*Math 2351 :

-**Instroctor:** Walaa A. Yasin

-**Text book:** Mathematical Applications for the Mangement, Life and Social Sciences, 10th edition, Harshbarger, Reynolds

← اسم الكتاب

• Sec 1.6 : Applications of Functions in Business and Economics. تطبيقات الاقترانات في التجارة والاقتصاد

*Review:

→A **function** f is a rule from a set A to a set B that assigns to each point x in A a unique element $y = f(x)$ in B.

اقتران

① **X**: independent variable -**input**-. المدخلات

② **y**: dependent variable -**output**-. المخرجات

③ The set of all values of **inputs** is called the **domain**. المجال

④ The set of all values of **outputs** is called the **range**. المدى

→**Linear functions:** A function of the form $y = f(x) = mx + b$ is called a linear function; m and b are constants.

① m is called the slope. « coefficient of X » معامل X -
= الميل

② b is called the y-intercept.

③ To graph a linear functions we find:-

→ x-intercept: set $y = 0 \rightarrow (x, 0)$.
→ y-intercept: set $x = 0 \rightarrow (0, y)$.
→ graph a straight line.

* If we have a line passes through (x_1, y_1) and (x_2, y_2) then:-

→ the slope « m » $= \frac{y_2 - y_1}{x_2 - x_1}$.
→ the equation of the line is: $y - y_1 = m(x - x_1)$.

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*linear Models: Cost, Revenue, Profit and Break Even:-

1 Total cost: اقتران التكاليف الكلية

A linear cost function expresses the total costs $C(x)$ as a linear function of the number of items produced x.

- The cost function is composed of 2 parts:-

① Fixed costs «F.C» التكاليف الثابتة

the costs that remain constand regardless of the number of units produced.

→ For example: rent تأجير, salaries رواتب, utilities خدمات, insurance التأمين and so on....

→ $F.C = b$

② Variable costs «V.C» التكاليف المتغيرة

the cost that directly related to the number of units produced.

→ let m be the cost per unit and x be the number of units, then:-

$$V.C = \text{cost per unit} * \text{number of units}$$

$$\therefore V.C = mX$$

So, the total costs function $C(x)$:-

$$C(x) = V.C + F.C$$
$$= mX + b$$; m: the cost per unit

X: number of units.

b: fixed costs.

*Remarks:-

① If the total cost function is linear, then the cost per unit «m» is the slope.

② The slope «m» is also called the marginal cost «$\overline{MC}$»
التكلفة الحدية

→ cost per unit «m» = slope = marginal cost «$\overline{MC}$»

the marginal cost is the cost of producing one additional unit at any level of production, that is,

$$C(x+1) - C(x) = \overline{MC}$$ «if $C(x)$ is linear»

تكلفة إنتاج وحدة إضافية في أي مرحلة من مراحل الإنتاج.

③ The fixed costs «b» is $C(0)$.

$\longrightarrow$ fixed costs «b» $= C(0) =$ C-intercept.

- Ex: The daily total cost to produce x units of a product is given by $C(x) = 10x + 200$ dollars. Find:

a) The marginal cost.

$\longrightarrow \overline{MC} =$ cost per unit $=$ slop $= b$.

b) The fixed cost.

$\longrightarrow F.C = b = 200$.

c) $C(0)$. « C-intercept»

$\longrightarrow C(0) = b = 200$.

note that, the fixed cost is the part of the cost that isn't affected by the number of units produced.

d) The daily cost to produce 15 units.

$\longrightarrow C(15) = 10(15) + 200 = 350\$$

e) The daily cost to produce 14 units.

$\rightarrow C(14) = 10(14) + 200 = 340\$$

note that $C(15) - C(14) = 10 = \overline{MC}$, so the cost of producing unit #15 is 10\$.

-Ex: Suppose that when a company produces its product, fixed costs are $12500 and variable cost per item is $75.

variable cost per unit «m» = 75, fixed cost «b» = $12500

a) Write the total cost function.

$$\rightarrow C(x) = mx + b$$
$$= 75x + 12500$$

b) Are fixed costs equal to $C(0)$?

→ Yes, $C(0) = b = 12500$.

-Ex: The cost of producing 50 units of a product is $1000, and the cost of producing 100 units of the same product is $1100. Suppose the total cost is linear.

$$\rightarrow C(50) = 1000 \qquad \therefore \quad (50, 1000)$$
$$C(100) = 1100 \qquad\qquad (100, 1100).$$

since the cost is linear, and we have $\underline{\underline{2}}$ points:-

$$\rightarrow \text{slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1100 - 1000}{100 - 50} = \frac{100}{50} = \underline{\underline{2}}$$

$$\rightarrow y - y_1 = m(x - x_1)$$
$$y - 1000 = 2(x - 50)$$
$$y - 1000 = 2x - 100$$
$$\quad +1000 \qquad\qquad +1000$$
$$y = 2x + 900 \longrightarrow C(x) = 2x + 900.$$

2 Total revenue اقتران العوائد

Revenue results from the sale of items.

→ If $R(x)$ is the revenue from selling x items at a price of p each, then:

Total revenue = selling price per unit * number of units sold.

$$\longrightarrow R(x) = px$$

*Remarks:

① If the total revenue is linear, then the selling price per unit «p» is the slope.

② the slope «p» is also called the marginal revenue. العائد الحدي

→ selling price per unit «p» = slope = marginal revenue «MR».

the marginal revenue is the revenue of selling one additional unit at any level of production, that is $R(x+1) - R(x) = \overline{MR}$ «if $R(x)$ is linear»

العائد الناتج من بيع وحدة اضافية في أي مرحلة من مراحل الانتاج

③ $R(0) = 0$. « 0 units produce 0 revenue ».

-Ex: Suppose a company sells its product at \$10 per unit. Find :-

a) The revenue function.

$\longrightarrow R(X) = pX$; p: selling price per unit = 10
$= 10$ X: # of units.

b) $R(7)$.

$\longrightarrow R(7) = 70$

c) $R(6)$.

$\longrightarrow R(6) = 60$

d) The revenue of producing and selling the seventh unit.

$\longrightarrow R(7) - R(6) = 10 = \overline{MR}$.

3 Total profit. اقتران الربح

The profit is the net proceeds.

→ Profit = Revenue − cost.

$P(x) = R(x) - C(x)$

* Remarks:-

معامل x

① If $P(x)$ is linear, then the coefficient of x is the marginal profit. الربح الحدي

→ the profit of producing and selling one additiona unit at any level of production. الربح الناتج من إنتاج وبيع وحدة إضافية في أي مرحلة من مراحل الإنتاج

② $P(0) = -$ fixed costs $= -b$

0 units yield a loss equal to fixed costs.

③ $P(x)$: +ve profit.

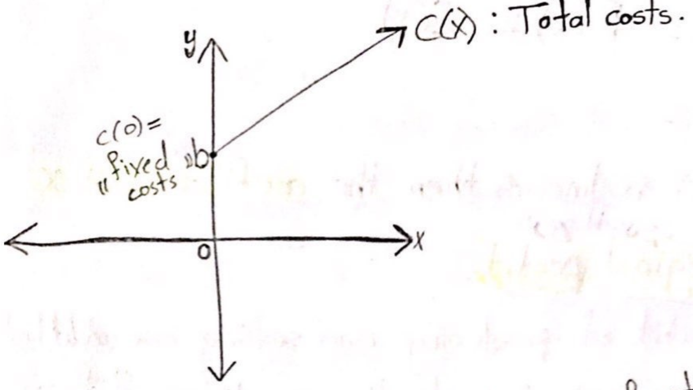
$P(x)$: −ve loss.

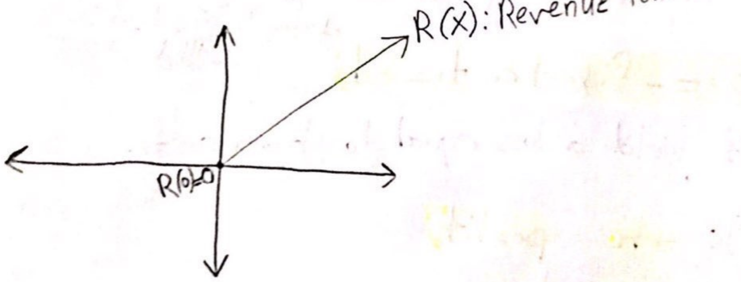
$P(x)$: 0 The break even "neither profit nor loss"

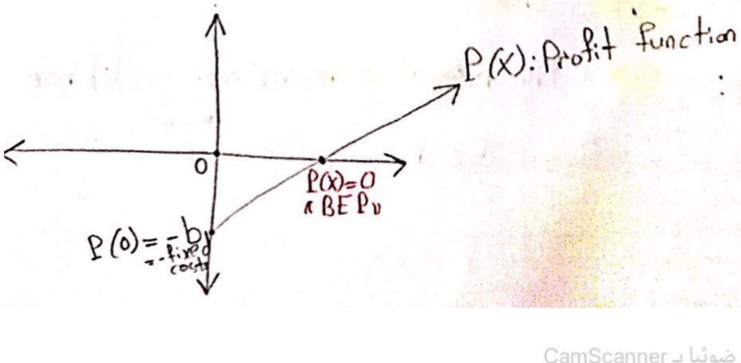
BEP: نقطة التعادل

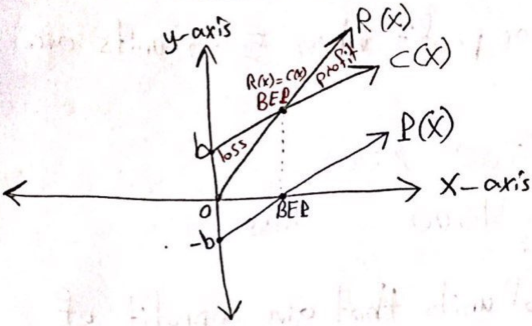
* Break even point (BEP) نقطة التعادل

If $\underline{P(X)=0}$, that is $\underline{R(X)=C(X)}$, then the BEP is $(X, R(X))=(X, C(X))$.









-Ex: If the variable cost per unit for a product is 24, the fixed costs are 8000, and the selling price per unit is $32. Find:-

① The profit function.

variable cost per unit «m»= 24 $
fixed costs «b»= 8000 $.

$$\longrightarrow C(X)=mx+b$$
$$=24x+8000$$

selling price per unit «p»= 32 $

$$\longrightarrow R(X)=pX$$
$$=32X$$

$$\longrightarrow P(X)=R(X)-C(X)$$
$$=32X-(24x+8000)$$
$$=8X-8000$$

« profit function ».

b) What is the loss or profit when 500 units are produced and sold?

$$\longrightarrow P(500) = 8(500) - 8000$$
$$= \$-4000 \quad \text{loss}$$

c) Find the number of units that give a profit of \$1600.

$$\longrightarrow P(x) = 1600$$
$$8x - 8000 = 1600$$
$$+8000 \quad +8000$$

$$\frac{8x}{8} = \frac{9600}{8}$$

$$\therefore \ x = 1200 \text{ units}$$

d) Find the break-even point.

$$\longrightarrow P(x) = 0$$
$$8x - 8000 = 0$$
$$+8000 \quad +8000$$

$$\frac{8x}{8} = \frac{8000}{8} \longrightarrow x = 1000 \text{ units.}$$

$$\therefore \text{The BEP is } \left(1000, \underset{=C(1000)}{R(1000)}\right) \longrightarrow$$

نعوض إما في
C(x) أو R(x)

$$= (1000, 32000).$$

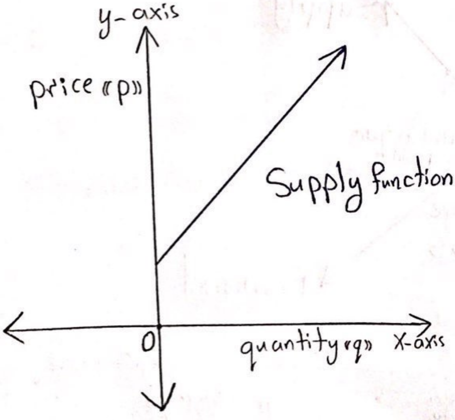
* linear Models: Demand ، Supply and Equilibrium points:

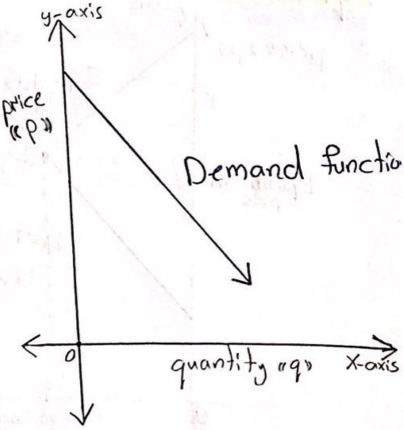
- **The law of supply**: قانون العرض

As price increases, the quantity supplied will increase. كلما زاد السعر، تزداد الكمية المعروضة

- **The law of demand**: قانون الطلب

the quantity demanded will increase as price decreases, and the quantity demanded will decrease as price increases. كلما زاد السعر، تقل الكمية المطلوبة، وكلما قل السعر، زادات الكمية المطلوبة.



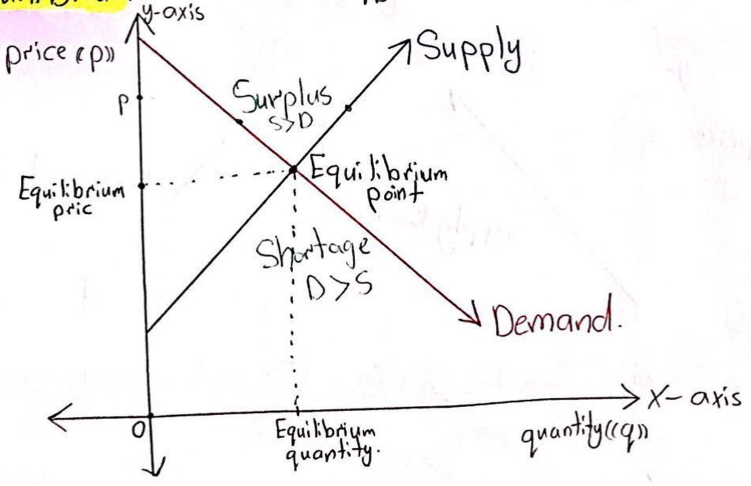


- Remark:

① If the quantity demanded "q_D" is greater than the quantity supplied "q_s", then there is a market **shortage**. "$q_D > q_s$" shortage عجز

② If the quantity demanded "q_D" is less than the quantity supplied, then there is a market **surplus**. "$q_D < q_s$" surplus فائض

③ If the quantity demanded is equal to the quantity supplied, then there is a market **equilibrium**. "$q_D = q_s$" equilibrium. توازن



*Additive Tax and Market Equilibrium:-

Often government imposes taxes on certain products in order to raise more revenue.

Suppose a supplier is taxed \$t per unit sold, and the tax is passed on to the consumers by adding \$t to the selling price of the product, (we should assume that the quantity demanded consumers depends only on the price alone, that is, the demand equation doesn't change).

If the original supply is given by: $p = f(q)$, then the new supply function after passing the tax on, is given by: $p = f(q) + t$.

عند فرض ضريبة فإن سعر العرض يتأثر بمقدار الضريبة.

- Ex: Suppose that consumers will demand 100 units of a product when the price is \$10 and 120 units when the price is \$8. Assuming, linear relationship. Find the price at which 150 units are demanded.

points: $(\underset{x_1}{\overset{q}{100}}, \underset{y_1}{\overset{p}{10}})$ $(\underset{x_2}{\overset{q}{120}}, \underset{y_2}{\overset{p}{8}})$ «linear»

$\longrightarrow$ slope $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{8-10}{120-100} = -0.1$

$\rightarrow$ $y - y_1 = m(x - x_1)$

$y - 10 = -0.1(x - 100)$

$y - 10 = -0.1x + 10$
$\quad +10 \qquad\qquad\quad +10$

$y = = -0.1x + 20$

نضع مكان x ← q «quantity» ومكان y ← price «p»

$\therefore$ the demand equation is: $p = -0.1q + 20$

$\longrightarrow$ when $q = 150$ units, the price:
$p = -0.1(150) + 20 = \$5$

- **Ex**: At a price of \$30 per unit, a company can supply 2000 units, wherease the demand is 2800 units. At a price of \$35, 400 more units can be supplied. However at this increased price, the demand reduced by 1000 units. Assuming linear relationships, determine the supply and demand functions.

$S: (\overset{x_1}{2000}, \overset{y_1}{30})$

400 more units ↓

$(\overset{x_2}{2400}, \overset{y_2}{35})$

$\rightarrow$ slope ((m)) $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{35-30}{2400-2000}$

$= 0.0125$

$\rightarrow y - y_1 = m(x - x_1)$

$y - 30 = 0.0125(x - 2000)$

$y - 30 = 0.0125x - 25$

$\quad +30 \qquad\qquad\qquad +30$

$\therefore$ the supply function is:-

$p = 0.0125q + 5$

$D: (\overset{x_1}{2800}, \overset{y_1}{30})$

reduced by 100 units ↓

$(\overset{x_2}{2700}, \overset{y_2}{35})$

$\rightarrow$ slope $= \dfrac{35-30}{2700-2800}$

$= -0.05$

$\rightarrow y - y_1 = m(x - x_1)$

$y - 30 = -0.05(x - 2800)$

$y - 30 = -0.05x + 140$

$\quad +30 \qquad\qquad\qquad +30$

$y = -0.05x + 170$

$\therefore$ the demand function is:

$p = -0.05q + 170$

-Ex: If the equilibrium price is \$15, then at $p=\$30$, Is there a market surplus, or shortage?

Note:

* If price < Equilibrium price, then there is a market shortage.

* If price > Equilibrium price, then there is a market surplus.

$\longrightarrow$ $p=30 > \text{Eq. price} = 15$

So there is a market surplus.

-Ex: Consider the following linear supply and demand:

$$D: p = 75 - 0.5q$$

$$S: p = 15 + 0.1q$$

Determine whether there is a shortage or surplus at the price of \$20.

→ At $p = \$20$

D: $$20 = 75 - 0.5q.$$

$$-75 \qquad -75$$

$$\frac{-55}{-0.5} = \frac{-0.5q}{-0.5} \longrightarrow q = 110 \text{ units}$$

$$\therefore \boxed{q_D = 110 \text{ units}}$$

S: $$20 = 15 + 0.1q$$

$$-15 \qquad -15$$

$$\frac{5}{0.1} = \frac{0.1q}{0.1} \longrightarrow q = 50 \text{ units}$$

$$\therefore \boxed{q_s = 50 \text{ units.}}$$

→ $q_D > q_s$ so there is a market shortage.

- Ex: The demand for a certain product is $5p+2x=200$, and the supply is $p=\frac{4}{5}x+10$.

$D: 5p+2x=200$

$\rightarrow p=\frac{200-2x}{5}$

$S: p = \frac{4}{5}x+10=\frac{4x+50}{5}$

a) Find the **equilibrium point**

Eq. point: D = S

$$\frac{200-2x}{5}=\frac{4x+50}{5}$$

$$\rightarrow 200-2x=4x+50$$

$$+2x \qquad +2x$$

$$200=6x+50$$

$$-50 \qquad -50$$

$$\frac{150}{6}=\frac{6x}{6} \longrightarrow \therefore \boxed{x=25} \text{ units}$$

$$\rightarrow p=\frac{200-2x}{5}=$$ نعوض X في D أو S

$$\therefore p=\frac{200-2(25)}{5}=\$30$$

∴ the equilibrium quantity is 25 and the equilibrium price is \$30 → the equilibrium point is (25, 30)

b) Find the equilibrium point after a tax of $6 per unit.

→ Demand doesn't change, only supply.

the new equilibrium point:

$$D = S_{new}$$

$$\frac{200-2X}{5} = \frac{4X+50}{5} + \overset{tax}{6}$$

$$\frac{200-2X}{5} = \frac{4X+50+30}{5}$$

$$\frac{200-2X}{5} = \frac{4X+80}{5}$$

$$\begin{aligned} 200 - 2X &= 4X + 80 \\ -80 \quad +2X \quad & \quad +2X \quad -80 \end{aligned}$$

$$\frac{120}{6} = \frac{6}{6}X \longrightarrow X = 20 \text{ units.}$$

نعوض في Snew أو D

$$\longrightarrow p = \frac{200-2X}{5}$$

$$= \frac{200-2(20)}{5} = \$32$$

∴ the eq. point after a tax of $6 per unit is (20, 32).

• Sec 2.3 : Quadratic Models.

