frac{-3}{4}} + 5$

Rule 5: $\frac{d}{dx}(f \cdot g) = f \cdot g' + g \cdot f'$

Ex: $\frac{d}{dx}(x^3 + 10x)(x^4 + 2x + 5)$

$= (x^3 + 10x)(4x^3 + 2) + (x^4 + 2x + 5)(3x^2 + 10)$

Rule 6: $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g \cdot f' - f \cdot g'}{g^2}$

Ex: $\frac{d}{dx}\left(\frac{x^2 + 10}{x^3 + 3x^2}\right) = \frac{(x^3 + 3x^2)(2x) - (x^2 + 10)(3x^2 + 6x)}{(x^3 + 3x^2)^2}$

③ $\frac{d}{dx}(\sqrt[5]{x^4}) = \frac{d}{dx}(x^{\frac{4}{5}}) = \frac{4}{5}x^{\frac{-1}{5}} = \frac{4}{5\sqrt[5]{x}}$

Rule 2: $\frac{d(c)}{dx} = 0$ (c is Constant)

$y = C$ is a horizontal line $\Rightarrow$ Slope $= 0$

Ex: $f(x) = 2020 \Rightarrow f'(x) = 0$

Rule 3: $\frac{d}{dx}[c f(x)] = c \cdot \frac{d}{dx}[f(x)] = c f'(x)$

Ex: ① $\frac{d}{dx}(5x^8) = (5)(8x^7) = 40x^7$

② $\frac{d}{dx}\left(\frac{5}{x^{10}}\right) = \frac{d}{dx}(5x^{-10}) = -50x^{-11}$

③ $\frac{d}{dx}\left(\frac{1}{4\sqrt[3]{x^5}}\right) = \frac{1}{4}\frac{d}{dx}\left(\frac{1}{x^{\frac{5}{3}}}\right)$

$= \frac{1}{4}\frac{d}{dx}(x^{-\frac{5}{3}})$

$= \frac{1}{4} \times \frac{-5}{3} x^{-\frac{8}{3}}$

$= \frac{-5}{12} x^{-\frac{8}{3}}$

* مهم جداً

⇒ Interpretations of the derivative Given $y = f(x)$ find $y' = f'(x) = \frac{dy}{dx}$ means: تفسير المشتقة الأولى

dependent متغير تابع — independent متغير مستقل

① The rate of change of y with respect to x

معدل التغير في y بالنسبة لـ X ⇐ $\frac{dy}{dx}$, معدل التغير اللحظي

② The velocity (Instantaneous)

السرعة اللحظية

③ The slope of the tangent line to the graph of $f(x)$

④ marginal (cost, revenue, profit)

المشتقة الأولى تعني الاقتصادي

$R'(x) \equiv \overline{MR} \equiv$ marginal revenue

$C'(x) \equiv \overline{MC} \equiv$ marginal cost

$P'(x) \equiv \overline{MP} \equiv$ marginal profit

* * * *

Rules: if f, g are differentiable functions of x, C is a constant, then:

Rule 1: $\frac{d(x^n)}{dx} = n x^{n-1}$, n is any real number

Ex: ① $\frac{d(x^{2020})}{dx} = 2020 x^{2019}$

② $\frac{d}{dx}\left(\frac{1}{x^{100}}\right) = \frac{d}{dx}(x^{-100}) = -100 x^{-101}$

* تلخيص الي قبل :

* $\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}$ = average rate of change

* $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ if exists is called the

معدل التغير اللحظي

instantaneous rate of change or simply the rate of change.

$\Rightarrow \lim_{h \to 0}$ average rate of change = rate of change

مش مطلوب منا

* if the limit above exists, it's called the

"المشتقة الأولى للاقتران"

first derivative of the function $f(x)$ with respect to x. It is denoted by $f'(x)$

* $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ limit exists

رموز المشتقة الأولى

* Notations : f', y', $\frac{dy}{dx}$, $\frac{df}{dx}$

* $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(x) \Big|_{x=a}$

* if the above limit exist, then $f(x)$ is said to be a differentrable function of x

اقتران قابل للاشتقاق

Ex①: $f(x) = \sqrt{2x+1}$ $x \in [0,4]$

Average rate of change $= \dfrac{f(4) - f(0)}{4-0}$

$= \dfrac{\sqrt{9} - \sqrt{1}}{4-0} = \dfrac{2}{4} = \dfrac{1}{2}$

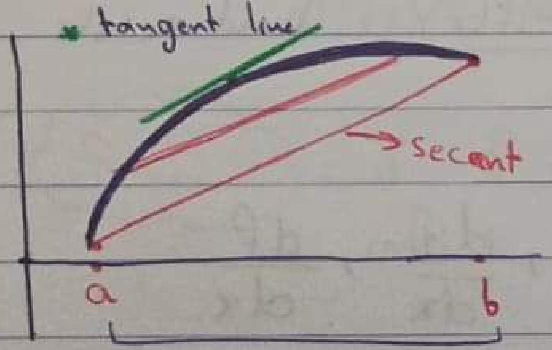
Ex②: the cost function for a product is given by $C(x) = x^2 + 5x + 100$, find the average rate of change of x from $x = 10$ to $x = 20$ units

average rate $= \dfrac{C(20) - C(10)}{20-10} = \dfrac{600 - 250}{10}$

$= 35$ dollars per unit

****

Average rate of change $= \dfrac{\Delta y}{\Delta x}$



$\Delta X = h$

كل ما قلّ ΔX بقل ال secant

لما $\Delta X \to 0$ بطلع عندي أشي جديد

هو tangent line "المماس" وبطل

عندي secant "بصير ال secant

بتحول لـ "tangent"

$= \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

$= \dfrac{f(x + h) - f(x)}{h}$ $\Delta x = h$

* as $\Delta x = h$ approaches 0, the slope of the secant approaches the slope of tangent.

That is $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \boxed{\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}}$

if exists is the slope of tangent line to $y = f(x)$

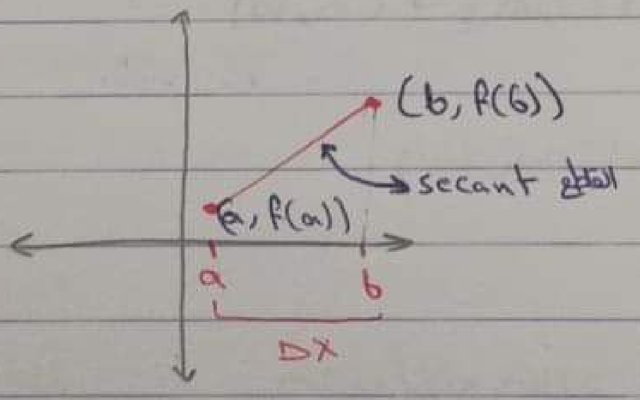
9.3 Rates of Changes and Derivatives

the slope "m" of a given line is defined by:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

* Definition: consider a function $y = F(x)$ defined over an interval $[a, b]$, the معدل التغير average rate of change of $f(x)$ from $x = a$ to $x = b$ is given by:

Average rate of change $= \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$



$\Delta x = b - a$

$b = a + \Delta x$

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

بشكل عام نحط x بدل a أو أي رمز تاني

The average rate of change is the slope of the secant line القاطع Joining $(a, f(a))$ and $(b, f(b))$

* إذا ~~[scribbled word]~~ درجة البسط أقل من درجة المقام يكون الجواب zero
* اذا درجة البسط تساوي درجة المقام يكون الجواب عدد ثابت
* اذا درجة البسط أكبر من درجة المقام يكون الجواب $\pm\infty$

****

Ex: (1) $\lim\limits_{x\to\infty} \dfrac{x^2+10x}{x^3-100} = 0$

(2) $\lim\limits_{x\to\infty} \dfrac{3x^2+2x+4}{5x^2+10} = \dfrac{3}{5}$ ← لو اشارة x الي فوق سالب بطلع الجواب $-\infty$