*** Quadratic functions:-** الاقترانات التربيعية

A function of the form $y=f(x)=ax^2+bx+c$ is called a quadratic function ; a, b, c are constants and $a\neq 0$.

① **x-intercept(s):** we set $y=0 \longrightarrow (x,0)$

$$0=ax^2+bx+c$$

we solve it for x, by using the quadratic formula:

$$x=\frac{-b\mp\sqrt{b^2-4ac}}{2a}$$; if $b^2-4ac<0$ then there is no real solutions.

② **y-intercept:** we set $x=0 \longrightarrow (0,y)$.

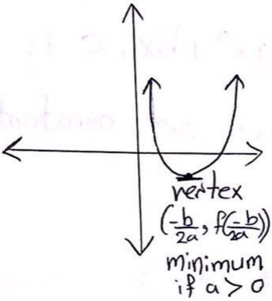
③ **the vertex:**

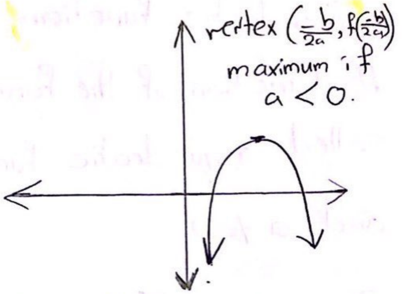
$$x=\frac{-b}{2a} \longrightarrow \text{the vertex is } \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right).$$

$\longrightarrow$ If $a>0$, th graph opens up. (a minimum point).

$\longrightarrow$ If $a<0$; the graph opens. down. (a maximum point).

④ The graph of a quadratic function is called a parabola. قطع مكافئ





In sec 1.6 we worked with revenue functions where every item x was sold at a fixed price p. Thus, the formula of the revenue is $R(x) = px$ is linear.

Also, in the previous section we worked with 2 price functions, the supply and demand price. Since we can only make a sale if the consumer is willing to buy, we typically use the demand price in computing revenue.

Our model is now: $R(x) =$ demand price $*$ quantity

If the demand price is a linear, then revenue is a quadratic function.

التكاليف

→ Total costs = Variable costs + Fixed costs.

$C(x) = mx + b$; m = cost per unit
b = fixed costs.

العوائد

→ $R(x)$ = demand price * # of units ,

$= p * X$; p: demand price = selling price per unit.

الربح

→ $P(x) = R(x) - C(x)$

نقطة التعادل

→ BEP: $P(x) = 0$ «that is, $R(x) = C(x)$»

→ max. revenue, maximum profit: vertex.

- Ex: The profit of selling x units of a product is given by the function $P(x) = 12X - 0.1X^2$? What is the maximum profit and how many units should be sold in order to earn this maximum profit.

→ $P(x) = 12X - 0.1X^2$ «quadratic function».

$a = \underset{<0}{-0.1}$, $b = 12$, $c = 0$

∴ vertex is maximum.

vertex $= \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$

$\therefore \; X = \frac{-b}{2a} = \frac{-12}{2(-0.1)} = 60$ units.

$f\left(\frac{-b}{2a}\right) = f(60) = -0.1(60)^2 + 12(60) = 360$ \$

∴ the maximum profit is \$360 when 60 units are produced and sold.

- Ex: If the total costs are $C(x) = 1600 + 1500x$, and total revenues are $R(x) = 1600x - x^2$.

① Find the break-even points.

$\rightarrow P(x) = 0$

$R(x) - C(x) = 0$

$(1600x - x^2) - (1600 + 1500x) = 0$

$100x - x^2 - 1600 = 0$ «we can solve it by factoring or by quadratic formulas»

$a = -1, \; b = 100, \; c = -1600$

$$\rightarrow x = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-100 \mp \sqrt{100^2 - 4(-1)(-1600)}}{2(-1)}$$

$$= \frac{-100 \mp \sqrt{3600}}{-2} \begin{cases} \rightarrow \frac{-100+60}{-2} = 20 \\ \rightarrow \frac{-100-60}{-2} = 80 \end{cases}$$

There are $\underline{\underline{2}}$ break even points : $(20, R(20)) = (20, 31600)$

$(80, R(80)) = (80, 121600)$

② Find maximum profit.

$P(x) = 100x - x^2 - 1600$

$a = -1$, $b = 100$, $c = -1600$

$\rightarrow$ vertex: $x = \frac{-b}{2a} = \frac{-100}{2(-1)} = 50$ units

$$f\left(\frac{-b}{2a}\right) = f(50) = 100(50) - 50^2 - 1600$$

$$= \$900$$

$\therefore$ the max. profit is \$900

③ Compare the level of production to maximize profit with the level to maximize revenue. Do they agree.

→ the level of production maximizes profit $= 50$. « from part 2 »

→ the level of production maximizes revenue

$R(x) = 1600x - x^2$

$a = -1$, $b = 1600$, $c = 0$

vertex: $x = \frac{-b}{2a} = \frac{-1600}{2(-1)} = 800$

We need to sell 800 to maximize revenue, but only 50 to maximize profit. These numbers are not equal.

- Ex: Suppose a company has fixed costs of \$360, and variable costs of $10 + 0.2x$ dollars per unit, where x is the number of units produced. Suppose further the demand price is $p = 50 - 0.2x$ dollars per unit.

a) Write the cost function.

$\rightarrow C(x) = mx+b$

$= (10+0.2x)x+360$

$= 10x+0.2x^2+360$

b) Write the revenue function.

$\rightarrow R(x) = p\,x$; p: demand price

$=(50-0.2x)x$: selling price per unit.

$= 50x-0.2x^2$

c) Write the profit function.

$\rightarrow P(x) = R(x) - C(x)$

$= (50x-0.2x^2)-(10x+0.2x^2+360)$

$= 40x-0.4x^2-360$

d) Find the max. profit.

$a=-0.4,\ b=40,\ c=-360$

vertex: $x = \frac{-b}{2a} = \frac{-40}{2(-0.4)} = 50$ units.

$\therefore$ max. profit $= P(50) = 40(50)-0.4(50)^2-360$

$= \$640$

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e) What price maximizes the profit ?

$$p = 50 - 0.2x$$

→ the price maximizes the profit :-

$$50 - 0.2(50) = \underline{\$40}$$

f) Find the sales levels where the product is not losing money.

$$\longrightarrow P(X) = 0$$

$$40X - 0.4X^2 - 360 = 0$$

$$a = -0.4,\ b = 40,\ c = 360$$

$$X = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} = \frac{-40 \mp \sqrt{40^2 - 4(-0.4)(-360)}}{2(-0.4)}$$

$$= \frac{-40 \mp \sqrt{1456}}{-0.8} \rightarrow 2.3,\ \rightarrow 97.7$$

∴ the sales levels where the product isn't losing money:- $\underline{2.3 \leq X \leq 97.7}$

- Ex: If the demand and supply functions for a product are $p^2+2q=1600$ and $200-p^2+2q=0$, respectively. Find the equilibrium price and quantity.

⟶ Eq. point: $D=S$; $D: p^2=1600-2q.$

$S: p^2=200+2q.$

$$\therefore \underset{\substack{-200\\+2q}}{1600-2q} = \underset{\substack{-200\\+2q}}{200+2q}$$

$$\frac{1400}{4} = \frac{4q}{4} \longrightarrow \therefore q = 350 \text{ units}$$

«Equilibrium quantity»

$$p^2 = 200+2q$$
$$= 200+2(350)$$
$$= 900$$

$\therefore p = 30$ «Equilibrium price»

⟶ the equilibrium point is $(350, 30)$

- Ex: If the demand and supply functions for a product are $pq = 100 + 20q$ and $2p - q = 50$, respectively

I Find the market equilibrium.

Eq. point: $D = S$; **D**: $pq = 100 + 20q$

$\rightarrow p = \dfrac{100 + 20q}{q}$

S: $2p - q = 50$

$\rightarrow p = \dfrac{50 + q}{2}$

$$\therefore \frac{100 + 20q}{q} = \frac{50 + q}{2}$$

$$2(100 + 20q) = q(50 + q)$$

$$200 + 40q = 50q + q^2$$

$$-200 \quad -40q \quad -40q \quad -200$$

$$\rightarrow q^2 + 10q - 200 = 0$$

$$(q + 20)(q - 10) = 0$$

$\therefore q = \underset{\times}{\underbrace{-20}}$ or $10 \rightarrow$ Equilibrium quantity $= 10$ units.

$q = \#$ of units مرفوضة

$\rightarrow$ Equilibrium price: $p = \dfrac{50 + q}{2} = \dfrac{50 + 10}{2} = 30$ \$

نعوض في D او S

$\therefore$ Eq. point: $(10, 30)$

$\boxed{2}$ If a $12.5 tax is placed on production and passed through the supplier, find the new equilibrium point.

→ Eq. point after a $12.5 tax

$$D = S_{new} \quad ; \; D: \; p = \frac{100+20q}{q}$$

$$S_{new}: \; p = \frac{50+q}{2} + \overset{tax}{12.5}$$

$$\frac{100+20q}{q} = \frac{50+q}{2} + \frac{12.5 \cdot 2}{1 \cdot 2}$$

$$\frac{100+20q}{q} = \frac{50+q+25}{2}$$

$$\frac{100+20q}{q} = \frac{75+q}{2}$$

$$2(100+20q) = q(75+q).$$

$$200+40q = 75q+q^2$$

$$-200 \quad -40q \qquad -40q \; -200$$

$$\rightarrow q^2+35q-200=0$$

$$(q+40)(q-5)=0 \rightarrow q = \underset{مرفوضة}{-40}, 5$$

$$\therefore \text{Eq. point}: (5, \underline{40})$$

نعوض في D أو S_{new}

• Sec 5.1 + 5.2: Exponential and logarithmic functions:-

الاقترانات الأسية واللوغارتمية

الاقتران الأسي

→ Exponential function: $y = a^{Kx}$; a positive and $a > 1$.

a: base , X: power, K: any number ((constant))

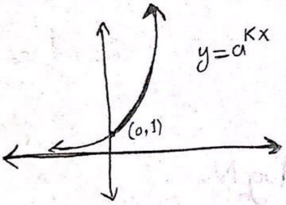
المجال

domain: ℝ ((all real numbers)) جميع الأعداد الحقيقية

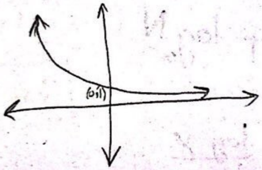
المدى

range: all positive numbers. جميع الأعداد الموجبة

① If $K > 0$:



② If $K < 0$:



الاقتران اللوغارتمي

→ logarithmic function: $y = \log_a X$; $a > 0$, $x > 0$

① $\ln x = \log_e X$; $e \cong 2.71……$

② $\log X = \log_{10} X$

* اللوغارتم عكس الأس

domain: all positive numbers.

range: all reall numbers.

* Some properties:-

① $a^0 = 1$

② $a^{-n} = \dfrac{1}{a^n}$

③ $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

④ $\log_a 1 = 0 \longrightarrow$ «$\ln 1 = 0$»

⑤ $\log_a a = 1 \longrightarrow$ «$\ln e = 1$»

⑥ $\log_a b^n = n \log_a b$

⑦ $\log_a \dfrac{M}{N} = \log_a M - \log_a N$.

⑧ $\log_a (MN) = \log_a M + \log_a N$.

⑨ $\log_a X = \dfrac{\ln X}{\ln a} = \dfrac{\log X}{\log a}$.

$\longrightarrow$ Calculator: استخدام الآلة الحاسبة

① $2^{100} \longrightarrow 2 \boxed{\cap} 100 =$

② $e^4 \longrightarrow \boxed{\text{shift}} \boxed{\ln} 4 =$ لا نضع إشارة $\boxed{\cap}$

③ $\ln 14 \longrightarrow \boxed{\ln} 14 =$

④ $\log 5 \longrightarrow \boxed{\log} 5 =$

⑤ $\log_3 4 \longrightarrow \boxed{\ln} 4 \div \boxed{\ln} 3 =$

*** Solving Equations:**

حل المعادلات

① $2^{x+4} = 5$

دائماً إذا كان الأس مجهول نأخذ ln للطرفين

$$\ln 2^{x+4} = \ln 5$$

$$\rightarrow (x+4)\ln 2 = \ln 5$$

$$x+4 = \frac{\ln 5}{\ln 2}$$

من الآلة

$$x+4 = 2.32$$

$$-4 \qquad -4$$

$$\therefore \boxed{x = -1.68}$$

② $e^x = 4$

$$\ln e^x = \ln 4$$

$$x \cancel{\ln e} = \ln 4$$

«$\ln e = 1$»

$$\therefore \boxed{x = \ln 4}$$

③ $\log_4 X = 2.1$; find x.

$\rightarrow \log_4 X = 2.1$

$4^{2.1} = X$ مع الآلة

الأساس قوة الجواب = مابداخل log

$\therefore \boxed{X = 18.379}$

④ $\log_4 X + \log_4 X - 3 = 1$

أولاً نحولهم داخل log واحد

تصبح ضرب

$\rightarrow \log_4 X \cdot (X-3) = 1$

$4^1 = X(X-3)$

$\underset{-4}{4} = X^2 - 3X_{-4}$

$\therefore X^2 - 3X - 4 = 0$

$(X-4)(X+1) = 0$

$X = 4 \;,\; \boxed{X = -1}$ ✗

نرفض السالب

$\therefore \boxed{X = 4}$

⑤ $\log_3(x-2) - \log_3 4 = 2$

$\longrightarrow \log_3\left(\frac{x-2}{4}\right) = 2$.

$$3^2 = \frac{x-2}{4}$$

$$9 = \frac{x-2}{4}$$

$$9(4) = x-2$$

$$\begin{array}{ccc} 36 & = & x-2 \\ +2 & & +2 \end{array} \longrightarrow \boxed{x=38}$$

⑥ $\log(x+7) = 2$

$\longrightarrow 10^2 = x+7$

$$\begin{array}{ccc} 100 & = & x+7 \\ -7 & & -7 \end{array}$$

$$\therefore \boxed{x=93}$$

⑦ $\ln(x-3)=4$

$$\longrightarrow e^4 = x-3$$

$$+3 \qquad +3$$

$$\therefore\ x = 3+e^4$$

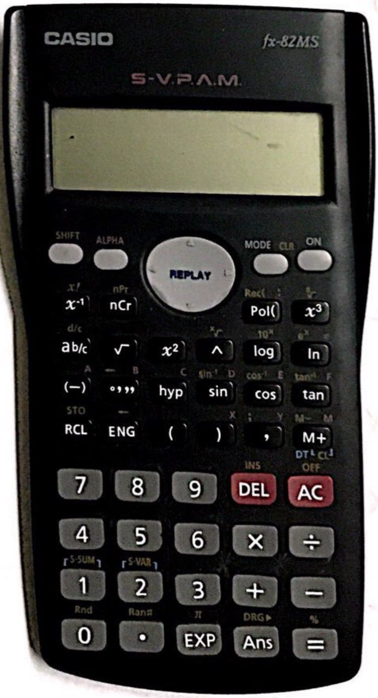
$$= 3+54.59$$

$$= 57.59$$

نجد e^4 من الآلة الحاسبة

$\boxed{\text{Shift}}\ \boxed{\ln}\ \boxed{4}\ \boxed{=}$

54.59



-Ex: Suppose the demand function is given by $p = 100\, e^{-x/10}$.

a) What is the total revenue.

$$\longrightarrow R(x) = p\, x \qquad ; p: \text{selling price} \equiv \text{demand}.$$

$$= \left(100 e^{-\frac{x}{10}}\right) x$$

$$= 100x\, e^{-\frac{x}{10}}$$

b) What would be the total revenue if 30 units were demanded and supplied?

$$\longrightarrow R(30) = 100\,(30)\, e^{-\frac{30}{10}}$$

$$= 3000\, e^{-3} \qquad \boxed{3000}\ \boxed{\text{shift}}\ \boxed{\ln}\ \boxed{-3}\ \boxed{=}$$

$$= 14.93$$

-Ex: Suppose the total cost function for a product is $C(x) = x^2 + \log(x+20) + 1000$.

a) Find the fixed cost.

$$\longrightarrow C(0) = 0^2 + \log(0+20) + 1000$$

$$= \log(20) + 1000 \qquad \boxed{\log}\ \boxed{20}\ \boxed{+}\ \boxed{1000}$$

$$= 1001.3$$

b) Find the total cost of producing 100 units.

$$\longrightarrow C(100) = 100^2 + \log(102) + 1000$$

$$= 11002 \$$$

• Sec 6.1: Simple Interest. الفائدة البسيطة

If a sum of money P (called principal) is invested for a time period t (frquently in years) at an interest rate r per period, the simple interest is given by the following formula.

→ The simple interest $I = Prt$; P = principal (present value)
r = annual interest rate
t = time (in years).

→ The future value
$$S = P + I$$
$$= P + Prt$$
$$= P(1 + rt).$$

في الفائدة البسيطة دائماً تكون على المبلغ الأصلي.

المبلغ الذي تم استثماره: P
نسبة الفائدة السنوية: r
عدد سنوات الاستثمار: t
قيمة الفائدة بعد t من السنوات: I
المبلغ كامل بعد t من السنوات: S

-Ex:

If \$8000 is invested for 2 years at an annual simple interest rate of 9%, how much interest will be received at the end of the 2 years?

$\longrightarrow$ P = \$8000 , t = 2 years , simple interest,

$r = 9\% = \frac{9}{100} = 0.09$.

$\longrightarrow$ Interest «I» $= Prt$

$= 8000(0.09)(2)$

$= \$1440$

-Ex:

If \$4000 is borrowed for 39 weeks at an annual simple interest rate of 15%, how much interest is due at the end of the 39 weeks?

$\longrightarrow$ P = \$4000, t = 39 weeks , $r = 15\% = \frac{15}{100} = 0.15$

$39 \times 7 =$ نجد t بالسنوات :-

1 year $\longrightarrow$ 365

t $\longrightarrow$ 273

$\therefore t = 0.7479... \approx 0.75$ years

$$\therefore\ I = Prt$$
$$= 4000(0.15)(0.75)$$
$$= \$450$$

-Ex:

a) If \$2000 is borrowed for one half year at a simple interest rate of 12% per year, what is the future value of the loan at the end of the half-year?

$\longrightarrow$ $P = \$2000$, $t = 0.5$ year , $r = 12\% = 0.12$

$$S = P + I \qquad ; \quad I = Prt$$
$$= 2000 + 120 \qquad\quad = 2000(0.12)(0.5)$$
$$= \$2120 \qquad\quad = \$120$$

b) An investor wants to have \$20000 in 9 months. If the available interest rate is 6.05% per year, how much must be invested now to yield the desired amount?

$\longrightarrow$ $S = \$20\,000$, $t = 9$ month , $r = 6.05\%$
$= 0.0605$

1 year $\longrightarrow$ 12 months: نحول t الى سنوات
$t \longrightarrow 9$

$\therefore\ t = \frac{12}{9} = 0.75$ years

$$S = P + I$$

$$20000 = P + Prt$$

$$20000 = P + P(0.0605)(0.75)$$

$$20000 = P(1 + 0.045375)$$

$$\frac{20000}{1.045375} = \frac{P(1.045375)}{1.045375} \rightarrow P = 19131.89\$$$

-Ex:

If \$1000 is invested at 5.8% simple interest, how long will it take to grow to \$1100?

$\longrightarrow P = \$1000$, $r = 5.8\% = \frac{5.8}{100} = 0.058$ «simple»

$S = \$1100$, $t = ??$

$\therefore S = P + I$; $I = Prt$

$= 1000(0.058)t$

$= 58t$

$1100 = 1000 + 58t$

$-1000 \quad -1000$

$\frac{100}{58} = \frac{58t}{58}$ $\therefore t = 1.72$ years.

-Ex:

What is the present value of an investment at 6% annual simple interest if it is worth \$832 in 8 months?

$\longrightarrow P = ??$, $r = 6\% = 0.06$ «simple»,

$S = \$832$, $t = 8$ months $= \frac{8}{12} = 0.667$ year.

$\therefore S = P + I$; $I = Prt = P(0.06)(0.667)$

$= 0.04P$

$832 = P + 0.04P$

$832 = P(1 + 0.04)$

$$\frac{832}{1.04} = \frac{1.04}{1.04}P \longrightarrow P = \$800$$

- Sec 6.2: Compound Interest. الربح المركب

In the previous section we discussed simple interest. A second method of paying interest is the compound interest where the interest for each period is added to the principal befor interest is calculated for the next period. With compound, both the interest added and the principal earn interest for the next period.