(3) $\lim\limits_{x\to\infty} \dfrac{x^3+10}{x^2+1} = +\infty$

(4) $\lim\limits_{x\to-\infty} \dfrac{x^3+10}{x^2+1} = -\infty$

****

Ex: $f(x) = \dfrac{x^2-1}{x^3+x^2}$ (4) $\lim\limits_{x\to\pm\infty} f(x) = 0$

(1) $\lim\limits_{x\to 1} f(x) = \dfrac{(1)^2-1}{1+1} = \dfrac{0}{2} = 0$

(2) $\lim\limits_{x\to -1} f(x) = \dfrac{-1^2-1}{-1^3+-1^2} = \dfrac{0}{0}$

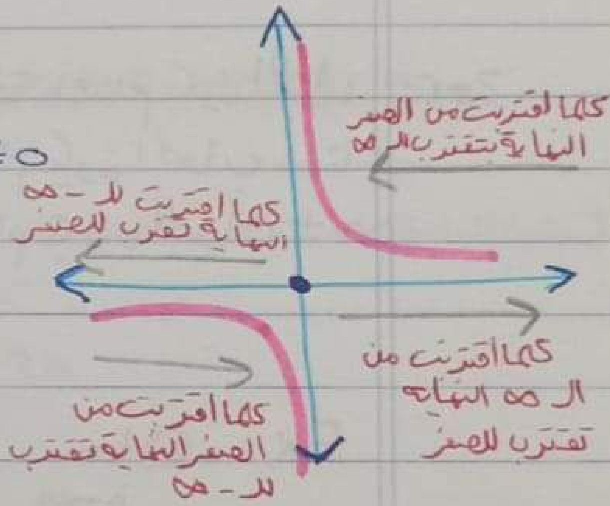
$\Rightarrow \lim\limits_{x\to-1} \dfrac{x^2-1}{x^3+x^2} = \lim\limits_{x\to-1} \dfrac{\cancel{(x+1)}(x-1)}{x^2\cancel{(x+1)}}$

$\lim\limits_{x\to-1} \dfrac{x-1}{x^2} = \dfrac{-1-1}{(-1)^2} = -2$

(3) $\lim\limits_{x\to 0} f(x) = \dfrac{0-1}{0} = \dfrac{-1}{0}$ lim DNE

limits at Infinity :-

Consider $y = f(x) = \frac{1}{x}$ $\quad x \neq 0$



$$\lim_{x \to \infty} \frac{1}{x} = \lim_{x \to -\infty} \frac{1}{x} = 0$$

يعني 1 تقسيم رقم كبير جداً Zero

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty \, , \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

Note:

① $\lim_{x \to \pm\infty} c = c$

② $\lim_{x \to \pm\infty} \frac{c}{x^n} = 0 \, , \; n > 0$

③ $\lim_{x \to \infty} x^n = +\infty \, , \; n > 0$

if $R(x) = \frac{p(x)}{q(x)}$, then

$$\lim_{x \to \infty} R(x) = \begin{cases} 0 & \deg p < \deg q \\ \text{constant} & \deg p = \deg q \\ \pm\infty & \deg p > \deg q \end{cases}$$

* deg : درجة X

"القوة"

Ex: $f(x) = \begin{cases} x^3 - 5 & x < 2 \\ 3 & x \geq 2 \end{cases}$

at $x = 2$ $\quad f(2) = 3$

$$\lim_{x \to 2^+} f(x) = 3$$

$$\lim_{x \to 2^-} f(x) = 2^3 - 5 = 3$$

$$f(2) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = 3$$

$\therefore f(x)$ is Continous for all x

Ex: Given $f(x) = \begin{cases} x^2 + 3 , & x \leq -1 \\ ax + 2 & x > -1 \end{cases}$

find the value of $\boxed{a}$ such $f(x)$ is Continous for all x.

$$f(-1) = \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x)$$

لو قلنا أصلاً هي حطيت النهاية من اليسار نفس الأصلي عادي

$$f(-1) = (-1)^2 + 3 = 4$$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} ax + 2 = 4$$

$$= a(-1) + 2 = 4$$

$$-a + \cancel{2} = 4$$

$$\cancel{-2} \quad -2$$

$$-a = 2 \Rightarrow \boxed{a = -2}$$

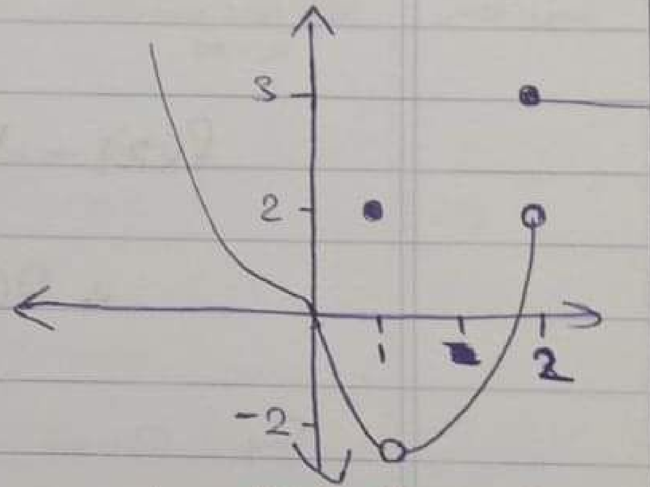
اقتران نسبي

② if $R(x) = \frac{f(x)}{g(x)}$ is a rational function then

$R(x)$ is continuous for x such that $g(x) \neq 0$

الاقتران النسبي متصل عند جميع قيم x ماعدا أصفار المقام ($g(x) = 0$)

Ex: Consider the graph of $f(x)$



* at $x=1$: $\lim_{x \to 1} f(x) = -2$

$f(1) = 2$

$\Rightarrow f(x)$ is discontinuous at $x=1$

* at $x = 2$ $f(2) = 3$

$\lim_{x \to 2^+} f(x) = 3 \neq \lim_{x \to 2^-} f(x) = 2$ $\lim_{x \to 2} f(x)$ D.N.E

$\Rightarrow f(x)$ is discontinuous at $x = 2$

Ex : $f(x) = \begin{cases} x^2 + 2x - 1 & x \geq 2 \\ \sqrt{2x+5} & x < 2 \end{cases}$ متصل لأنه كثير حدود / متصل على المجال تأكد

at $x=2$ * $f(2) = 2^2 + 2 \times 2 - 1 = 7$

* $\lim_{x \to 2^+} f(x) = 7$

* $\lim_{x \to 2^-} f(x) = 3$

$\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$ $\therefore \lim_{x \to 2} f(x)$ D.N.E

$\Rightarrow f(x)$ is discontinuous at $x=2$

Q: $f(x) = \begin{cases} 12 - \frac{3}{4}x , & x \le 4 \\ x^2 - 7 , & x > 4 \end{cases}$

* $\lim_{x \to 4} f(x) \Rightarrow \lim_{x \to 4^+} f(x) = \lim_{x \to 4^-} f(x) = 9$

$\lim_{x \to 4^+} f(x) = (4)^2 - 7 = 9$

$\lim_{x \to 4^-} f(x) = 12 - \frac{3}{\cancel{4}} \times \cancel{4} = 9$

* * * * *

9.2 Continuous functions .... الاقترانات المتصلة

تعريف Def : the functions $f(x)$ is continuous at $x = c$

يوجد ٣ شروط للاتصال يجب أن تتحقق جميعها

if ① $\lim_{x \to c} f(x)$ exists

② $f(c)$ exists (defined)

③ $\lim_{x \to c} f(x) = f(c)$

* إذا لم يتحقق شرط واحد على الأقل يسمى الاقتران discontinuous

Note:

① if $f(x)$ is a polynumail (كثير حدود), then $f(x)$ is continuous for all x

الاقتران كثير الحدود يكون متصلاً دائماً ($\lim_{x \to c} f(x) = P(c)$ section 9.1)

Q: $\lim_{x \to 5} \left(f(x) - g(x)\right) = 8$, $\lim_{x \to 5} g(x) = 2$

① $\lim_{x \to 5} f(x) \Rightarrow \lim_{x \to 5} f(x) - \lim_{x \to 5} g(x) = 8$

$$\lim_{x \to 5} f(x) - 2 = 8$$

$$\Rightarrow \lim_{x \to 5} f(x) = 10$$

② $\lim_{x \to 5} \left((g(x))^2 - f(x)\right) = \left(\lim_{x \to 5} g(x)\right)^2 - \lim_{x \to 5} f(x)$

$$(2)^2 - 10 = -6$$

③ $\lim_{x \to 5} \left(\dfrac{2x \cdot g(x)}{4 - f(x)}\right) = \dfrac{\lim_{x \to 5} 2x \cdot g(x)}{\lim_{x \to 5} (4 - f(x))}$

$$= \frac{\lim_{x \to 5} 2x \cdot \lim_{x \to 5} g(x)}{\lim_{x \to 5} 4 - \lim_{x \to 5} f(x)}$$

$$\frac{10 - 2}{4 - 10} = \frac{8}{-6} = \frac{-4}{3}$$

Ex: $f(x) = \frac{x^2 + x}{x^2 - 1}$

① $\lim_{x \to 0} f(x) = \frac{0+0}{0-1} = 0$

② $\lim_{x \to 2} f(x) = \frac{4+2}{4-1} = 2$

③ $\lim_{x \to -1} f(x) = \frac{1-1}{1-1} = \frac{0}{0}$

$\lim_{x \to -1} \frac{x^2 + x}{x^2 - 1} = \lim_{x \to -1} \frac{x\cancel{(x+1)}}{(x-1)\cancel{(x+1)}}$