في الربح البسيط تكون قيمة الفائدة على المبلغ الأصلي «P»، أما في الربح المركب تكون قيمة الفائدة على المبلغ الذي تم تحصيله.

* If \$P is invested for t years at a nominal interest rate r, compounded m times per year, then the future value «S» $= P\left(1+\frac{r}{m}\right)^{mt}$;

P: present value.
r: compound interest.
t: # of year's.
m: # of

→ compounded annually:

$m = 1$

$\therefore S = P(1+r)^t$

→ Compounded semiannually:

$m = 2$

$\therefore S = P\left(1+\frac{r}{2}\right)^{2t}$

→ Compounded quarterly:

$m=4$

$\therefore S = P(1+\frac{r}{4})^{4t}$

→ Compounded monthly:

$m=12$

$\therefore S = P(1+\frac{r}{12})^{12t}$

→ Compounded continuously:-

$m \longrightarrow \infty$

$\therefore S = Pe^{rt}$

* In the compound interest, the interest value.

$I = S - P$; S: futur value.

P: present value.

-Ex:

If \$1500 is invested at an annual rate of interest of 4%. What is the amount after 7 years if the compound take place compounding:-

a) Annually.

$\longrightarrow P = \$1500, \ r = 4\% = 0.04, \ t = 7$ years,

$m = 1$

$$S = P(1+r)^{t}$$
$$= 1500(1+0.04)^{7}$$
$$= 1500(1.04)^{7}$$
$$= 1973.89 \ \$$$

b) Semiannually.

$m = 2.$

$$S = P\left(1+\frac{r}{2}\right)^{2t}$$
$$= 1500\left(1+\frac{0.04}{2}\right)^{2(7)}$$
$$= 1500(1.02)^{14}$$
$$= 1979.22 \ \$$$

c) Quarterly.

$m=4.$

$$\therefore S = P\left(1+\frac{r}{4}\right)^{4t}$$

$$=1500\left(1+\frac{0.04}{4}\right)^{4(7)}$$

$$=1500\,(1.01)^{28}$$

$$=1981.94\ \$$$

d) Continuously.

$$S = Pe^{rt}$$

$$=1500e^{0.04(7)}$$

$$=1500e^{0.28}$$

$$=1984.69\ \$$$

-Ex:

How much should be deposited (present value) in an account paying 3.5% compounded semiannually in order to have an amount of \$1800 in 4 years.

$\rightarrow P = ??$, $r = 3.5\% = 0.035$, $m = 2$, $S = \$1800$

$t = 4$ years.

$$S = P\left(1 + \frac{r}{2}\right)^{2t}$$

$$1800 = P\left(1 + \frac{0.035}{2}\right)^{2(4)}$$

$$1800 = P(1.0175)^{8}$$

$$\frac{1800}{1.15} = \frac{P(1.15)}{1.15}$$

$\therefore P = 1565.22$ \$

- Ex:

How long would it take an investment to double (يتضاعف) if it is invested at:-

a) 4.8% simple interest

$t = ??$, $S = 2P$, $P = ??$, $r = 4.8\% = 0.048$

$$\longrightarrow S = P(1 +$$

$$= P(- + rt)$$

$$\frac{2P}{P} = \frac{P}{P}(1 + 0.048t)$$

$$2 = 1 + 0.048t$$
$$-1 \quad -1$$

$$\frac{1}{0.048} = \frac{0.048t}{0.048} \longrightarrow t = 20.83 \text{ years.}$$

b) 4.8% compounded annually?

$$S = P(1+r)^t$$

$$2P = P(1+0.048)^t$$

$$\frac{2P}{P} = \frac{P}{P}(1.048)^t$$

$$2 = (1.048)^t$$

$$\ln 2 = \ln 1.048^t$$

$$\frac{\ln 2}{\ln 1.048} = t \frac{\ln 1.048}{\ln 1.048}$$

$$\therefore t = \frac{\ln 2}{\ln 1.048} = 14.78 \text{ years.}$$

c) 4.8% compounded quarterly?

$$S = P\left(1+\frac{r}{4}\right)^{4t}$$

$$2P = P\left(1+\frac{0.048}{4}\right)^{4t}$$

$$\frac{2P}{P} = \frac{P}{P}(1.012)^{4t}$$

$$2 = (1.012)^{4t}$$

$$\ln 2 = \ln 1.012^{4t}$$

$$\ln 2 = 4t \ln 1.012$$

$$\frac{\ln 2}{0.048} = \frac{0.048}{0.048} t$$

$\therefore t = 14.44$ years

d) 4.8% compounded continuously.

$$S = Pe^{rt}$$

$$\frac{2P}{P} = \frac{P}{P} e^{0.048t}$$

$$2 = e^{0.048t}$$

$$\ln 2 = \ln e^{0.048t}$$

$$\ln 2 = 0.048t \underset{1}{\cancel{\ln e}}$$

$$\frac{\ln 2}{0.048} = \frac{0.048}{0.048}t$$

$\therefore t = 14.44$ years.

-Ex: **How long** would \$700 have to be invested at 11.9%, compounded continuously, to earn \$300 interest?

$t = ??$, $P = 700$, $I = 300$, $r = 11.9\% = 0.119$

$\therefore S = P + I = 1000$

$$\longrightarrow S = Pe^{rt}$$

$$\frac{1000}{700} = \frac{700}{700} e^{0.119t}$$

$$1.43 = e^{0.119t}$$

$$\ln 1.43 = \ln e^{0.119t}$$

$$\frac{\ln 1.43}{0.119} = \frac{0.119t}{0.119} \underset{1}{\cancel{\ln e}}$$

$\therefore t = 3$ years

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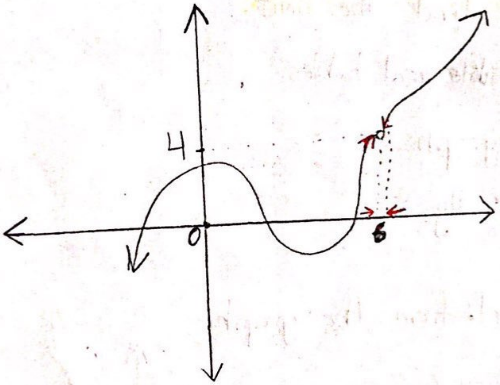
*Ch9: Derivatives:- المشتقات

• Sec 9.1 : Limits النهايات

- Concept of limits: مفهوم النهاية

We have used the notation $f(c)$ to indicate the value of a function at $x = c$. If we need to discuss a value that $f(x)$ approaches as x approaches c, we use the idea of a limit.



-Def : let $f(x)$ be a function defined on an open interval containing c, except perhaps at c. Then:-

$$\lim_{x \to c} f(x) = L$$

is read " the limit of $f(x)$ as x approaches c equals L".

The number L exists if we can make values of $f(x)$ as

الممسوحة ضوئيا بـ CamScanner

close to L as we desire by choosing values of X sufficiently close to c. When the values of f(X) do not approach a single finite value L as X approaches c, we say the limit does not exist.

⟶ A limit as $X \to c$ can exist only if the function approaches a single finite value as X approches c from both the left and right of c.

* We can find the limit:

① Using numerical tables.

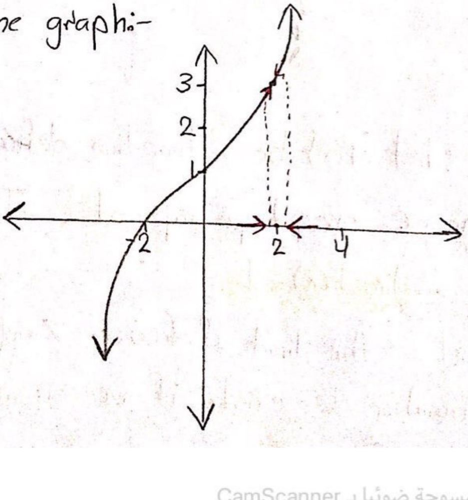
② Using graphs.

③ Algebraically.

-Ex: Find from the graph:-

① $\lim_{X \to 2} f(X) = 3$

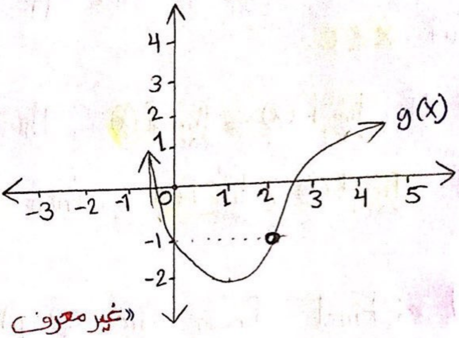
② $f(2) = 3$



- Ex: Find from the graph:-

① $\lim_{x \to 2} g(x)$

$= -1$

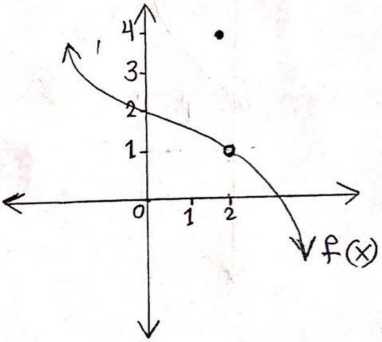


② $f(2) =$ undifined «غير معروف»

- Ex: Find from the graph:

① $\lim_{x \to 2} f(x)$

$= 1$



② $f(2) = 4$

* One Sided limits :

① limit from the right : $\lim_{x \to c^+} f(x) = L$

means the values of $f(x)$ approach the value L as $x \to c$ but $x > c$.

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② limit from left: $\lim_{x \to c^-} f(x) = M$

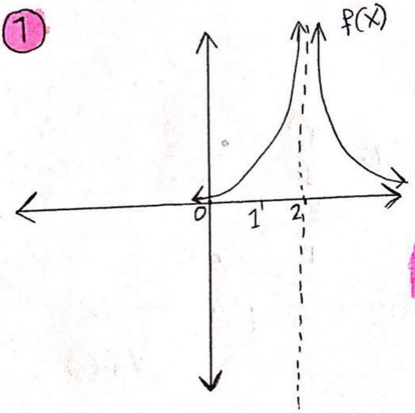
means the values of f(x) approach the value M as $x \to c$ but $x < c$.

③ If $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$, then $\lim_{x \to c} f(x) = \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$.

If $\lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x)$, then $\lim_{x \to c} f(x)$ does not exist.

-Ex: Find from the graph:-

①



a) $\lim_{x \to 2} f(x)$

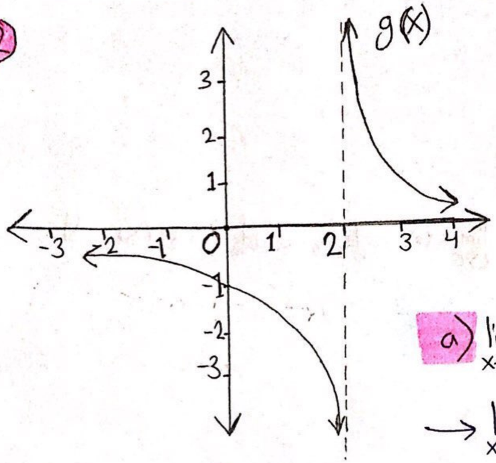
$\longrightarrow \lim_{x \to 2^+} f(x) = \infty$

$\lim_{x \to 2^-} f(x) = \infty$

$\therefore \lim_{x \to 2} f(x) = \infty$

b) $f(2) =$ undefined.

②



a) $\lim_{x \to 2} g(x)$

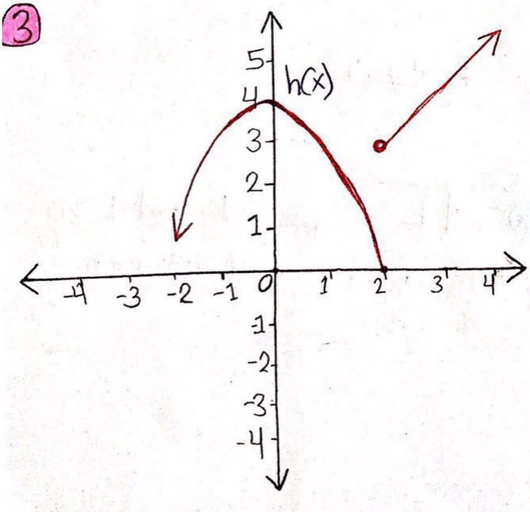
$\rightarrow \lim_{x \to 2^+} g(x) = \infty$

$\lim_{x \to 2^-} g(x) = -\infty$

$\therefore \lim_{x \to 2} g(x) =$ DNE «does not exist»

b) $f(2) =$ undefined.

③



a) $\lim_{x \to 2} h(x)$

$\rightarrow \lim_{x \to 2^+} h(x) = 3$

$\lim_{x \to 2^-} h(x) = 0$

$\therefore \lim_{x \to 2} h(x) =$ DNE

b) $h(2) = 0$

* Properties of limits:

If K is a constant, $\lim_{x \to c} f(x) = L$, and $\lim_{x \to c} g(x) = M$, then the following are true:-

① $\lim_{x \to c} K = K$

② $\lim_{x \to c} x = c$

③ $\lim_{x \to c} \left[f(x) \mp g(x) \right] = L \mp M$

④ $\lim_{x \to c} \left[f(x) \cdot g(x) \right] = L \cdot M$

⑤ $\lim_{x \to c} \left[\frac{f(x)}{g(x)} \right] = \frac{L}{M}$ if $M \neq 0$

⑥ $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)} = \sqrt[n]{L}$ provided that $L > 0$ when n is even.

- Ex: Find the following limits, if they exist :-

a) $\lim_{x\to -1}(x^3 - 2x)$

$= \lim_{x\to -1}(x^3 - 2x) = (-1)^3 - 2(-1) = 1$

b) $\lim_{x\to 4} \frac{x^2 - 4x}{x-2}$

$= \frac{4^2 - 4(4)}{4-2} = \frac{0}{2} = 0$

* To find $\lim_{x\to c} \frac{f(x)}{g(x)}$ «rational functions»:

① Substitution. « put $x = c$ »

② We have $\underline{3}$ cases:-

$\longrightarrow$ If $\frac{f(c)}{g(c)} = L$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = L$

$\longrightarrow$ If $\frac{f(c)}{g(c)} = \frac{L}{0}$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \infty$ or $-\infty$ or DNE

$\longrightarrow$ If $\frac{f(c)}{g(c)} = \frac{0}{0}$, factoring.

- Ex: Evaluate the following limits, if they exist:-

① $\lim\limits_{x\to 2} \frac{x^2-4}{x-2}$ «$\frac{0}{0}$» factoring.

$$= \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = \lim_{x\to 2}(x+2) = 4$$

② $\lim\limits_{x\to 1} \frac{x^2-3x+2}{x^2-1}$ «$\frac{0}{0}$» factoring.

$$= \lim_{x\to 1} \frac{(x-2)\cancel{(x-1)}}{(x+1)\cancel{(x-1)}} = \lim_{x\to 1} \frac{x-2}{x+1} = \frac{-1}{2}.$$

③ $\lim\limits_{x\to 2} \frac{1}{x-2}$ «$\frac{1}{0}$»

$$\longrightarrow \lim_{x\to 2^+} \frac{1}{x-2} = \infty$$

$$\lim_{x\to 2^-} \frac{1}{x-2} = -\infty$$

$$\therefore \lim \frac{1}{x-2} = \text{D-NE}$$

الممسوحة ضوئيا بـ CamScanner

* limits of a piecewise defined function :-

To see how we evaluate a limit involving a piecewise defined function, consider the following example.

- Ex: let $f(x) = \begin{cases} x^2+1 & ; x \leqslant 1 \\ x+2 & ; x > 1 \end{cases}$

Find $\lim_{x \to 1} f(x)$.

$\longrightarrow \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x+2) = 3$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2+1) = 2$

$\therefore \lim_{x \to 1} f(x) = \text{DNE}.$ « since $\lim_{x \to 1^+} f(x) \neq \lim_{x \to 1^-} f(x)$ »

- Ex: let $f(x) = \begin{cases} 4 - x^2 & ; x \leqslant -2 \\ x^2+2x & ; x > -2 \end{cases}$; Find $\lim_{x \to -2} f(x)$.

$\longrightarrow \lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (x^2+2x) = 0$

$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} (4 - x^2) = 0$

$\therefore \lim_{x \to -2} f(x) = 0.$ « since $\lim_{x \to -2^+} f(x) = \lim_{x \to -2^-} f(x) = 0$ »

– Sec 9.2: Continuous Functions, limits at Infinity.

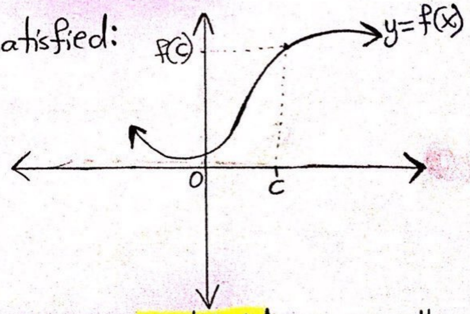
اتصال الاقترانات، النهايات عند ∞.

• Continuous Functions: الاقترانات المتصلة

The graphs of such functions can be drawn without lifting the pencil from the paper, and graphs of others may have holes, vertical asymptotes, or jumps that make it impossible to draw them without lifting the pencil.

• Def: Continuity at a point:

The function f is continuous at $x=c$ if all of the following conditions are satisfied:



① $f(c)$ defined.

② $\lim\limits_{x\to c} f(x)$ exists.

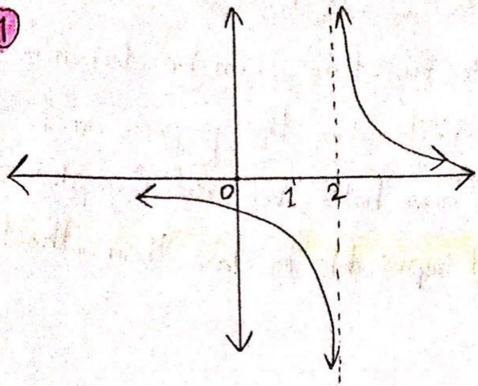
③ $\lim\limits_{x\to c} f(x) = f(c)$

If one or more of the conditions don't hold, we say the function is discontinuous at $x=c$.

غير متصل

*The below figures shows graphs of some functions - that are discontinuous at $x=2$.

①

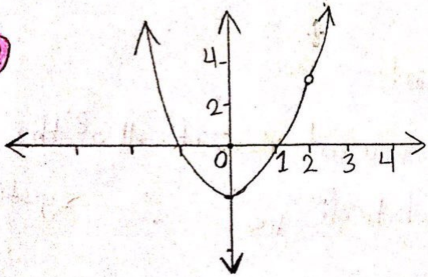


$f(x)$ isn't continuous at $x=2$ since:

$f(2)$ is undefined.

$\lim_{x\to 2} f(x)$ DNE

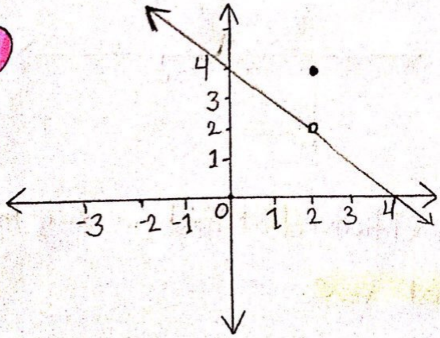
②



$f(x)$ isn't continuous at $x=2$, since:

$f(2)$ is undefined.

③



$f(x)$ is discontinuous at $x=2$ since:

$f(2)=4$

$\lim_{x\to 2} f(x)=2$

$\rightarrow \lim_{x\to 2} f(x) \neq f(2)$

* Polynomial and Rational functions:-

- Every polynomial function is continuous for all weal numbers.
- Every rational function is continuous at all values of x except those that make the denomenator 0.

-Ex: For what values of x, if any, are the following functions continuous?

a) $h(x) = \frac{3x+2}{4x-6}$.

$$\longrightarrow 4x - 6 = 0$$
$$\quad +6 \quad +6$$

$$\frac{4x}{4} = \frac{6}{4}$$

$$\therefore x = \frac{3}{2}$$

h(x) is continuous at all values of x except at $x = \frac{3}{2}$

b) $f(x) = \frac{x^2 - x - 2}{x^2 - 4}$.

$$\longrightarrow x^2 - 4 = 0$$
$$\quad +4 \quad +4$$

$$x^2 = 4$$

$$\therefore x = 2, -2$$

$f(x)$ is continuous everywhere except at $x=2,-2$.

c) $g(x)= x^3-3x+1$.

→ $g(x)$ is polynomial, so $g(x)$ is continuous for all real numbers.

*** Piecewise defined functions:-** الاقتران متعدد القاعدة

If the pieces of a piecewise defined function are polynomials, the only values of x where the function might be discontinuous are those at which the defintion of the function changes.

من الممكن أن يكون غير متصل عند تفرعات أي عند النقاط التي يغيّر الاقتران من تعريفه عندها.

-Ex: Determine the values of x, for which the following functions are discontinuous:-

a) $$g(x)=\begin{cases}(x+2)^3+1 & ; x\leq -1\\ 3 & ; x>-1\end{cases}$$

at $x=-1$:

$\lim_{x\to -1^+} g(x) = 3$

$\lim_{x\to -1^-} g(x) = 2$

$\therefore \lim_{x\to -1} g(x) = DNE$

so $g(x)$ is discontinuous at $x=-1$.

b) $f(x)=\begin{cases} 4-x^2 & ; x<2 \\ x-2 & ; x\geqslant 2 \end{cases}$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(4-x^2) = 0$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(x-2) = 0$$

$$\therefore \lim_{x\to 2} f(x) = 0$$

$\rightarrow f(x)$ is continuous for all real numbers.

c) $f(x)=\begin{cases} x^2+4 & ; x\neq 4 \\ 8 & ; x=4 \end{cases}$

$$\lim_{x\to 4} f(x) = \lim_{x\to 4}(x^2+4) = 12$$

هنا لا نبحث lim من
لليمين واليسار، نجد lim
مرة واحدة لأنه لا يتغير
التعريف على يمينها ويسارها (> و <)

$$f(4) = 8$$

$$\longrightarrow \lim_{x\to 4} f(x) \neq f(4)$$

$\therefore f(x)$ isn't continuous at $x=4$.

*Limits at Infinity:-

النهايات عند ∞

If c is any constant, then:

① $\lim_{x \to \infty} c = c$ and $\lim_{x \to -\infty} c = c$

② $\lim_{x \to \infty} \frac{c}{x^p} = 0$, where $p > 0$.

③ $\lim_{x \to -\infty} \frac{c}{x^n} = 0$; n is positive integer.

-Ex: Find each of the following limits:-

① $\lim_{x \to \infty} \frac{3}{x} = 0$

② $\lim_{x \to -\infty} \frac{4}{x^3} = 0$

③ $\lim_{x \to \infty} x^2 = \infty$

-Note:

To find $\lim_{x \to \mp\infty} \frac{f(x)}{g(x)}$:

① If the degree of f(x) is less than the degree of g(x), then $\lim_{x \to \mp\infty} \frac{f(x)}{g(x)} = 0$.

② If the degree of f(x) equals the degree of g(x), then $\lim_{x\to\mp\infty}\frac{f(x)}{g(x)} = \frac{\text{coefficient of the greatest degree of f}}{\text{coefficient of the greatest degree of g}}$

③ If the degree of f(x) is greater than the degree of g(x) then $\lim_{x\to\mp\infty}\frac{f(x)}{g(x)}$ either ∞ or −∞.

Ex: Find each of the following limit:-

a) $\lim_{x\to\infty}\frac{2x-1}{x+2}$ «degree f= degree g»

$=\frac{2}{1}$

b) $\lim_{x\to\mp\infty}\frac{x}{x^3-2x+1} =$ «degree f < degree g»

$=0$

c) $\lim_{x\to\mp\infty}\frac{2x-3x^2+1}{4+5x^2+x}$ «degree f = degree g»

$=\frac{-3}{5}$.

d) $\lim_{x\to\mp\infty}\frac{x^2+3}{1-x}$ «degree f > degree g»

$=\lim_{x\to\mp\infty}\frac{x^2}{-x} = \lim_{x\to-\infty} - x = \infty$

* Horizontal Asymptotes :-

If $\lim\limits_{x \to \mp\infty} f(x) = b$, then $y = b$ is a horizontal asymptote.

-Ex.: Find the horizontal asymptotes :-

a) $f(x) = \dfrac{x^2 - 4}{2x^2 - 7}$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^2 - 4}{2x^2 - 7} = \frac{1}{2}$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x^2 - 4}{2x^2 - 7} = \frac{1}{2}$$

$\therefore y = \frac{1}{2}$ is a H. A .

b) $f(x) = \dfrac{x - 4}{1 - x^2}$

$$\lim_{x \to \mp\infty} f(x) = \lim_{x \to \infty} \frac{x - 4}{1 - x^2} = 0$$

$\therefore y = 0$ is a H.A

c) $f(x) = \dfrac{1 - 3x^2}{x - 4}$

$$\lim_{x \to \infty} \frac{1 - 3x^2}{x - 4} = \lim_{x \to \infty} \frac{-3x^2}{x} = \lim_{x \to \infty} -3x = -\infty$$

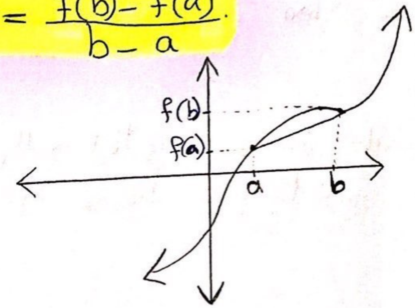
$\therefore$ No H.A

- Sec 9.3: Rates of change and derivatives :-

• Average Rate of Change: متوسط التغير

The average rate of change of a function $y=f(x)$ from $x=a$ to $x=b$ is defined by:

$$\text{Average rate of change} = \frac{f(b)-f(a)}{b-a}.$$



-Ex: Suppose a company's total cost in dollars to produce x units of its product is given by:

$$C(x) = 0.01x^2 + 25x + 1500$$

Find the average rate of change of total cost from $x=100$ to $x=200$.

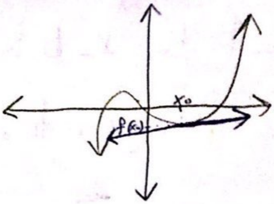
$$\longrightarrow \text{Average rate of change} = \frac{C(200)-C(100)}{200-100}$$

$$C(200) = 0.01(200)^2 + 25(200) + 1500 = 6900.$$
$$C(100) = 0.01(100)^2 + 25(100) + 1500 = 4100.$$

∴ Average rate of change $= \frac{2800}{100} = 28$ $

* Instataneous Rates of change «Velocity, derivative, or slope of the tangent»

$$V' = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$



سنأخذ قواعد الاشتقاق التي تم التوصل إليها من خلال هذه العلاقة
جاهزة في 6 sec 9.4 ودون التطرق لكيفية استخدام القانون.

- Sec 9.4: Derivative Formulas قواعد الاشتقاق

• The derivative (المشتقة) of a function can be used to find the rate of change of the function. We will discuss formulas that will make it easier to find certain derivative.
«$f'(x)$, $\frac{dy}{dx}$»

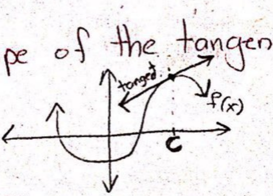
① Poweres of x rule:
If $f(x) = x^n$; n is a real number, then $f'(x) = n x^{n-1}$

② Constant function rule:
If $f(x) = c$, where c is a constant, then $f'(x) = 0$.

③ Coefficient rule:
If $f(x) = c.U(x)$, where c is a constant and $U(x)$ is a differentiable function of x, then $f'(x) = c.U'(x)$.

④ Sum and difference rule:-
If $f(x) = U(x) \mp V(x)$, where U and V are differentiable functions of x, then $f'(x) = U'(x) \mp V'(x)$

-Note: «$f'(x)$, $\frac{dy}{dx}$»
The derivative at $c = f'(c) =$ «The slope of the tangent at $x = c$» = The rate of change.



- Ex: Find the derivatives of the following functions:-

a) $y = x^{14}$

$\rightarrow \frac{dy}{dx} = 14x^{13}$

b) $f(x) = x^{-2}$

$\rightarrow f'(x) = -2x^{-3} = \frac{-2}{x^3}$

c) $y = x$

$\rightarrow \frac{dy}{dx} = 1$

d) $g(x) = x^{1/3}$

$\rightarrow g'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$

e) $u(s) = s^8$

$\rightarrow u'(s) = 8s^7$

f) $p = q^{2/3}$

$\rightarrow \frac{dp}{dq} = \frac{2}{3}q^{-\frac{1}{3}} = \frac{2}{3q^{\frac{1}{3}}} = \frac{2}{3\sqrt[3]{q}}$

g) $C(t) = \sqrt{t}$

$C(t) = t^{\frac{1}{2}}$

$\longrightarrow C'(t) = \frac{1}{2} t^{-\frac{1}{2}} = \frac{1}{2t^{\frac{1}{2}}} = \frac{1}{2\sqrt{t}}$

h) $S = \frac{1}{\sqrt{t}}$

$S = t^{\frac{-1}{2}}$

$\longrightarrow S' = \frac{-1}{2} t^{-\frac{3}{2}} = \frac{-1}{2\ ^{\frac{3}{2}}} = \frac{-1}{2\sqrt{t^3}}$

i) $y = 4$

$\longrightarrow \frac{dy}{dx} = 0$

J) $f(x) = 4x^5$

$\longrightarrow f'(x) = 4.5X^4$

$= 20X^4$

k) $g(t) = \frac{1}{2} t^2$

$\longrightarrow g'(t) = \frac{1}{2} \cdot 2t$

$= t$

l) $p = \frac{5}{\sqrt{q}}$

$p = 5\,q^{\frac{-1}{2}}$

$\longrightarrow p' = 5 \cdot \frac{-1}{2} q^{\frac{-3}{2}}$

$= \frac{-5}{2q^{\frac{3}{2}}} = \frac{-5}{2\sqrt{q^3}}$

m) $y = 3x + 5$

$\longrightarrow \frac{dy}{dx} = 3$

n) $p = \frac{1}{3}q^3 + 2q^2 - 3$

$\longrightarrow \frac{dp}{dq} = q^2 + 4q$

o) $f(x) = 4x^3 - 2x^2 + 5x - 3$

$\longrightarrow f'(x) = 12x^2 - 4x + 5$

p) $u(x) = 5x^4 + x^{1/3}$

$\longrightarrow u'(x) = 20x^3 + \frac{1}{3}x^{\frac{-2}{3}}$

$= 20x^3 + \frac{1}{3x^{\frac{2}{3}}}$

9) $y = 4x^3 + \sqrt{x}$

$\rightarrow \frac{dy}{dx} = 12x^2 + \frac{1}{2}x^{\frac{-1}{2}}$

$= 12x^2 + \frac{1}{2\sqrt{x}}$

١٠) $S = 5t^6 - \frac{1}{t^2}$

$S = 5t^6 - t^{-2}$

$\longrightarrow S' = 30t^5 - -2t^{-3}$

$= 30t^5 + \frac{2}{t^3}$

*Note:

ميل المماس

① slope of the tangent line at $x=a$ is $f'(a)$.

معادلة المماس

② The equation of the tangent line at $x=a$ is:

$y - y_0 = m(x - x_0)$; $(x_0, y_0) = (a, f(a))$

m: slope $= f'(a)$.

المماس أفقي او موازي لمحور السينات

③ The tangent line is horizontal « parallel to the x-axis» if $f'(a) = 0$.

- Ex: Write the equation of the tangent line to each curve at the indicated point:-

$y = x^3 - 5x^2 + 7$ at $x = 1$.

$\longrightarrow \quad y - y_0 = m(x - x_0) \quad ; \quad (x_0, y_0) = (a, f(a))$

$= (1, f(1))$

$m: f'(1)$

صورة العدد في الاقتران الأصلي

$f(1) = 1^3 - 5(1)^2 + 7 = 3$

$f'(x) = 3x^2 - 10x$

$\longrightarrow f'(1) = 3(1)^2 - 10(1) = -7$

$\therefore$ the equation of the tangent line at $x = 1$ is:

$y - 3 = -7(x - 1)$

$\underset{+3}{y - 3} = \underset{+3}{-7x + 7} \longrightarrow y = -7x + 10$

- Ex: Find all points on the graph of:

$f(x) = x^3 + 3x^2 - 45x + 4$

where the tangent line is horizontal.

يكون مماس أفقي عندما $f'(x) = 0$

$\longrightarrow f'(x) = 0$

$3x^2 + 6x - 45 = 0$

نحل هذه المعادلة بالتحليل أو القانون العام.

$\frac{3(x^2 + 2x - 15)}{3} = \frac{0}{3}$

$$X^2+2X-15=0$$

$$\longrightarrow (X+5)(X-3)=0$$

$$\therefore\ X=-5,3$$

So the points on the graph where the tangent is horizontal are:

$$(-5, f(-5)) = (-5, 179)$$

$$(3, f(3)) = (3, -77).$$

* Suppose that the total revenue function for a product is given by $R(x)$. Then the marginal revenue at X units is $\overline{MR} = R'(x)$.

-Ex: Suppose that an oil company's revenue is given by the equation $R(x) = 100x - x^2$ where X is the number of thousands of barrels of oil sold each day.

a) Find the function that gives the marginal revenue at any value of X.

$$\overline{MR} = R'(x)$$

$$= 100 - 2X$$

b) Find the marginal revenue when 20000 barrels are sold. (that is, at $X = 20$) لأن X تُمثل عدد الوحدات بالألوف

$$\rightarrow R'(20) = 100 - 2(20)$$
$$= 60$$

* The marginal revenue is used to approximate the revenue from the sale of 1 additional unit, that is,

$R'(x)$: the expected revenue from the sale of the next unit will be approximately $R'(x)$.

*The (exact) actual revenue from the sale of the next unit is

$R(x+1) - R(x)$.

$$* \ \underset{\text{exact}}{R(x+1) - R(x)} \cong \underset{\text{approximated}}{R'(x)}$$

-Ex: Suppose that a manufacturer of a product knows that because of the demand for this product, his revenue is given by: $R(x) = 1500x - 0.02x^2$; $0 \leqslant x \leqslant 10$
where x is the number of units sold and $R(x)$ is in dollars.

a) Find the marginal revenue at $x = 500$.

$$\rightarrow \overline{MR} = R'(x) = 1500 - 0.04x$$
$$R'(500) = 1500 - 0.04(500)$$
$$= 1480 \ \$$$

b) Find the change in revenue caused by the increase in sales from 500 to 501.

exact chang.
التغير الفعلي في R(x)

$$\longrightarrow R(501) - R(500)$$

$$= 746479.98 - 745000$$

$$= 1479.98$$

↳ Notice that the marginal revenue at $X = 500$ is a good estimate of the revenue from the 501st unit.

الممسوحة ضوئيا بـ CamScanner

- Sec 9.5: The Product Rule and the Quotien Rule
قاعدة الضرب والقسمة

*** The product rule:** قاعدة الضرب

If $f(x) = U(x) \cdot V(x)$, where U and V are differentiable functions of x, then:

$$f'(x) = U(x)V' + V(x) \cdot U'(x)$$

*** The quotient rule:** قاعدة القسمة

If $f(x) = \frac{U(x)}{V(x)}$, where U and V are differentiable ف

then $f'(x) = \frac{V(x)U' - U(x)V'(x)}{(V(x))^2}$.

- Ex: ① Find the derivative of the following function:

$$y = (2x^3+3x+1)(x^2+4).$$

$$\longrightarrow \frac{dy}{dx} = (2x^3+3x+1)(2x)+(x^2+4)(6x^2+3).$$

$$= \underline{4x^3} + \underline{6x^2} + \underline{\underline{2x}} + \underline{6x^4} + \underline{\underline{3x^2}} + \underline{\underline{24x^2}} + \underline{\underline{12}}$$

$$= 10x^4 + 33x^2 + 2x + 12$$

② Find the slope of the tangent to the graph of

$y = f(x) = (4x^3+5x^2-6x+5)(x^3-4x^2+1)$ at $x=1$

$\longrightarrow$ slope of the tangent at $x=1$ is $f'(1)$.

$$f'(x)=(4x^3+5x^2-6x+5)(3x^2-8x)+(x^3-4x^2+1)(12x^2+10x-6)$$

$$f'(1)=(4+5-6+5)(3-8)+(1-4+1)(12+10-6)$$

$$=(8)(-5)+(-2)(16)=-72.$$

$\therefore$ slope of the tangent at $1=-72$.

③ If $f(x)=\dfrac{x^2-4x}{x+5}$, find $f'(x)$.

$$\longrightarrow f'(x)=\frac{(x+5)(2x-4)-(x^2-4x)(1)}{(x+5)^2}$$

$$=\frac{2x^2-4x+10x-20-x^2+4x}{(x+5)^2}$$

$$=\frac{x^2+10x-20}{(x+5)^2}$$

④ If $f(x)=\dfrac{x^3-3x^2+2}{x^2-4}$, find the instantaneous rate of change of $f(x)$ at $x=3$.

$$\longrightarrow f'(x)=\frac{(x^2-4)(3x^2-6x)-(x^3-3x^2+2)(2x)}{(x^2-4)^2}$$

$$f'(3)=\frac{(9-4)(27-18)-(27-27+2)(6)}{(9-4)^2}=\frac{33}{25}$$

$$=1.32$$

-Ex: Find the derivative of $y = \frac{1}{x^3}$.

→ we can find $\frac{dy}{dx}$ by the Quotient rule of by the power rule:

- Quotient rule: $\frac{dy}{dx} = \frac{x^3(0) - 1(3x^2)}{(x^3)^2} = \frac{-3x^2}{x^6} = \frac{-3}{x^4}$

- Power rule: $\frac{dy}{dx} = -3x^{-3-1} = -3x^{-4} = \frac{-3}{x^4}$

-Ex: Suppose the revenue function for a flash derive is given by: $R(x) = 10x + \frac{100x}{3x+5}$. Find the margin revenue when $x = 15$.

$$\longrightarrow \overline{MR} = R'(15)$$

$$R'(x) = 10 + \frac{(3x+5)(100) - (100x)(3)}{(3x+5)^2}$$

$$= 10 + \frac{3\cancel{0}0x + 500 - 3\cancel{0}0}{(3x+5)^2}$$

$$= 10 + \frac{500}{(3x+5)^2}$$

$$\therefore \overline{MR} = R'(15) = 10 + \frac{500}{(45+5)^2} = 10.2 \text{ dollars.}$$

الممسوحة ضوئيا بـ CamScanner

- Sec 9.6: The chain rule and the power rule.

قاعدة السلسلة والقوى

- If f and g are functions, then the composite function «denoted by $f \circ g$» is defined as:

$$(f \circ g)(x) = f(g(x))$$

- Ex: If $f(x) = 3x^2$, $g(x) = 2x - 1$. Find $(f \circ g)(x)$.

$$\longrightarrow (f \circ g)(x) = f(g(x))$$
$$= f(2x-1)$$
$$= 3(2x-1)^2$$

- Chain rule:

If f and g are differentiable functions then:

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

- Ex: If $y = \sqrt{x^2 - 1}$, find $\frac{dy}{dx}$.

$$\longrightarrow y = (x^2 - 1)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(x^2 - 1)^{-\frac{1}{2}}(2x)$$

$$= (x^2 - 1)^{-\frac{1}{2}} \cdot x = \frac{x}{\sqrt{x^2 - 1}}$$

- Ex: If $y=(x^2-4x)^6$, find $\frac{dy}{dx}$.

$$\longrightarrow \frac{dy}{dx} = 6\left(x^2-4x\right)^5\left(2x-4\right).$$

- Ex: If $p=\frac{4}{3q^2+1}$, find $\frac{dp}{dq}$

$$\longrightarrow p = 4\left(3q^2+1\right)^{-1}$$

$$\frac{dp}{dq} = 4\cdot(-1)\left(3q^2+1\right)^{-2}(6q)$$

$$= \frac{-24q}{\left(3q^2+1\right)^2}$$

- Ex: Let $h(x) = (f\circ g)(x)$, and $f(-1)=2$, $f'(-1)=1$, $g(-1)=3$, $g'(-1)=2$, $f(3)=5$, $f'(3)=4$. Find $h'(-1)$.

$$\longrightarrow h'(x) = f'(g(x))\cdot g'(x)$$

$$h'(-1) = f'(g(-1))\cdot g'(-1)$$

$$= f'(3)\, g'(-1)$$

$$= 4(2)$$

$$= 8$$

الممسوحة ضوئيا بـ CamScanner

-Ex: Find the derivatives of:

a) $y = 3\sqrt[3]{x^2-3x+1} = 3(x^2-3x+1)^{\frac{1}{3}}$

$$\rightarrow \frac{dy}{dx} = 3\left(\frac{1}{3}\right)(x^2-3x+1)^{-\frac{2}{3}}(2x-3)$$

$$= (2x-3)(x^2-3x+1)^{-\frac{2}{3}}$$

$$= \frac{2x-3}{(x^2-3x+1)^{\frac{2}{3}}}$$

b) $y = \frac{1}{\sqrt{(x^2+1)^3}} = (x^2+1)^{-\frac{3}{2}}$

$$\rightarrow \frac{dy}{dx} = \frac{-3}{2}(x^2+1)^{-\frac{5}{2}}(2x)$$

$$= \frac{-3x}{(x^2+1)^{\frac{5}{2}}} = \frac{-3x}{\sqrt{(x^2+1)^5}}$$

-Ex: The demand for x hundred units of a product is given by $x = 98(2p+1)^{-\frac{1}{2}} - 1$, where p is the price per unit in dollars. Find the rate of change of the demand with respect to price when $p = 24$.

$$\longrightarrow \frac{dx}{dp} = 98(2p+1)^{-\frac{3}{2}}(2)$$

$$= \frac{98}{(3p+1)^{\frac{3}{2}}}$$

at $p = 24\$$:

$$\left.\frac{dx}{dp}\right|_{p=24} = \frac{98}{(2\cdot 24+1)^{\frac{3}{2}}} = \frac{98}{49^{3/2}} = 0.286$$

If the price change by 1$ at $p = 24$, demand will decrease by 28.6 units.