$= \frac{-1}{-2} = \frac{1}{2}$

④ $\lim_{x \to 1} f(x) = \frac{1+1}{1-1} = \frac{2}{0}$ D.N.E

Q:

$\lim_{h \to 0} \frac{2(x+h)^2 - 2x^2}{h}$ $\left(\frac{0}{0}\right)$

$= 2 \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$

$= 2 \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$

$= 2(2x+0) = 4x$

Ex :

① $\lim\limits_{x \to -1} 10 = 10$

② $\lim\limits_{x \to 0} \dfrac{x^2+3x}{x+5} = \dfrac{0^2+3\times 0}{0+5} = 0$

③ $\lim\limits_{x \to 2} \sqrt{x^3+1} = \sqrt{8+1} = 3$ أو $\sqrt{\lim\limits_{x \to 2} x^3+1} = (9)^{\frac{1}{2}} = 3$

****

* Rational functions $\left(\frac{0}{0}\right)$ "indeterminate form" قيمة غير معينة

⇐ إذا طلعت النهاية = $\dfrac{\text{صفر}}{\text{صفر}}$ "هاي بدنا نشتغل عليها"

⇐ " " " = $\dfrac{\text{صفر}}{\text{عدد}}$ = 0

⇐ " " " = $\dfrac{\text{عدد}}{\text{صفر}}$ = النهاية غير موجودة "D.N.E"

Ex : $f(x) = \dfrac{x^2-2x-15}{x-5}$ $\quad x \neq 5$

① $\lim\limits_{x \to 0} f(x) = \dfrac{0-0-15}{0-5} = 3$

② $\lim\limits_{x \to -3} f(x) = \dfrac{0}{-8} = 0$

③ $\lim\limits_{x \to 5} f(x) = \dfrac{0}{0}$ indeterminate form

$\lim\limits_{x \to 5} \left(\dfrac{x^2-2x-15}{x-5}\right) = \lim\limits_{x \to 5} \dfrac{\cancel{(x-5)}(x+3)}{\cancel{x-5}}$

$\Rightarrow \lim\limits_{x \to 5} x+3 = 5+3 = 8$ exists

اذا كانت n زوجية لازم يكون تحت الجذر صفراً أو عدد موجب

(7) $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)} = L^{\frac{1}{n}}$

كثير حدود

(8) if f(x) is a polynomial of degree n

$$f(x) = \underline{a_n} x^n + \underline{a_{n-1}} x^{n-1} + \dots + \underline{a_1} x + \underline{a_0}$$

ثوابت

then $\lim_{x \to c} f(x) = f(c)$

يعني منعوض ⇐

"إذا بدي أوجد النهاية لأي اقتران سواء كان كثير حدود أو لا أول إشي بعوض"

اقتران نسبي

(9) if f(x) is a rational function

then $\lim_{x \to c} f(x) = \dfrac{\lim_{x \to c} g(x)}{\lim_{x \to c} K(x)} = \dfrac{g(c)}{K(c)}$ $K(c) \neq 0$

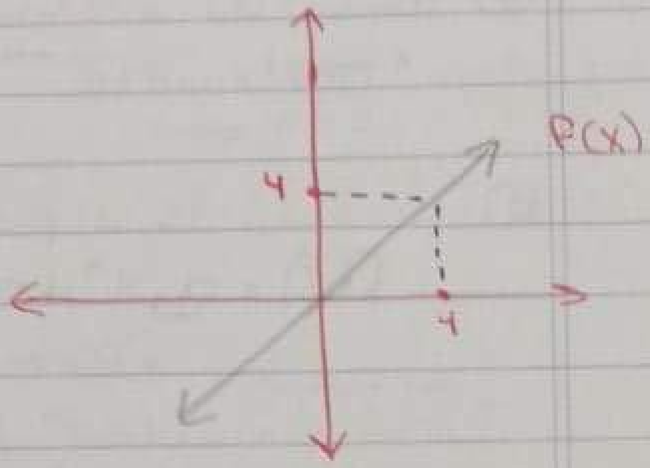
$f(x) = \dfrac{g(x)}{K(x)}$

* g(x), K(x) are both polynomials

Ex:

$$f(x) = x$$

$$\lim_{x \to 4} f(x) = 4$$



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Properties of limits : صفات النهايات