الممسوحة ضوئيا بـ CamScanner

-Sec 9.7: Using Derivative Formulas

استخدام قواعد المشتقات

In this section, we use derivative formulas separately and in combination with each other. دمج أكثر من قاعدة في مسألة واحدة.

-Ex: If $y=\left(\frac{x^2}{x-1}\right)^5$, **find y'.**

$$\longrightarrow y' = 5\left(\frac{x^2}{x-1}\right)^4\left(\frac{(x-1)(2x)-(x^2)(1)}{(x-1)^2}\right)$$

$$= 5\left(\frac{x^2}{x-1}\right)^4\left(\frac{x^2-2x}{(x-1)^2}\right)$$

$$= \frac{5x^8(x^2-2x)}{(x-1)^6}$$

$$= \frac{5x^{10}-10x^9}{(x-1)^6}$$

-Ex: Find $f'(x)$ if $f(x)=\frac{(x-1)^2}{(x^4+3)^3}$.

$$\longrightarrow f'(x) = \frac{(x^4+3)^3 \cdot 2(x-1)(1) - (x-1)^2 \cdot 3(x^4+3)^2(4x^3)}{\left((x^4+3)^3\right)^2}$$

$$= \frac{2(x-1)(x^4+3)^3 - 12x^3(x-1)^2(x^4+3)^2}{(x^4+3)^6}$$

-Ex: Find $f'(x)$ if $f(x)=(x^2-1)\sqrt{3-x^2}$.

$$\longrightarrow f'(x)=(x^2-1)\cdot\frac{1}{2}(3-x^2)^{\frac{-1}{2}}(0-2x)+\sqrt{3-x^2}\,(2x)$$

$$=\frac{1}{\cancel{2}}(x^2-1)(3-x^2)^{\frac{-1}{2}}(-\cancel{2}x)+2x\sqrt{3-x^2}$$

$$=\frac{-x(x^2-1)}{\sqrt{3-x^2}}+\frac{2x\sqrt{3-x^2}\cdot\sqrt{3-x^2}}{1\cdot\sqrt{3-x^2}}$$ توحيد مقامات

$$=\frac{-x^3+x}{\sqrt{3-x^2}}+\frac{2x(3-x^2)}{\sqrt{3-x^2}}$$

$$=\frac{-x^3+x+6x-2x^3}{\sqrt{3-x^2}}$$

$$=\frac{-3x^3+7x}{\sqrt{3-x^2}}$$

* Summary of derivative formulas:

① If $f(x)=c$, then $f'(x)=0$.

② If $f(x)=x^n$; then $f'(x)=n\,x^{n-1}$

③ If $f(x)=cU(x)$; then $f'(x)=c\cdot U'(x)$

④ If $f(x)=U(x)\mp V(x)$; then $f'(x)=U'(x)\mp V'(x)$.

⑤ If $f(x)=U(x)\cdot V(x)$; then $f'(x)=U(x)\cdot V'(x)+V(x)\cdot U'(x)$

⑥ If $f(x)=\frac{U(x)}{V(x)}$; then $f'(x)=\frac{V(x)\cdot U'(x)-U(x)\cdot V'(x)}{V^2(x)}$

⑦ If $y=(f\circ g)(x)=f(g(x))$; then $y'=f'(g(x))\cdot g'(x)$.

-Ex: Find the derivative of the following functions:-

① $f(x) = 14$.

$\rightarrow f'(x) = 0$

② $f(x) = x^4$.

$\rightarrow f'(x) = 4x^3$

③ $f(x) = 5x^3$

$\rightarrow f'(x) = 15x^2$

④ a) $f(x) = 3x^2 + 4x$.

$\rightarrow f'(x) = 6x + 4$.

b) $f(x) = 3x^2 - 7x + 5$

$\rightarrow f'(x) = 6x - 7$

⑤ $y = (x^2 - 2)(x+4)$.

$$\rightarrow \frac{dy}{dx} = (x^2 - 2)(1) + (x+4)(2x)$$

$$= x^2 - 2 + 2x^2 + 8x$$

$$= 3x^2 + 8x - 2$$

⑥ $f(x) = \frac{x^3}{x^2 + 1}$

$$\rightarrow f'(x) = \frac{(x^2+1)(3x^2) - (x^3)(2x)}{(x^2+1)^2}$$

$$= \frac{x^4 + 3x^2}{(x^2+1)^2}$$

⑦ If $y = (x^3 - 4x)^{10}$

$\longrightarrow y' = 10(x^3 - 4x)^9 (3x^2 - 4)$.

- Sec 9.8: Higher- Order Derivatives. المشتقات العليا.

We can take a derivative of the derivative. The derivative of a first derivative is called a second derivative. We can also find third, fourth, and higher derivatives.

$\longrightarrow$ the second derivative: $f''(x)$, $\frac{d^2y}{dx^2}$

$\longrightarrow$ the third derivative: $f'''(x)$, $\frac{d^3y}{dx^3}$

$\longrightarrow$ the fourth derivative: $f^{(4)}(x)$; $\frac{d^4y}{dx^4}$

$\vdots$

- Ex: Find the fourth derivative of $f(x)= 4x^3+5x^2+3$.

$\longrightarrow f'(x)= 12x^2+10x$

$f''(x)= 24x+10$

$f'''(x)= 24$

$f^{(4)}(x)= 0$

- Ex: let $f(x)= 3x^4+6x^3-3x^2+4$.

a) How fast is $f(x)$ changing at $(1,10)$.

$\longrightarrow f'(x)= 12x^3+18x^2-6x$

$f'(1)= 12+18-6 = \boxed{24}$

b) How fast is $f'(x)$ changing at $(1;10)$.

$\longrightarrow f''(x) = 36x^2 + 36x - 6$

$f''(1) = 36 + 36 - 6$

$= 66$

- Ex: If $y = 36x^2 - 6x^3 + x$, what is the rate of change of y' at $(1, 31)$.

$\longrightarrow y' = 72x - 18x^2 + 1$

$y'' = 72 - 36x$

$y''\Big|_{\text{at } x=1} = 72 - 36 = \boxed{36}$

- Ex: The revenue from the sale of x units of a certain product can be described by:-

$R(x) = 100x - 0.01x^2$.

Find the instantaneous rate of change of the marginal revenue.

$\longrightarrow R'(x) = 100 - 0.02x$

$R''(x) = -0.02$

ممكن أن يكون السؤال How fast is the marginal revenue changing when $x = 2$. « $R''(2)$ »

- Sec 9.9: Applications: Marginals and Derivatives.

تطبيقات على المشتقة

* If $R = R(x)$ is the total revenue function for a commodity, the the marginal revenue is $\overline{MR} = R'(x)$.

- Ex: If the demand for a product is given by:

$p = 16 - 0.02X$ where X is the number of units and p is the price per unit. Find the marginal revenue for this product at $X = 40$.

$$\longrightarrow R(X) = p \cdot X$$
$$= (16 - 0.02X)X$$
$$= 16X - 0.02X^2$$

$$\rightarrow \overline{MR} = R'(X) = 16 - 0.04X$$

$$R'(40) = 16 - 0.04(40)$$
$$= \$14.4$$

the approximated «estimated» revenue gained from the sale of the 41st item is \$14.4

ملاحظة: actual «exact» revenue from the sale 41st item is $R(41) - R(40)$

* If $C = C(x)$ is a total cost function for a product then the marginal cost is $\overline{MC} = C'(x)$.

* If $P = P(x)$ is the profit function for a commodity, then the marginal profit function is $\overline{MP} = P'(x)$.

-Ex: If the total profit, in thousands of dollars, for a product is given by $P(x) = 20\sqrt{x+1} - 2x - 22$, what is the marginal profit at a production level of 15 units.

$$\longrightarrow \overline{MP} = P'(x) = 20\left(\frac{1}{2}\right)(x+1)^{\frac{-1}{2}}(1) - 2$$

$$= \frac{10}{\sqrt{x+1}} - 2$$

$$\therefore P'(15) = \frac{10}{\sqrt{16}} - 2 = 0.5 \ \$$$

the profit from the sale of the 16th unit is approximately 0.5

-Ex: Suppose that the total revenue function is given by $R(x) = 46X$ and the total cost function is given by $C(x) = 100 + 30x + \frac{1}{10}x^2$.

a) Find $P(100)$.

$$\rightarrow P(x) = R(x) - C(x)$$
$$= 46X - (100 + 30x + \frac{1}{10}x^2)$$
$$= 16X - 100 - \frac{1}{10}x^2$$

$$\therefore P(100) = 16(100) - 100 - \frac{1}{10}(100)^2$$
$$= 500 \ \$$$

b) Find the marginal profit at $X = 100$ and explain.

$$\rightarrow \overline{MP} = P'(x) = 16 - \frac{2}{10}X$$
$$= 16 - \frac{1}{5}X$$

$$\therefore P'(100) = 16 - \frac{1}{5}(100)$$
$$= -4$$

the approximated profit of 101st unit is -4.
«loss»

c) Find $P(101) - P(100)$, and explain.

$$\rightarrow P(101) = 16(101) - 100 - \frac{1}{10}(101)^2 = 495.9$$

$P(100) = 500$

$\therefore P(101) - P(100) = -4.1$

the exact profit from the 101st unit is $\underset{\text{(loss)}}{\underline{-4.1}}$

- Ex: If the total profit function $P(x) = 30x - x^2 - 200$

a) Find the approximated profit from the sale of the 11th unit.

$$\longrightarrow P'(x) = 30 - 2x$$

$$P'(10) = 10 \ \$$$

b) Find the exact profit from the sale of the 11th unit.

$$\longrightarrow P(11) - P(10)$$

$$= (30(11) - 11^2 - 200) - (30(10) - 10^2 - 200)$$

$$= 9 \ \$$$

*** Note :**

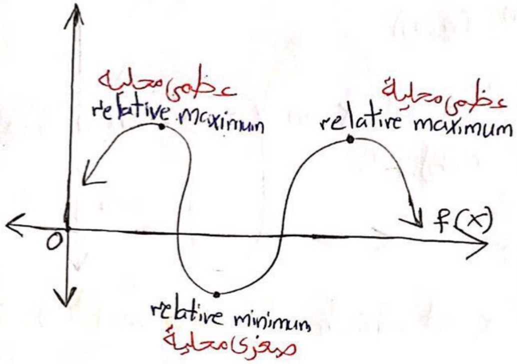
In general ① $R(x+1) - R(x) \cong R'(x)$.

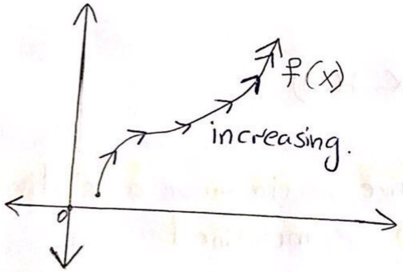
② $P(x+1) - P(x) \cong P'(x)$.

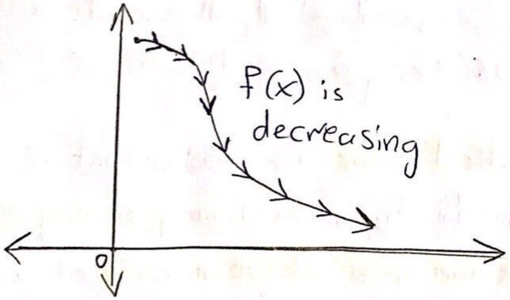
③ $C(x+1) - C(x) \cong C'(x)$.

*Ch 10: Applications of Derivatives

-Sec 10.1: Relative Maxima and Minima:







* Increasing and Decreasing Functions.

If f is a differentiable function on interval (a,b)

① if $f'(x) > 0$ for all x in (a,b), then $f(x)$ is increasing on (a,b).

② if $f'(x) < 0$ for all x in (a,b), then $f(x)$ is decreasing on (a,b).

* Critical values:

The values of x at which $f'(x)=0$ or $f'(x)$ is undefined.

⟶ critical point: $(x, f(x))$

* If f has a relative maximum or a relative minimum at c, then $f'(c)=0$ or undefined.

يعني أن جميع القيم العظمى والصغرى تكون عندها المشتقة صفراً و
غير معرفة (أي تكون قيم حرجة) ولكن العكس غير صحيح.

* If $x=c$ is a critical value for $f(x)$, «that is $f'(c)=0$ or undefined» then $f(x)$ may or may not have a relative maximum or a relative minimum at $x=c$.

* We will use the first derivative test to determine the relative maxima and minima and the intervals for which f(x) increasing or decreasing.

• First derivative test:

To find relative maxima nd minima:

① Find $f'(x)$.

② Find critical values by setting $f'(x) = 0$ and $f'(x)$ undefined.

③ Creat the sign diagram for $f'(x)$.

$\longrightarrow f'(x) \; \underset{c}{\overset{+\,+\,+\;\overset{0}{|}\;-\,-\,-}{\longleftrightarrow}}$ «if c is critical value»

$\therefore$ f(x) is increasing $x < c$
f(x) is decreasing $x > c$
(c, f(c)) is local maximum.
relative

$\longrightarrow f'(x) \; \underset{c}{\overset{-\,-\,-\,-\,-\;\overset{0}{|}\;+\,+\,+\,+}{\longleftrightarrow}}$

$\therefore$ $f(x)$ is increasing $x > c$
$f(x)$ is decreasing $x < c$
$(c, f(c))$ is local minimum.

$\rightarrow$ $f'(x)$ $\xleftrightarrow{+ + + + + \, | \, + + + +}$ or $\xleftrightarrow{- - - - \, | \, - - - -}$
$\qquad\qquad\quad c \qquad\qquad\qquad\qquad c$

$(c, f(c))$ neither maximum nor minimum and is called horizontal point of inflection.

- Ex: Find the relative maxima and minima of $f(x)$, and the intervals where $f(x)$ is increasing, or decreasing:- $f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2 + 8$.

$\rightarrow f'(x) = x^3 - x^2 - 6x$.

$\rightarrow$ Set: ① $f'(x) = 0$

$x^3 - x^2 - 6x = 0$

$x(x^2 - x - 6) = 0$

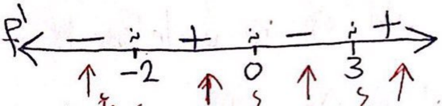
$x(x-3)(x+2) = 0$

نخرج عامل مشترك
بعدها نحلل أو عالقانون
العام.

either $x = 0$, $x = 3$ or $x = -2$

② $f'(x)$ undefined: no values.

→ Sign diagram: f'

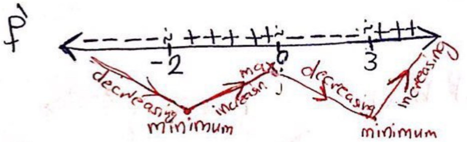


نأخذ عدد في هذه المنطقة مثلاً -3 ونعوضه في f' لمعرفة الإشارة فإذا كان الناتج + بغض النظر عن قيمة التعويض تكون الإشارة + . وإذا كان التعويض سالب تكون الإشارة -.

نعوض هنا مثلاً العدد -1 في f' لمعرفة الإشارة.

نعوض هنا مثلاً العدد 1 في f'.

نعوض مثلاً العدد 4 في f'.



$\therefore$ $f(x)$ is increasing on $(-2, 0) \cup (3, \infty)$ } intervals
$f(x)$ is decreasing on $(-\infty, -2) \cup (0, 3)$ } فترات.

$f(x)$ has a relative maximum : $(0, f(0)) = (0, 8)$

$f(x)$ has a relative minimum : $(-2, f(-2)) = (-2, \frac{8}{3})$

$(3, f(3)) = (3, \frac{-31}{4})$

-Ex: Find the relative maxima, minima, and horizontal points of inflection, and horizonal points of inflection of $h(x) = \frac{1}{4}x^4 - \frac{2}{3}x^3$.

$\rightarrow h'(x) = x^3 - 2x^2$.

$\longrightarrow$ ① $h'(x)$ is undefined: no values.

② $h'(x) = 0$

$x^3 - 2x^2 = 0$

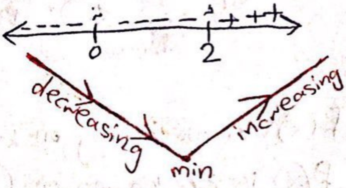
$x^2(x-2) = 0$

$\therefore x^2 = 0 \rightarrow \boxed{x=0}$

or $x - 2 = 0 \rightarrow \boxed{x=2}$

} critical values
↓
critical points:-
$(0, f(0)) = (0, 0)$
$(2, f(2)) = (2, \frac{-4}{3})$

$\longrightarrow$ Sign diagram of $h'(x)$:



$\therefore f(x)$ is increasing on $(2, \infty)$.

$f(x)$ is decreasing on $(-\infty, 0) \cup (0, 2)$

relative maxima: None

relative minima: $(2, f(2)) = (2, -\frac{4}{3})$.

horizontal point of inflection: $(0, f(0)) = (0, 0)$

-Ex: Find the relative maxima and minima (if any) of the graph $y=(x+2)^{\frac{2}{3}}$.

$\longrightarrow y' = \frac{2}{3}(x+2)^{-\frac{1}{3}}(1) = \frac{2}{3(x+2)^{1/3}}$

$\longrightarrow y' = 0$

$\frac{2}{3(x+2)^{\frac{1}{3}}} = 0$

$2 = 0$ ∴ No values.

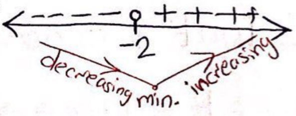
$\longrightarrow$ y' is undefined:

$\frac{3}{3}(x+2)^{\frac{1}{3}} = \frac{0}{3}$

$\left((x+2)^{\frac{1}{3}}\right)^3 = (0)^3$

$x+2=0$ ∴ $\boxed{x=-2}$
$\quad -2 \quad -2$

$\longrightarrow$ sign diagram of y'



∴ f(x) is increasing on $(-2,\infty)$
f(x) is decreasing on $(-\infty,-2)$
relative min. : $(-2, f(-2)) = (-2,0)$.

-Ex: The weekly sales S of a product during an advertising campaign are given by:-

$$S = \frac{100t}{t^2+100} \; ; \; 0 \leqslant t \leqslant 20$$

where t is the number of weeks.

a) Over what interval are sales increasing, decreasing?

$$\longrightarrow S' = \frac{(t^2+100)(100) - 100t\,(2t)}{(t^2+100)^2}$$

$$= \frac{100t^2 + 10000 - 200t^2}{(t^2+100)^2}$$

$$= \frac{-200t^2 + 10000}{(t^2+100)^2}$$

$$\longrightarrow \quad S' = 0$$

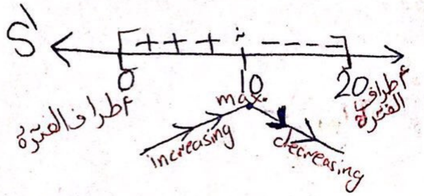
$$\frac{-100t^2 + 10000}{(t^2+100)^2} = 0$$

$$-100t^2 + 10000 = 0$$

$$\therefore t^2 = 100 \longrightarrow t = 10 \, , \, \cancel{-10} \; \text{rejected.}$$

$\longrightarrow$ S' is undefined:

$$(t^2+100)^2 = 0 \longrightarrow t^2 + 100 = 0 \longrightarrow \underset{\times}{t^2 = -100}$$



∴ $f(x)$ is increasing on $[0, 10)$

$f(x)$ is decreasing on $(10, 20]$

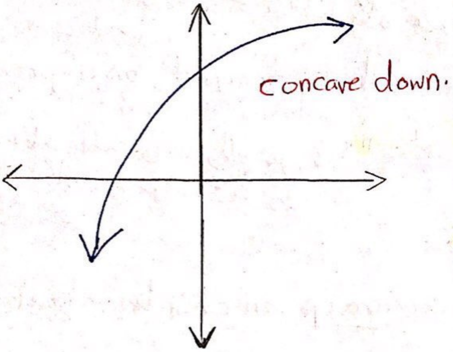
b) What are the maximum weekly sales ?

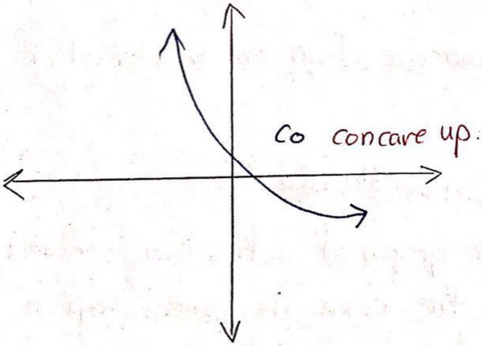
$$(10 , S(10)) = (10 , 5)$$

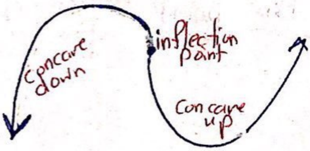
∴ maximum weekly sales $= 5$.

- Sec 10.2 : Concavity ; Points of Inflection

التقعر ونقاط الانعطاف







*Concavity:

- A curve is said to be concave up on (a, b) if each point on the interval the curve is above its tangent at the point. يكون المنحنى مقعر للأعلى ، إذا كان جميع مماساته تحت المنحنى.

- If the curve is below all its tangent on a given interval, it is concave down. يكون المنحنى مقعر للأسفل ، إذا كان جميع مماساته فوق المنحنى.

* Concave up and down:

① The function f is concave up on an interval, if $f''(x) > 0$ on I.

② The function f is concave down on an interval, if $f''(x) < 0$ on I.

③ Point of inflection: نقطة الانعطاف

A point (x_0, y_0) on the graph of a function is called a point of inflection if the curve is concave up on one side and concave down on the other side.

The second derivative at this point $f''(x_0) = 0$ or undefined.

*Findind points of inflection and concavity:-

① Find $f''(x)$.

② Set $f''(x)=0$, and $f''(x)$ undefined.

③ Sign diagram fo $f''(x)$.

→ $f''(x)$ + concave up.

$f''(x)$ − concave up.

→ If $f''(x)$ has opposit signs on the 2 sides of one of the values, a point of inflection occurs.

-Ex: Find the points of inflection and concavity of $y=\frac{x^4}{4}-x^3+5$.

→ $f'(x)=x^3-3x^2$

→ $f''(x)=3x^2-6x$

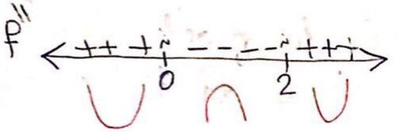
① $f''(x)$ is undefined: No values.

② $f''(x)=0$

$3x^2-6x=0$

$3x(x-2)=0$

$\therefore \boxed{x=0}$ or $\boxed{x=2}$



$\therefore$ $f(x)$ is concave up on $(-\infty, 0) \cup (2, \infty)$

$f(x)$ is concave down on $(0, 2)$.

inflection points: $(0, f(0)) = (0, 5)$

$(2, f(2)) = (2, 5)$

* Diminishing point of returns:-

The point where the rate of change of f changes from increasing to decreasing, that is $f'(x)$ is maximized at this point.

$\therefore$ Diminishing returns = inflection point f'' + | −

-Ex: Suppose the annual profit for a store is given by $P(x) = -0.2x^3 + 3x^2 + 6$, find the point of diminishing returns for the profit.

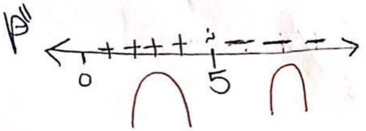
$\longrightarrow$ $P'(x) = -0.6x^2 + 6x$.

$P''(x) = -1.2x + 6$

$\longrightarrow$ $P''(x)$ undefined: No values.

$P''(x) = 0$

$-1.2x + 6 = 0 \longrightarrow \boxed{x = 5}$



inflection point = diminishing point of returns $= (5, f(5))$

$= (5, 56)$

***Second derivative test:** اختبار المشتقة الثانية

لإيجاد max, min ولكن لا يمكننا دائماً استخدامه.

If $f'(x) = 0$ at $x = a$, and $f''(a) > 0$, then $(a, f(a))$ is a relative minimum.

If $f'(x) = 0$ at $x = a$ and $f''(a) < 0$, then $(a, f(a))$ is a relative maximum.

أحياناً بفشل هذا الاختبار في تحديد max, min لذا الأفضل استخدام 1st test

→ The second derivative test sometimes fails to give results. (when $f'(a)$ undefined or when $f'(a) = 0$ and $f''(a) = 0$).

-Ex: Find critical values, relative maxima, relative minima, and points of inflection:-

$$y = x^{\frac{4}{3}}(x - 7).$$

$$\longrightarrow y' = x^{\frac{4}{3}}(1) + (x-7)\cdot\frac{4}{3}x^{\frac{1}{3}}$$

$$= x^{\frac{1}{3}}\left(x + \frac{4}{3}(x-7)\right)$$

$$= x^{\frac{1}{3}}\left(\frac{7x-28}{3}\right)$$

$$\longrightarrow y'' = x^{\frac{1}{3}}\left(\frac{7}{3}\right) + \left(\frac{7x-28}{3}\right)\cdot\frac{1}{3}x^{-\frac{2}{3}}$$

$$= \frac{7}{3}\left[\frac{3x^{\frac{2}{3}}\cdot x^{\frac{1}{3}}}{3\cdot x^{\frac{2}{3}}} + \frac{x-4}{3x^{\frac{2}{3}}}\right]$$

$$= \frac{7}{3}\left[\frac{3x + x - 4}{3\,x^{\frac{2}{3}}}\right]$$

$$= \frac{7}{9}\left[\frac{4x-4}{3\,x^{2/3}}\right]$$

① $y' = 0$

$$x^{\frac{1}{3}}\left(\frac{7x-28}{3}\right) = 0$$

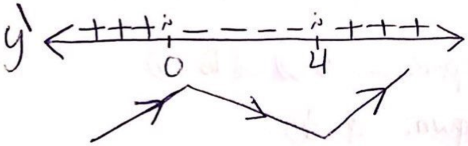
either $x^{\frac{1}{3}} = 0 \longrightarrow \left(x^{\frac{1}{3}}\right)^3 = (0)^3 \longrightarrow x = 0.$

or $\frac{7x-28}{3}=0 \longrightarrow 7x-28=0$

$+28 \quad +28$

$\frac{7}{7}x=\frac{28}{7}$ ∴ $\boxed{x=4}$

y' undefined: No values.



∴ f(x) is increasing $(-\infty,0)\cup(4,\infty)$.

f(x) is decreasing $(0,4)$.

relative maxima: $(0, f(0)) = (0,0)$

relative minima: $(4, f(4)) = (4,$

② $y''=0$

$$\frac{7}{9}\left[\frac{4x-4}{3x^{2/3}}\right]=0$$

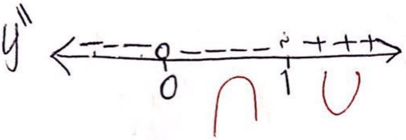
$\frac{7}{7}(4x-4)=\frac{0}{7} \longrightarrow 4x-4=0$

$+4 \quad +4$

$\frac{4}{4}x=\frac{4}{4}$ ∴ $\boxed{x=1}$

y'' is undefined.

$$\frac{3x^{\frac{2}{3}}}{3} = \frac{0}{3} \longrightarrow \left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} = 0^{\frac{3}{2}} \longrightarrow \boxed{x=0}$$



f(x) is concare down: $(-\infty, 0) \cup (0, 1)$.

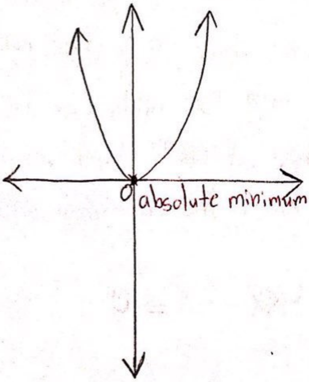
f(x) is concare up : $(1, \infty)$.

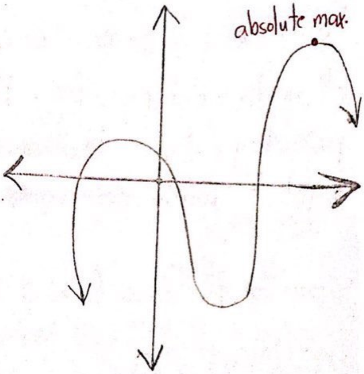
inflection points : $(1, f(1)) = (1, -6)$

-Sec 10.3: Optimization in Business and Economics

***Absolute Extrema:** القيم القصوى المطلقة

- The value $f(a)$ is the **absolute maximum** (عظمى مطلقة) of f if $f(a) \geqslant f(x)$ for all x in the domain of f.
- The value $f(b)$ is the **absolute minimum** (صغرى مطلقة) of f if $f(b) \leqslant f(x)$ for all x in the domain of f.





-Ex: Find the absolute maxima and minima for

$f(x) = x^3 - 3x + 3$; $[-3, 1.5]$

$\longrightarrow f'(x) = 3x^2 - 3$

① $f'(x)$ undefined: No values.

② $f'(x)=0$

$$3x^2 \underset{+3}{-} 3 = \underset{+3}{0} \longrightarrow \frac{3x^2}{3} = \frac{3}{3} \longrightarrow x^2 = 1 \longrightarrow \boxed{x = \mp 1}$$

$x = -1 \longrightarrow f(-1) = 5$ absolute max.

$x = 1 \longrightarrow f(1) = 1$

$x = -3 \longrightarrow f(-3) = -15$ absolute min.

$x = 1.5 \longrightarrow f(1.5) = 1.875$

-Ex: The total revenue in dollars for a firm is given by: $R(x) = 8000x - 40x^2 - x^3$; where x is the number of units sold per day. If only 50 units can be sold per day, find the number of units that must be sold to maximize revenue. Find the maximum revenue.

$\longrightarrow R'(x) = 8000 - 80x - 3x^2 = 0$

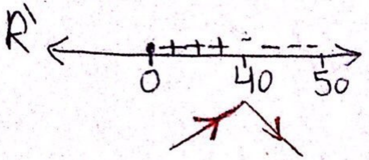
① $R'(x) = 0$

$8000 - 80x - 3x^2 = 0$

$a = -3,\ b = -80,\ c = 8000$

$$x = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} = \frac{80 \mp \sqrt{(-80)^2 - 4(-3)(8000)}}{2(-3)}$$

$$= 40,\ \boxed{\frac{-200}{3}} \rightarrow$$ مرفوضة عدد الوحدات لا يمكن أن يكون



$\therefore$ max. revenue : $R(40) = 8000(40) - 40(40)^2 - 40^3$
$= 192000$ \$

* Average Cost:

If the total cost function is $C(X)$, then the per unit average cost function is $\overline{C} = \frac{C(X)}{X}$.

Note that the average cost per unit is undefined if no units are produced.

لأن $C(x)$ دائماً متزايد، نستطيع ايجاد عدد الوحدات التي تجعل متوسط التكاليف أقل ما يمكن.

-Ex: If the total cost for a product is given by $C = \frac{1}{4}X^2 + 4X + 100$, where X is the number of units, producing how many units will result in a minimum average cost per unit? Find the minimum average cost?

$\longrightarrow$ Average cost «$\overline{C}$» $= \frac{C(X)}{X} = \frac{1}{4}X + 4 + \frac{100}{X}$

$\overline{C}' = \frac{1}{4} - \frac{100}{X^2}$

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$\overline{C}' = 0$

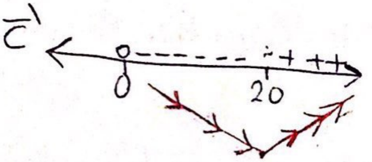
$$\frac{1}{4} - \frac{100}{x^2} = 0 \longrightarrow \frac{1}{4} = \frac{100}{x^2} \rightarrow x^2 = 400$$

$$+\frac{100}{x^2} \qquad +\frac{100}{x^2}$$

$\therefore x = 20, \enclose{circle}{-20}\, x$

$\overline{C}'$ undefined:- $x^2 = 0 \rightarrow \boxed{x = 0}$

$\overline{C}'$



the number of units that will minimize the average cost is 20.

$\longrightarrow$ minimum average cost $= \overline{C}(20)$
$= \$14$

-Ex: Suppose that the production capacity for a certain product can't exceed 30. If the total profit $P(x) = 4x^3 - 210x^2 + 3600x - 200$, where x is the number of units, find the number of items that will maximize profit.

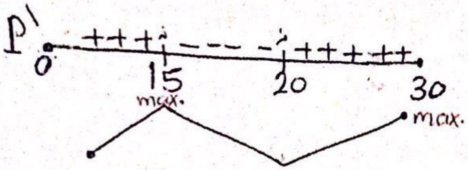
$$\longrightarrow P'(x) = 12x^2 - 420x + 3600 = 0$$

$$12(x^2 - 35x + 300) = 0$$

$$12(x-15)(x-20) = 0$$

$$\therefore \boxed{x = 15} \text{ or } \boxed{x = 20}$$

$\longrightarrow$ $P'(x)$ undefined: no values



$(15, P(15)) = (15, 20050)$

$(30, P(30)) = \underline{(30, 26800)}$ ∴ absolute maximum.

$\longrightarrow$ max. profit is \$26800 at $X = 30$.

-Ex: The price of a product is related to the number of units x demanded daily by $p = 168 - 0.2x$.

A monopolist finds that the daily average cost for this product $\overline{C} = 120 + x$.

a) How many units must be sold to maximize profit?

$\longrightarrow P(x) = R(x) - C(x)$,

$R(x) = p\,X$

$= (168 - 0.2)X = 168X - 0.2X^2$

$C(x) = \overline{C} \cdot X$

$= (120 + X)X = 120X + X^2$

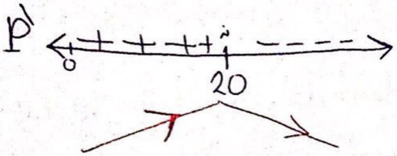
$\therefore P(x) = 48x - 1.2x^2$

$P'(x) = 48 - 2.4x$

① $P'(x) = 0$

$48 - 2.4x = 0 \longrightarrow \frac{48}{2.4} = \frac{2.4x}{2.4} \longrightarrow \boxed{x = 20}$

② $P'(x)$ undefined: no values.



20 units will maximize profit.

b) What is the maximum profit?

$\longrightarrow P(20) = 48(20) - 1.2(20)^2$
$= \$480$

c) What is the selling price at this optimal level of production?

price نعوض X في

$p = 168 - 0.2(20) = 164$

*Ch11: Derivatives Continued

- Sec 11.1: Derivatives of logarithmic functions

• Derivative of $y=\ln u$:

If $y=\ln u$, then $\frac{dy}{dx}=\frac{u'}{u}$

• Derivative of $y=\log_a u$:

If $y=\log_a u$, then $\frac{dy}{dx}=\frac{u'}{u\ln a}$.

- Ex: Find the **derivative** of the following functions:

a) If $y=x^3+3\ln x$.

$$\longrightarrow \frac{dy}{dx}=3x^2+3\cdot\frac{1}{x}$$

$$=3x^2+\frac{3}{x}.$$

b) $y=x^2\ln x$.

$$\longrightarrow y'=x^2\cdot\frac{1}{x}+\ln x\cdot(2x)$$

$$=x+2x\ln x.$$

c) $f(x)=\ln(x^4-3x+7)$.

$$\longrightarrow f'(x)=\frac{4x^3-3}{x^4-3x+7}$$

d) $f(x)=\frac{1}{3}\ln(2x^6-3x+2)$.

$$\longrightarrow f'(x)=\frac{1}{3}\cdot\frac{12x^5-3}{2x^6-3x+2}=\frac{\cancel{3}(4x^5-1)}{\cancel{3}(2x^6-3x+2)}=\frac{4x^5-1}{2x^6-3x+2}$$

e) $g(x)=\frac{\ln(2x+1)}{2x+1}$.

$$\longrightarrow g'(x)=\frac{(2x+1)\left(\frac{2}{(2x+1)}\right)-\ln(2x+1)(2)}{(2x+1)^2}$$

$$=\frac{2-2\ln(2x+1)}{(2x+1)^2}$$

f) $y=\ln\left[x(x^5-2)^{10}\right]$

$$\longrightarrow y=\ln x+\ln(x^5-2)^{10}$$

$$=\ln x+10\ln(x^5-2)$$

$$\longrightarrow y'=\frac{1}{x}+10\cdot\frac{5x^4}{x^5-2}$$

$$=\frac{1}{x}+\frac{50x^4}{x^5-2}.$$

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g) $f(x) = \ln\left(\frac{\sqrt[3]{3x+5}}{x^2+11}\right)^4$

$\longrightarrow f(x) = 4\ln\left(\frac{\sqrt[3]{\quad 5}}{x^2+11}\right)$

$= 4\left[\ln\sqrt[3]{3x+5} - \ln(x^2+11)\right]$

$= 4\left[\frac{1}{3}\ln(3x+5) - \ln(x^2+11)\right]$

$= \frac{4}{3} \cdot \frac{3}{3x+5} - 4\frac{2x}{x^2+11}$

$= \frac{4}{3x+5} - \frac{8x}{x^2+11}$

h) $y = \log_4(x^3+1)$

$\longrightarrow \frac{dy}{dx} = \frac{3x^2}{(x^3+1)\ln 4}$

i) $y = (\ln x)^{-1}$

$\longrightarrow \frac{dy}{dx} = -1(\ln x)^{-2}\left(\frac{1}{x}\right)$

$= \frac{-1}{x(\ln x)^2}$

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-Sec 11.2: Derivatives of exponential functions:

*** Derivative of $y=e^u$:-**

If $y=e^u$, then $\frac{dy}{dx}=u'e^u$.

*** Derivative of $y=a^u$; $a>0$ and $a\neq 1$:-**

If $y=a^u$, then $\frac{dy}{dx}=u'.a^u.\ln a$

-Ex: Find the derivative of the following functions:

a) $f(x)=e^{4x^3}$

$\longrightarrow f'(x)=12x^2e^{4x^3}$

b) $S=3t\,e^{3t^2+5t}$

$\longrightarrow S'=3t.(6t+5)e^{3t^2+5t}+3e^{3t^2+5t}$

$=(18t^2+15t)e^{3t^2+5t}+3e^{3t^2+5t}$

$=e^{3t^2+5t}(18t^2+15t+3)$

c) $u=\frac{w}{e^{3w}}$

$\longrightarrow u'=\frac{e^{3w}(1)-w(3e^{3w})}{(e^{3w})^2}=\frac{e^{3w}(1-3w)}{e^{6w}}=\frac{1-3w}{e^{3w}}$

d) $y = e^{\ln x^2}$

$\longrightarrow y = \cancel{e^{\ln}} x^2 = x^2$

$\longrightarrow y' = 2x$

e) $y = 4^x$

$\longrightarrow y' = 1 . 4^x \ln 4$

$= 4^x \ln 4$

f) $y = 5^{x^2+x}$

$\longrightarrow y' = (2x+1) . 5^{x^2+x} . \ln 5$

-Ex: let $R(x) = 250x\, e^{(1-0.01x)}$ be the ~~to~~total revenue function. Find the marginal revenue when 75 are sold.

$\longrightarrow \overline{MR} = R'(x) = 250x . (-0.01)\, e^{(1-0.01x)} + 250\, e^{(1-0.01x)}$

$= 250\, e^{(1-0.01x)}(-0.01x + 1)$

$\therefore R'(75) = 250\, e^{(1-0.01(75))} (-0.01(75)+1)$

$= 250\, e^{0.25}(0.25)$

$\cong 80.25$

$250 \boxed{\text{shift}} \boxed{\ln} (0.25) \times .25$

$=$

باستخدام الآلة

-Sec 11.3: Implicit Differentiation

الاشتقاق الضمني

Functions involving x and y have been written in the form $y = f(x)$, defining y as an explicit function of x. However, not all equations involving x and y can be written in the form $y = f(x)$, and we need technique for taking their derivatives. This technique is called implicit differentiation. We find $\frac{dy}{dx}$ by differentiating both sides of the equation with respect to x and then solving for $\frac{dy}{dx}$.