$\Rightarrow$ if $\lim_{x \to c} f(x) = L$, $\lim_{x \to c} g(x) = M$, K : constant ثابت

① $\lim_{x \to c} K = K$ نهـا ث = ث (س ← م)

② $\lim_{x \to c} x = c$ نهـا س = م (س ← م)

③ $\lim_{x \to c} (f \pm g) = L \pm M$ توزيع النهاية على الجمع ⇐

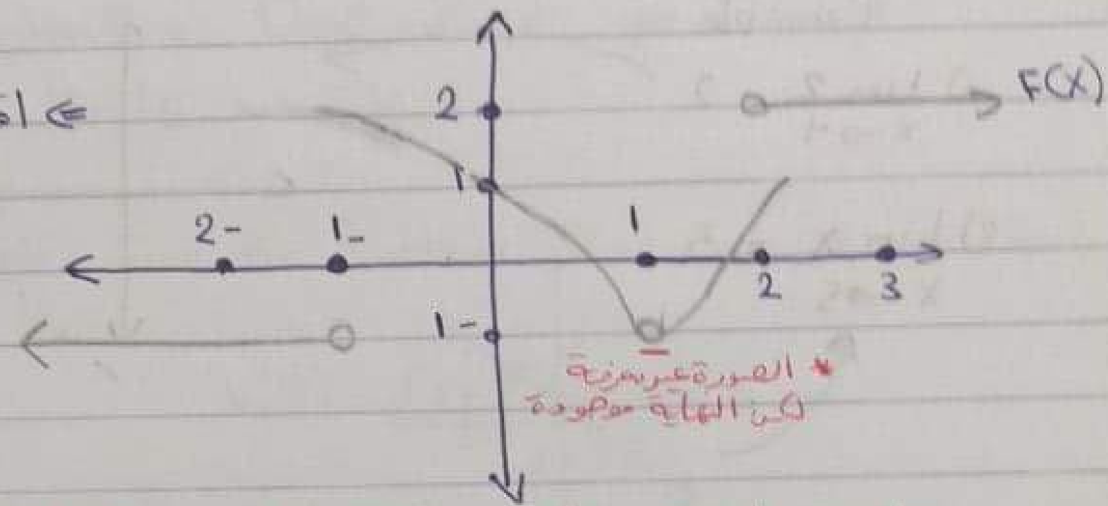
④ $\lim_{x \to c} K f(x) = K . L$

⑤ $\lim_{x \to c} (f(x) . g(x)) = L . M$ توزيع النهاية على الضرب ⇐

⑥ $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ $M \neq 0$ توزيع النهاية

Ex:

اقتران متعدد القاعدة ⇐



* $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x) = 1$ limit exists

* $\lim\limits_{x \to -1^+} f(x) = 2$, $\lim\limits_{x \to -1^-} f(x) = -1$

$\lim\limits_{x \to -1^+} f(x) \neq \lim\limits_{x \to -1^-} f(x)$ $\therefore \lim\limits_{x \to -1} f(x)$ does not exists

D.N.E

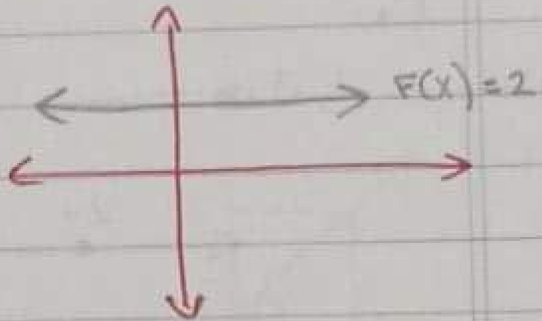
* $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^-} f(x) = -1$

* $\lim\limits_{x \to 2^-} f(x) = 1$, $\lim\limits_{x \to 2^+} f(x) = 2$

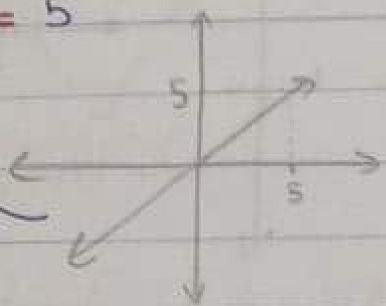
$\lim\limits_{x \to 2^-} f(x) \neq \lim\limits_{x \to 2^+} f(x)$ $\therefore \lim\limits_{x \to 2} f(x)$ D.N.E

Example :

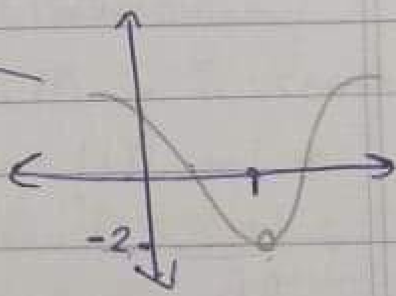
1) $\lim\limits_{x \to -1} 2 = 2$



2) $\lim\limits_{x \to 5} x = 5$



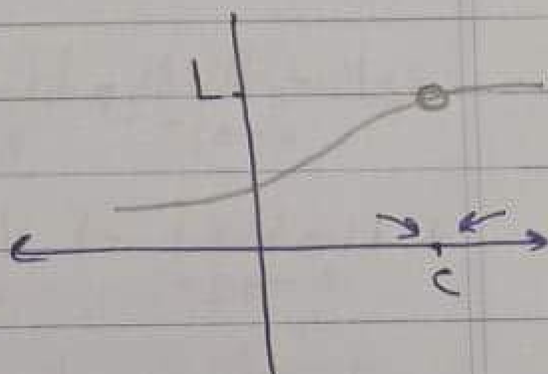
3) $\lim\limits_{x \to 1} F(x) = -2$



ONe side Limits

* Limit from right

$\lim\limits_{x \to c^+} f(x) = L$ "قيم أكبر من"



* Limit from Left

$\lim\limits_{x \to c^-} f(x) = L$ "قيم أقل من"

* حتى تكون النهاية موجودة لازم تكون النهاية من اليمين موجودة والنهاية من اليسار موجودة و يكونوا متساويات

$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$$

Both limits exist and equal

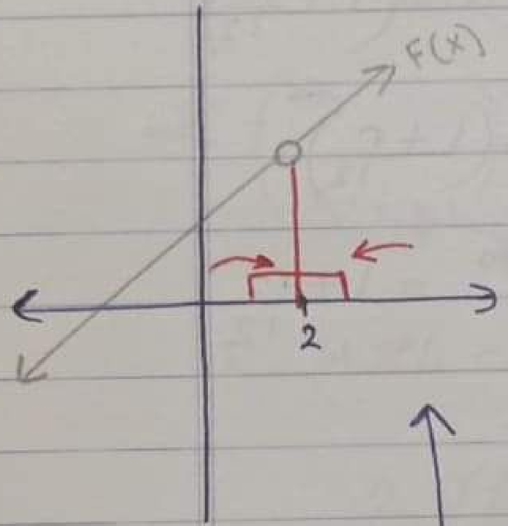
Chapter 9 : Derivatives ⇐ الإشتقاق

9.1 limits ⇐ النهايات

Example : Give the function $F(x) = \frac{x^2+2x-8}{x-2}$, $x \neq 2$

$$F(x) = \frac{\cancel{(x-2)}(x+4)}{\cancel{(x-2)}}$$

$$F(x) = x+4 \qquad x \neq 2$$



x	F(x)
1.9	5.9
1.95	5.95
1.99	5.99
2.01	6.01
2.05	6.05
2.1	6.1

تقترب

⇒ As x approaches 2

f(x) approaches 6

$x \to 2 \Rightarrow f(x) \to 6$

we say that the limit of (f(x)) as (x) approaches (2) equals (6) and we write

$$* \lim_{x \to 2} f(x) = 6 \equiv \lim_{x \to 2} \frac{x^2+2x-8}{x-2} = 6$$

النهاية موجودة $\boxed{\text{the limit exists}}$