عندما تكون y معرفة ضمنياً ، بمعنى أن
x و y على طرف واحد ، نشتق ضمني.

-Ex: ① Use implicit differentiation to find $\frac{dy}{dx}$ for

$y^2 = x$.

$\longrightarrow \quad y^2 = x$ «differentiate both sides»

$2y \cdot y' = 1$

$\therefore y' = \frac{1}{2y}$

② Find the slope of the tangent to the graph of $y^2 = x$ at the points $(4,2)$ and $(4,-2)$.

$\longrightarrow$ Slope of the tangent at $(4,2)$:-

$$y'\Big|_{(4,2)} = \frac{1}{2y}\Big|_{(4,2)} = \frac{1}{2(2)} = \frac{1}{4}.$$

$\longrightarrow$ Slope of the tangent at $(4,-2)$:

$$y'\Big|_{(4,-2)} = \frac{1}{2y}\Big|_{(4,-2)} = \frac{1}{2(-2)} = \frac{-1}{4}.$$

_Ex: Find the slope of the tangent to the graph of $x^2+y^2-9=0$ at $(\sqrt{5},2)$.

$$\longrightarrow 2x+2y\cdot y' = 0$$

$$y' = \frac{-2x}{2y} = \frac{-x}{y}$$

$\therefore$ Slope of the tangent at $(\sqrt{5},2)$:

$$y'\Big|_{(\sqrt{5},2)} = \frac{-x}{y}\Big|_{(\sqrt{5},2)} = \frac{-\sqrt{5}}{2}.$$

-Ex: Write the equation of the tangent to the graph of $x^3+xy+4=0$ at the point $(2,-6)$.

$$\longrightarrow 3x^2+x\cdot y'+y\cdot 1=0$$

$$\underset{-3x^2}{3x^2}+xy'+\underset{-y}{y}=0 \quad -3x^2-y$$

$$\frac{\cancel{x}y'}{\cancel{x}}=\frac{-3x^2-y}{x}$$

$$\therefore y'=\frac{-3x^2-y}{x}$$

$\longrightarrow$ Slope of the tangent at $(2,-6)$

$$y'\Big|_{(2,-6)}=\frac{-3(2)^2--6}{2}=-3$$

$\longrightarrow$ the equation of the tangent at $(2,-6)$:

$$y-y_0=m(x-x_0)$$

$$y--6=-3(x-2)$$

$$y+\underset{-6}{6}=-3x+\underset{-6}{6}$$

$$\therefore \boxed{y=-3x}$$

-Ex: Find $\frac{dy}{dx}$ if $x^4+5xy^4=2y^2+x-1.$

$$\longrightarrow 4x^3+5x\cdot 4y^3\cdot y'+y^4\cdot 5=4y\cdot y'+1$$

$$4x^3+20xy^3y'+5y^4=4yy'+1$$

$$-4x^3 \quad -4yy' \quad -5y^4 \quad -4yy' \quad -4x^3-5y^4$$

$$20xy^3y'-4yy'=1-4x^3-5y^4$$

$$\frac{y'(20xy^3-4y)}{20xy^3-4y}=\frac{1-4x^3-5y^4}{20xy^3-4y}$$

$$\boxed{\therefore\ y'=\frac{1-4x^3-5y^4}{20xy^3-4y}}$$

-Note:

① Horizontal tangents will occure where $y'=0$.

② Vertical tangents will occure where the derivative is undefined. (المقام في المشتقة = صفر)

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- Ex: a) At what point(s) does $x^2+4y^2-2x+4y-2=0$ have a horizontal tangent. «$y'=0$»

$$\longrightarrow 2x+\underset{\sim}{8y\cdot y'}-2+\underset{\sim}{4y'}=0$$

$$2x+0-2+0=0$$

$$2x-2=0$$

$$\therefore \boxed{x=1}$$

نجد y من المعادلة الأصلية: (بتعويض X في المعادلة)

$$\longrightarrow x^2+4y^2-2x+4y-2=0$$

$$1+4y^2-2+4y-2=0$$

$$4y^2+4y-3=0$$

$$(2y-1)(2y+3)=0$$

$$\therefore \boxed{y=\frac{1}{2}} \text{ or } \boxed{y=\frac{-3}{2}}$$

The horizontal tangents occure at $(1,\frac{1}{2}),(1,\frac{-3}{2})$.

b) At what point(s) does $x^2+4y^2-2x+4y-2=0$ have a vertical tangent.

$$\longrightarrow \underset{-2x}{2x}+8y\,y'\underset{+2}{-2}+4y'=0_{-2x+2}$$

$$8yy'+4y'=-2x+2$$

$$\frac{y'(8y+4)}{8y+4} = \frac{-2x+2}{8y+4}$$

$$\therefore\ y' = \frac{-2x+2}{8y+4}$$

$\longrightarrow y'$ is undefined: $\underset{-4}{8y+4} = \underset{-4}{0} \rightarrow \frac{8y}{8} = \frac{-4}{8}$

$$\boxed{y = \frac{-1}{2}}$$

نعوض قيمة y في المعادلة الأصلية لإيجاد x:

$$x^2 + 4\left(\frac{-1}{2}\right)^2 - 2x + 4\left(\frac{-1}{2}\right) - 2 = 0$$

$$x^2 + 1 - 2x - 2 - 2 = 0$$

$$x^2 - 2x - 3 = 0$$

$$\Rightarrow (x-3)(x+1) = 0$$

$$\therefore x = 3, -1$$

The vertical tangents occure at $(3, -\frac{1}{2})$ and $(-1, -\frac{1}{2})$.

- Ex: Find $\frac{dy}{dx}$ for each of the following:

a) $\ln(xy) = 6$

$$\rightarrow \ln x + \ln y = 6$$

$$\frac{1}{x} + \frac{y'}{y} = 0$$

$$\cancel{\frac{1}{x}} + \frac{y'}{y} = 0$$

$$\cancel{-\frac{1}{x}} \qquad\qquad -\frac{1}{x}$$

$$y \cdot \frac{y'}{y} = \frac{-1}{x} \cdot y$$

$$\therefore \boxed{y' = \frac{-y}{x}}$$

b) $4x^2 + e^{xy} = 6y$

$$\rightarrow 8x + (x \cdot y' + y \cdot 1)e^{xy} = 6y'$$

$$8x + xy'e^{xy} + ye^{xy} = 6y'$$

$$xy'e^{xy} - 6y' = -8x - ye^{xy}$$

$$\frac{y'\cancel{(xe^{xy} - 6)}}{\cancel{xe^{xy} - 6}} = \frac{-8x - ye^{xy}}{xe^{xy} - 6}$$

$$\therefore \boxed{y' = \frac{-8x - ye^{xy}}{xe^{xy} - 6}}$$

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-Sec 11.5: Applications in Business and Economics

We Know from the law of demand that consumers will respond to changes in prices; if prices increase, the quantity demanded will decrease. But the degree of responsiveness of the consumers to price changes will vary widely for different products. When the response to price changes is considerable, we say the demand is elastic. When price changes cause relatively small changes in demand for aproduct, the demand is said to be inelastic for that product.

* Elasticity:

The elasticity of demand at the point (q_0, p_0) is

$$\eta = E_d = \frac{-p}{q} \cdot \underbrace{\frac{dq}{dp}}_{q'}\Bigg|_{(q_0, p_0)}$$

① If $\eta > 1$, the demand is elastic, and the percent decrease in demand is greater than the corresponding percent increase in price.

② If $\eta<1$, the demand is inelastic, and the percent decrease in demand is less than the corresponding percent increase in price.

③ If $\eta=1$, the demand is unitary elastic, and the percent decrease in demand is approximately equal to the corresponding percent increase in price.

- Ex: Find the elasticity of the demand $p+5q=100$ when:

a) the price is \$40.

$$\longrightarrow \eta=\frac{-p}{q}\cdot q'$$

نجد q :-

$$p+5q=100 \quad \text{«implicit differentiation»}$$

$$\underset{-1}{1}+5q'=\underset{-1}{0}$$

$$\frac{5q'}{5}=\frac{-1}{5} \qquad \therefore \boxed{q'=\frac{-1}{5}}$$

to find q: نعوض p في المعادلة الأصلية

$$\underset{-40}{40}+5q=\underset{-40}{100}$$

$$\frac{5q}{5}=\frac{60}{5} \qquad \therefore \boxed{q=12}$$

$$\longrightarrow \eta = \frac{-40}{12} \cdot \left(\frac{-1}{5}\right) = 0.667 < 1$$

∴ the demand is inelastic.

- Ex: The demand for a certain product:

$$p = \frac{1000}{(q+1)^2}$$

Find the elasticity of demand when $q = 19$.

$$\longrightarrow \eta = \frac{-p}{q} \cdot q'$$

لإيجاد q' نشتق ضمني :-

$$1 = 1000(-2)(q+1)^{-3} \cdot q'$$

$$1 = \frac{-2000}{(q+1)^3} q'$$

$$1 = \frac{-2000}{20^3} \cdot q'$$

$$\frac{1}{-0.25} = \frac{-0.25q'}{-0.25} \longrightarrow ∴ \boxed{q' = -4}$$

لإيجاد p نعوض في المعادلة الأصلية :-

$$p = \frac{1000}{(19+1)^2} = 2.5$$

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$\therefore \eta = \frac{-2.5}{19} \cdot (-4) = 0.526 < 1$

the demand is inelastic.

* Elasticity and Revenue:

① If $\eta > 1$ means $\frac{dR}{dp} < 0$:-

If price increases, revenue decreases, and if price decreases, revenue increases. P↑ R↓ or P↓ R↑

② If $\eta < 1$ means $\frac{dR}{dp} > 0$:-

If price increases, revenue increases and if price decreases, revenue decreases. P↑ R↑ or P↓ R↓

③ If $\eta = 1$ mean $\frac{dR}{dp} = 0$:

An increase or decrease in price will not change revenue. Revenue is optimized at this point. (Revenue is max.).

-Ex: The demand for a product is given by

$$p = 10\sqrt{100-q} \;;\; 0 \leqslant q \leqslant 100$$

a) Find the point at which demand is of unitary elasticity and find intervals in which the demand is inelastic and in which it is elastic.

$$\longrightarrow \eta = \frac{-p}{q} \cdot \frac{dq}{dp}$$

لإيجاد q' نشتق ضمني :-

$$1 = 10\left(\tfrac{1}{2}\right)(100-q)^{\frac{-1}{2}}(-1q')$$

$$1 = \frac{-5q'}{\sqrt{100-q}}$$

$$\frac{\sqrt{100-q}}{-5} = \frac{-5q'}{-5}$$

$$\therefore\; q' = \frac{-\sqrt{100-q}}{5}$$

$$\longrightarrow \eta = \frac{-p}{q} \cdot \frac{\sqrt{100-q}}{-5}$$

$$= \frac{-10\sqrt{100-q}}{q} \cdot \frac{\sqrt{100-q}}{-5} = \frac{2(100-q)}{q}$$

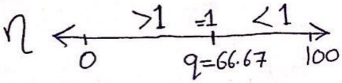
$$\therefore \eta = \frac{200-2q}{q}$$

① the demand is unitary elastic

$$1 = \frac{200-2q}{q}$$

$$\begin{aligned} q &= 200-2q \\ +2q & \qquad\quad +2q \end{aligned}$$

$$\frac{3q}{3} = \frac{200}{3} \qquad \therefore \boxed{q = 66.67}$$



② the demand is elastic:

$$0 < q < 66.67$$

③ the demand is inelastic:

$$66.67 < q < 100$$

b) Find the max. revenue.

⟶ Revenue is maximized where $\eta = 1$

$\therefore$ at $q = 66.67$

$$\therefore \text{max. revenue} = p \cdot q = 10\sqrt{100-q} \cdot q$$
$$= 10\sqrt{100-66.67}\,(66.67) = \$3849$$

* Taxation in Competitive Market:-

Many taxes imposed by governments. Suppose the government imposes a tax of $t on each unit, the demand function will not change, the tax will change the supply function.

We can use the following procedure for maximizing the total tax revenue.

① We find the tax per item:

$$t = D - S$$

② Find the total tax revenue:

$$T = t \cdot q$$

③ Find T', and set $T' = 0$ $\longleftrightarrow$

④ Substitute the value of q that maximizes T, in the equation for t and T.

-Ex: The demand and supply functions for a product are $p=900-20q-\frac{1}{3}q^2$ and $p=200+10q$. Find the tax per unit that will maximize the tax revenue T.

$\rightarrow t = D - S$ «tax per item»

$= (900-20q-\frac{1}{3}q^2)-(200+10q)$

$= 700-30q-\frac{1}{3}q^2$

$\rightarrow$ Total tax: $T = tq$

$= 700q-30q^2-\frac{1}{3}q^3$

$\rightarrow T' = 700-60q-q^2$

$\rightarrow T' = 0$

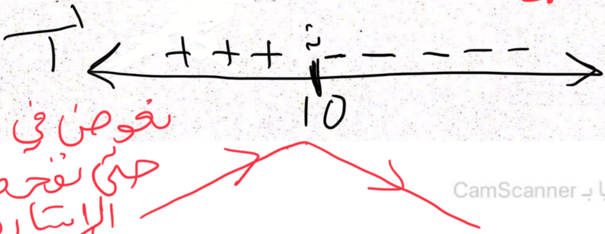
$-q^2-60q+700=0 \xrightarrow{\text{نضرب ب 1 - حتى يسهل التحليل}} q^2+60q-700=0$

$(q+70)(q-10)=0$

$\therefore q=-70$ or $q=10$

مرفوضة لأنها سالب

نحلها على القانون العام أو بالتحليل



نعوض في T' حتى نفحص الإشارة

$T' = 700-60q+700$

$\therefore$ maximum tax revenue $= T(10)$

$$= 700(10) - 30(10)^2 - \frac{1}{3}(10)^3$$

$$= \$3666.67$$

$\therefore$ the tax per unit that maximizes $T = t(10)$

$$= 700 - 30(10) - \frac{1}{3}(10)^2$$

$$= 366.67$$

*Ch12: Indefinite Integrals

التكامل غيرالمحدود

-Sec12.1: Indefinite Integrals.

In our study of the theory of the firm, we have worked with total cost, total revenue and profit functions and have found their marginal functions. In practice, it is often easier for a company to measure marginal functions and use these data to form functions.

يكون من الأسهل أحياناً جمع بيانات عن marginal
ومنها نكوّن اقترانات الربح والعوائد والتكاليف.

- We need to be able to reverse the process of differentiation. This reverse process is called antidifferentiation, or integration.

- The function $F(x)$ is called the antiderivative of $f(x)$ if $F'(x) = f(x)$. → «that is, $\int f(x)dx = F(x)$».

- $\int f(x)dx$:

$f(x)$: is the integrand function.

$\int$: is the integral sign.

dx : indicates that the integral is to be taken with respect to x.

- Ex: If $f'(x) = 3x^2$. What is $f(x)$?

الاقتران الذي مشتقته $3x^2$.

$\longrightarrow \int 3x^2 \, dx = x^3 + C.$; نضيف C لأن مشتقة الثابت = صفر

* Integration Formulas:

① $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C.$; $n \neq -1$, C: the constant of integration. ثابت التكامل

② $\int 1 \, dx = x + C.$

③ $\int c \cdot u(x) \, dx = c \int u(x) \, dx$

④ $\int [u(x) \mp v(x)] \, dx = \int u(x) \, dx \mp \int v(x) \, dx.$

- Ex: Evaluate the following integrals:-

① $\int x^3 \, dx$

$= \dfrac{x^4}{4} + C.$

② $\int \dfrac{1}{\sqrt{x}} \, dx$

$= \int x^{-\frac{1}{2}} \, dx = \dfrac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = \dfrac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = 2\sqrt{x} + C.$

③ $\int \sqrt[3]{x} \, dx$

$= \int x^{\frac{1}{3}} \, dx = \dfrac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C = \dfrac{x^{\frac{4}{3}}}{\frac{4}{3}} + C = \dfrac{3}{4} \sqrt[3]{x^4} + C.$

④ $\int \frac{1}{x^2} dx.$

$= \int x^{-2} dx = \frac{x^{-1}}{-1} + C = \frac{-1}{x} + C.$

⑤ $\int 4 \, dx$

$= 4x + C.$

⑥ $\int (x^3 + 4x) \, dx$

$= \frac{x^4}{4} + 4 \cdot \frac{x^2}{2} + C = \frac{x^4}{4} + 2x^2 + C.$

⑦ $\int (x^2 - 4)^2 dx$

$= \int (x^4 - 4x^2 + 16) dx = \frac{x^5}{5} - 4 \frac{x^3}{3} + 16x + C.$

⑧ $\int \frac{1}{3x^2} dx$

$= \frac{1}{3} \int x^{-2} dx = \frac{1}{3} \cdot \frac{x^{-1}}{-1} + C = \frac{-1}{3x} + C.$

⑨ $\int \frac{x+1}{x^3} dx$

$= \int \frac{x}{x^3} + \frac{1}{x^3} dx$

$= \int (x^{-2} + x^{-3}) dx = \frac{x^{-1}}{-1} + \frac{x^{-2}}{-2} + C = \frac{-1}{x} - \frac{2}{x^2} + C.$

-Sec 12.2: The power rule

قاعدة القوى

***Power rule:**

If $u = u(x)$, then $\int u^n du = \frac{u^{n+1}}{n+1} + C$; $n \neq -1$.

-Ex: Evaluate $\int (3x^2+4)^5 6x \, dx$.

$u' = 6x$

$$= \frac{(3x^2+4)^6}{6} + C.$$

-Ex: Evaluate $\int \sqrt{2x+3} \cdot 2 \, dx$

$$= \int (2x+3)^{\frac{1}{2}} \cdot 2 \, dx$$

$u' = 2$

$$= \frac{(2x+3)^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}(2x+3)^{\frac{3}{2}} + C.$$

الممسوحة ضوئيا بـ CamScanner

-Ex: $\int \underset{\sim}{x^3}\underbrace{(5x^4+11)^9}_{\downarrow\; u'=20x^3} dx.$

$$= \frac{1}{20}\int 20x^3(5x^4+11)^9 dx$$

$$= \frac{1}{20}\cdot\frac{(5x^4+11)^{10}}{10} + C$$

$$= \frac{1}{200}(5x^4+11)^{10} + C.$$

-Ex: $\int 5\underset{\sim}{x^2}\sqrt{\underbrace{x^3-4}_{\downarrow\; 3x^2}}\, dx$

$$= \frac{5}{3}\int 3x^2(x^3-4)^{\frac{1}{2}} dx$$

$$= \frac{5}{3}\cdot\frac{(x^3-4)^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{10}{9}(x^3-4)^{\frac{3}{2}} + C.$$

-Ex: Evaluate $\int (\underbrace{x^2+4}_{\downarrow\; 2\underset{\sim}{x}})^2 dx$

هنا لأن المشتقة x غير موجودة ، نفك الأقواس.

$$= \int (x^4+8x^2+16) dx$$

$$= \frac{x^5}{5} + \frac{8x^3}{3} + 16x + C.$$

-Ex: Evaluate:-

① $\int (\underbrace{2x^2-4x}_{\downarrow \; 4x-4})^2 (x-1)\,dx$

$$= \frac{1}{4}\int (2x^2-4x)^2 \, 4(x-1)\,dx$$

$$= \frac{1}{4} \cdot \frac{(2x^2-4x)^3}{3} + C.$$

$$= \frac{1}{12}(2x^2-4x)^3 + C.$$

② $\int \frac{x^2-1}{(\underbrace{x^3-3x}_{\downarrow \; 3x^2-3})^3}\,dx$

$$= \frac{1}{3}\int 3(x^2-1)(x^3-3x)^{-3}\,dx$$

$$= \frac{1}{3} \cdot \frac{(x^3-3x)^{-2}}{-2} + C.$$

$$= \frac{-1}{6}(x^3-3x)^{-2} + C.$$

$$= \frac{-1}{6(x^3-3x)^2} + C.$$

الممسوحة ضوئيا بـ CamScanner

- **Ex:** Suppose that the marginal revenue for a product is given by $\overline{MR} = \frac{600}{\sqrt{3x+1}} + 2$. **Find the total revenue function.**

$$R(x) = \int \frac{600}{\sqrt{3x+1}} + 2 \, dx$$

$$= \int 600(\underbrace{3x+1}_{3})^{-\frac{1}{2}} \, dx + \int 2 \, dx$$

$$= \frac{600}{3} \int 3(3x+1)^{-\frac{1}{2}} \, dx + 2x + c.$$

$$= 200 \frac{(3x+1)^{\frac{1}{2}}}{1/2} + 2x + c.$$

$$= 400(3x+1)^{\frac{1}{2}} + 2x + c.$$

To find c, We Know that $R(0) = 0$

$$0 = 400(0+1)^{\frac{1}{2}} + 2(0) + c$$

$$0 = 400 + c.$$

$$-400 \qquad -400$$

$$\therefore c = -400$$

$$\longrightarrow R(x) = 400\sqrt{3x+1} + 2x + c.$$

-Sec 12.3: Integrals involving exponential and logarithmic functions

تكاملات الاقترانات الأسية
والتي تتضمن لوغاريتمات في جوابها.

• Exponential formula:

If u is a function of x,

$$\int e^{u} \cdot u' \, dx = e^{u} + C.$$

In particular, $\int e^{x} dx = e^{x} + C.$

-Ex: Evaluate $\int 5e^{x} dx$.

$$\longrightarrow \int 5e^{\boxed{x}} dx = 5\int e^{x} dx = 5e^{x} + C.$$

$u' = 1$

-Ex: Evaluate:

a) $\int 2x e^{x^2} dx$

$$\longrightarrow \int \underline{2x}\, e^{\boxed{x^2}} dx = e^{x^2} + C.$$

$u' = 2x$

b) $\int \frac{x^2}{e^{x^3}} dx$

$$\longrightarrow \int \underline{x^2}\, e^{\boxed{-x^3}} dx = \frac{-1}{3}\int -3x^2 e^{-x^3} dx = \frac{-1}{3} e^{-x^3} + C.$$

$u' = -3x^2$

c) $\int \frac{dx}{e^{3x}}$

$$\rightarrow \int e^{-3x} dx = \frac{1}{-3}\int -3e^{-3x} dx$$

$u' = -3$

$$= \frac{-1}{3}e^{-3x} + C = \frac{-1}{3e^{3x}} + C.$$

- **Logarithmic Formula:**

If u is a function of x, then

$$\int u^{-1} u' \, dx = \int \frac{u'}{u} dx = \ln|u| + C.$$

-Ex: Find $\int \frac{1}{x} dx$

$$\rightarrow \int \frac{1}{x} dx = \ln|x| + C.$$

$u' = 1$

-Ex: Evaluate $\int \frac{4}{4x+8} dx$

$$\rightarrow \int \frac{4}{4x+8} dx = \ln|4x+8| + C.$$

$u' = 4$

الممسوحة ضوئيا بـ CamScanner

-Ex: Evaluate $\int \frac{x-3}{x^2-6x+1}\,dx$

$$\longrightarrow \int \frac{x-3}{\underbrace{x^2-6x+1}_{u'=2x-6}}\,dx = \frac{1}{2}\int \frac{2(x-3)}{x^2-6x+1}\,dx$$

$$= \frac{1}{2}\ln|x^2-6x+1| + C.$$

-Ex: $\int \frac{x^4-2x^3+4x^2-7x-1}{x^2-2x}\,dx$

درجة البسط أكبر من المقام ، إذن نقسم قسمة طويلة

$$\text{الناتج} + \frac{\text{الباقي}}{\text{المقسوم عليه}}$$

$$\begin{array}{r|l} & x^2+4 \\ \hline x^2-2x & \cancel{x^4}-\cancel{2x^3}+4x^2-7x-1 \\ & -\cancel{x^4} \pm \cancel{2x^3} \\ \hline & \cancel{4x^2}-7x-1 \\ & -\cancel{4x^2} \pm 8x \\ \hline & x-1 \quad \text{stop} \end{array}$$

$$= \int (x^2+4)\,dx + \int \frac{x-1}{x^2-2x}\,dx$$

$$= \int x^2+4\,dx + \frac{1}{2}\int \frac{2(x-1)}{\underbrace{x^2-2x}_{u'=2x-2}}\,dx$$

$$= \frac{x^3}{3} + 4x + \frac{1}{2}\ln|x^2-2x| + C.$$

-Sec 12.4: Applications of the Indefinite Integral in Business and Economics:

تطبيقات على التكامل غير المحدود

- If we have the marginal cost function, we can integrate to find the total cost. That is, $C(X)=\int \overline{MC}\,dX$.

تعتمد على الكلف الثابتة

The value of the constant of integration depends on the fixed costs. Thus, we can't determine the total cost from the marginal cost unless additional information is available.

- $R(X)=\int \overline{MR}\,dX$

To find C (the constant of integration) we use $R(0)=0$

- Optimal level of production:- $\overline{MR}=\overline{MC}$

-Ex: Suppose the marginal cost function for a month for a certain product is $\overline{MC}=3X+50$, where X is the number of units. If the fixed costs related to the product amount to \$10000 per month, find the total cost function for the month.

$$\longrightarrow C(X)=\int \overline{MC}\,dX$$

$$=\int 3X+50\,dX$$

$$=\frac{3X^2}{2}+50X+K$$

fixed costs $= C(0) = 10000$.

$\longrightarrow C(0) = \frac{3(0)^2}{2} + 50(0) + K$

$\boxed{10000 = K}$

$\therefore C(x) = \frac{3x^2}{2} + 50x + 10000$

-Ex: Suppose monthly records show that the rate of change of the cost for a product is $\overline{MC} = 3(2x+25)^{\frac{1}{2}}$. If the fixed costs for the month are \$11125, what would be the total cost of producing 300 items per month?

$\longrightarrow C(x) = \int \overline{MC}\, dx$

$= \int 3(\underbrace{2x+25}_{u'=2})^{\frac{1}{2}} dx$

$= \frac{3}{2} \int 2(2x+25)^{\frac{1}{2}} dx$

$= \frac{3}{2} \cdot \frac{(2x+25)^{\frac{3}{2}}}{\frac{3}{2}} + K$

$= (2x+25)^{\frac{3}{2}} + K$

$\longrightarrow$ fixed costs $= C(0) = \$11125$

$C(0) = (25)^{\frac{3}{2}} + K$

$11125 = 125 + K \qquad \therefore \boxed{K = 11000}$

$-125 \quad -125$

$\longrightarrow C(x) = (2x+25)^{\frac{3}{2}} + 11000 \longrightarrow C(300) = (2 \cdot 300 + 25)^{\frac{3}{2}} + 11000$

$= 26625\ \$$

-Ex: Suppose that the marginal revenue from the sale of X units of a product is $\overline{MR} = 6e^{0.01X}$. What is the revenue from the sale of 100 units of the product?

$$\longrightarrow R(X) = \int \overline{MR}\, dX$$

$$= \int 6\, \underset{\substack{\downarrow \\ u' = 0.01}}{e^{\boxed{0.01X}}}\, dX$$

$$= \frac{6}{0.01} \int \underbrace{0.01 e^{0.01X}}\, dX$$

$$= 600 e^{0.01X} + C$$

We Know, $R(0) = 0$

$$0 = 600 \boxed{e^{0.01(0)}} + C \longrightarrow e^{0} = 1$$

$$0 = 600 + C$$

$$\therefore \boxed{C = -600}$$

$$\longrightarrow R(X) = 600 e^{0.01X} - 600$$

-Ex: Given that $\overline{MR} = 200 - 4X$, $\overline{MC} = 50 + 2X$, what is the optimal level of production?

$$\longrightarrow \overline{MR} = \overline{MC}$$

$$200 - 4X = 50 + 2X$$

$$ +4X \qquad +4X$$

$$200 = 50 + 6X$$

$$-50 \quad -50$$

$$\frac{150}{6} = \frac{6}{6} X \qquad \therefore X = 25 \text{ units.}$$

*Ch 13: Definite Integrals

-Sec 13.2: The Definite Integral: The Fundamental theorem of calculus.

التكامل المحدود والنظرية الأساسية في التفاضل والتكامل.

•The Fundamental Theorem of Calculus:-

Let f be a continuous function on the closed interval $[a,b]$; and then the definite integral of f exists on this interval,

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any function such that $F'(x) = f(x)$ for all x in $[a,b]$.

-Ex: Evaluate $\int_2^4 (x^3+4)\,dx$.

$$\longrightarrow \int_2^4 (x^3+4)\,dx = \frac{x^4}{4} + \frac{4x^2}{2}\Big|_2^4$$

$$= (64+32) - (4+8)$$

$$= 96 - 12 = 84$$

***Properties of definite integrals:-**

① $\int_a^b [f(x) \mp g(x)]\,dx = \int_a^b f(x)\,dx \mp \int_a^b g(x)\,dx$.

② $\int_a^b K f(x)\,dx = K \int_a^b f(x)\,dx$.

③ $\int_a^a f(x)\,dx = 0$.

الممسوحة ضوئيا بـ CamScanner

④ $\int_a^b f(x)dx = -\int_b^a f(x)\,dx$

⑤ $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$

- Ex: If $\int_5^2 f(x)dx = -3$ and $\int_2^5 2g(x)dx = 8.$

Find the following:-

$\int_2^5 2g(x)\,dx = 8$
$\rightarrow \int_2^5 g(x)\,dx = 4$

$\int_5^2 f(x)dx = -3$
$\rightarrow \int_2^5 f(x)dx = 3$

a) $\int_2^5 [2f(x) - 3g(x) + x]dx$

$= 2\int_2^5 f(x)dx - 3\int_2^5 g(x)dx + \int_2^5 x\,dx$

$= 2(3) - 3(4) + \left(\frac{x^2}{2}\Big|_2^5\right)$

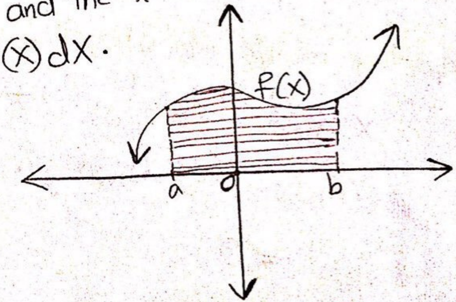
$= 6 - 12 + \left(\frac{25}{2} - \frac{4}{2}\right)$

$= -6 + \frac{21}{2} = \frac{9}{2}$

المساحة تحت المنحنى

***Area Under a curve:**

① If f is continuous on $[a, b]$ and $f(x) \geq 0$ then the area between $y = f(x)$ and the x-axis from $x=a$ to $x=b$ is given by $A = \int_a^b f(x)\,dx$.



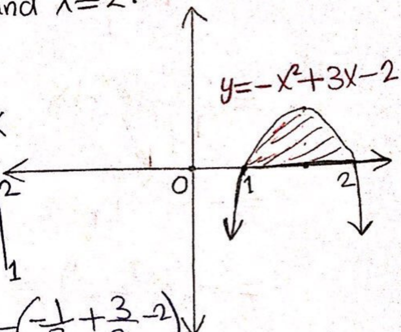
الممسوحة ضوئيا بـ CamScanner

② If $f(x) \leqslant 0$ for all x in $[a, b]$, then:-

$$A = -\int_a^b f(x)\,dx$$

اذا كان جواب التكامل بالسالب، فعكس الإشارة في المساحة حتى تصبح موجبة.

-Ex: Find the area between the curve $y = -x^2 + 3x - 2$ and the x-axis from $x = 1$ and $x = 2$.



$$\longrightarrow A = \int_1^2 (-x^2 + 3x - 2)\,dx$$

$$= \frac{-x^3}{3} + \frac{3x^2}{2} - 2x \Big|_1^2$$

$$= \left(\frac{-8}{3} + \frac{12}{2} - 4\right) - \left(\frac{-1}{3} + \frac{3}{2} - 2\right)$$

$$= \frac{-2}{3} - \frac{-5}{6} = \boxed{\frac{1}{6}}$$

-Ex: Find the area between the curve $y = x e^{x^2}$ and the x-axis from $x = 1$ to $x = 3$.

$$\longrightarrow A = \int_1^3 x e^{x^2}\,dx$$

$$u' = 2x$$

$$= \frac{1}{2} \int_1^3 \underset{\sim}{2x}\, e^{x^2}\,dx$$

$$= \frac{1}{2} e^{x^2} \Big|_1^3 = \frac{1}{2}(e^9 - e).$$

-Ex: Find the area between the curve $y = e^{-x}$ and the x-axis from $x = -1$ to $x = 1$.

$$\longrightarrow A = \int_{-1}^{1} e^{\underbrace{-x}_{u' = -1}} \, dx$$

$$= \frac{1}{-1} \int_{-1}^{1} \underbrace{-1 e^{-x}}\, dx$$

$$= - e^{-x} \Big|_{-1}^{1}$$

$$= -\left(e^{-1} - e^{1}\right) = -e^{-1} + e$$

$$= e - e^{-1}$$

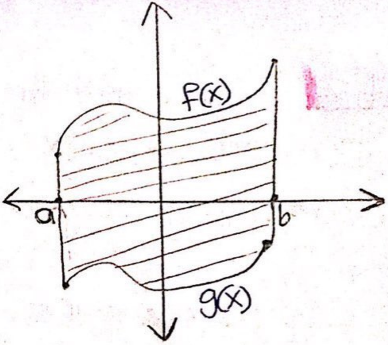
$$= e - \frac{1}{e}.$$

- Sec 13.3: Area between 2 curves:-

المساحة بين منحنيين

* If f and g are continuous functions on $[a,b]$ and if $f(x) \geq g(x)$, then the area of the region bounded by $y=f(x)$, $y=g(x)$, $x=a$ and $x=b$:-

$$\longrightarrow A = \int_a^b [f(x) - g(x)]\, dx$$



- Ex: Find the area enclosed by $y=x^2$ and $y=2x+3$.

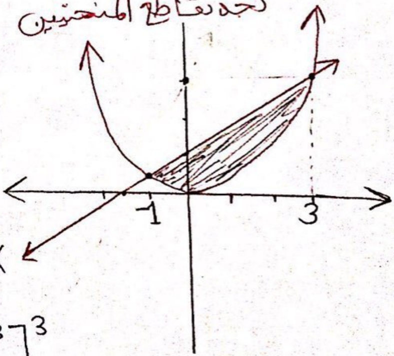
نجد تقاطع المنحنيين

$$\longrightarrow x^2 = 2x+3$$

$$x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0$$

$$x = 3, -1$$



$$\longrightarrow A = \int_{-1}^{3} (2x+3) - x^2 \, dx$$

$$= \left[\frac{2x^2}{2} + 3x - \frac{x^3}{3} \right]_{-1}^{3}$$

$$= (9+9-9) - (1-3+\frac{1}{3}) = \frac{32}{3} \text{ square units.}$$

*Average Value:

متوسط قيمة الاقتران

The average value of a continuous function $y=f(x)$ over the interval $[a,b]$ is:

$$\longrightarrow \text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

-Ex: Suppose that the cost in dollars is given by $C(x)=400+x+0.3x^2$. What is the average value of $C(x)$ for 10 to 20 units?

$$\longrightarrow \text{Average value} = \frac{1}{b-a}\int_a^b C(x)\,dx$$

$$= \frac{1}{20-10}\int_{10}^{20} 400+x+0.3x^2\,dx$$

$$= \frac{1}{10}\left[400x+\frac{x^2}{2}+\frac{0.3x^3}{3}\right]_{10}^{20}$$

$$= \frac{1}{10}\left[(8000+200+800)-(4000+50+100)\right]$$

$$= \frac{1}{10}\left[4850\right] = 485$$

CamScanner الممسوحة ضوئيا بـ

-Sec13.4: Applications of Definite Integrals in Business and Economics

تطبيقات التكامل المحدود.

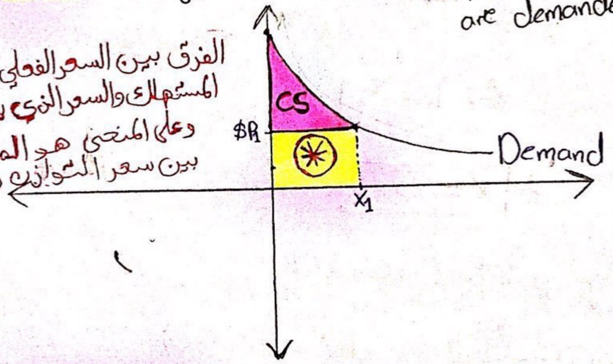
***Consumer's Surplus:**

If the demand has equation $p=f(x)$, the consumer's surplus is given by the area between $f(x)$ and the x-axis from 0 to X_1 minus the area of the rectangle ⊛:-

$$\longrightarrow CS=\int_0^{x_1} f(x)-P_1X_1 .$$

و P_1: the corresponding price if X_1 units are demanded.

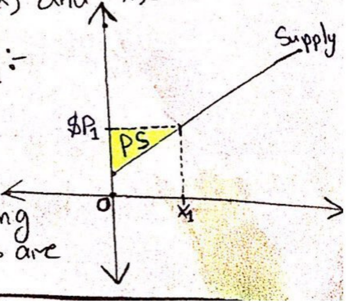
الفرق بين السعر الفعلي الذي يدفعه المستهلك والسعر الذي ينوي دفعه وعلى المنحنى هو المنطقة الواقعة بين سعر التوازن ومنحنى الطلب



***Producer's Surplus:**

If the supply function is $p=g(x)$, the producer's surplus is given by the area between $g(x)$ and the x-axis from 0 to X_1 subtracted from P_1X_1:-

$$\longrightarrow PS=P_1X_1-\int_0^{X_1} g(x)$$



و P_1: the corresponding price if X_1 units are supplied.

الفرق بين المبلغ الذي يكون الشخص على استعداد لقبوله مقابل كمية معينة وبين المبلغ الذي تباع به السلعة.

- Ex: The demand function for x units of a product is $p = \frac{1020}{x+1}$ dollars. If the equilibrium price is \$20, what is the consumer's surplus?

$\longrightarrow$ Equilibrium point: $p = \$20$

$$20 = \frac{1020}{x+1}$$

$$\frac{20(x+1)}{20} = \frac{1020}{20}$$

$$\begin{aligned} x+1 &= 51 \\ -1 \quad & \quad -1 \end{aligned}$$

$$\therefore \boxed{x = 50}$$

$\therefore$ Eq point $= (50, 20)$

$$\longrightarrow CS = \int_0^{50} \frac{1020}{x+1}\,dx - 20(50)$$

$$= 1020\int_0^{50} \frac{dx}{\underbrace{x+1}_{u' = 1}} - 1000$$

$$= 1020\ln(x+1)\Big|_0^{50} - 1000$$

$$= 1020\ln(51) - \underbrace{\cancel{1020\ln 1}}_{0} - 1000$$

$$= 1020\ln(51) - 1000$$

-Ex: The demand function for a product is $p=280-4x-x^2$ - and the supply function for it is $p=160+4x+x^2$. Find the producer's surplus at the equilibrium point.

→ Equilibrium point:

$$D=S$$

$$280-4x-x^2=160+4x+x^2$$

$$-280+4x+x^2 \quad -280+4x+x^2$$

$$0=2x^2+8x-120$$

$$a=2,\ b=8,\ c=-120$$

$$x=\frac{-8\mp\sqrt{64-4(2)(-120)}}{4}$$

$$=\frac{-8\mp 32}{4} \begin{cases} \rightarrow \dfrac{-8+32}{4}=6 \\ \rightarrow \dfrac{-8-32}{4}=-10\ ✗. \end{cases}$$

∴ Equilibrium point $=(6,\underset{\downarrow}{220})$

نعوض $X=6$ في معادلة
D أو S.

→ $PS=6(220)-\int_0^6(160+4x+x^2)\,dx$

$$=1320-\left[160x+2x^2+\frac{x^3}{3}\right]_0^6$$

$$=1320-[960+72+72]=216$$

-Ex: Suppose a monopoly has its total cost for a product given by $C(X) = 60 + 2X^2$. Suppose also that demand is given by $p = 30 - X$, where p is in dollars and X is the number of units. Find the consumer's surplus at the point where the monopoly has maximum profit.

$\longrightarrow$ We must first find the point where the profit function is maximized.

$P(X) = R(X) - C(X) \quad ; \quad R(X) = pX = (30-X)X$

$= (30X - X^2) - (60 + 2X^2)$

$= 30X - 3X^2 - 60$

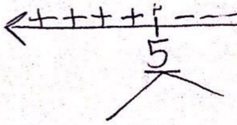
$P'(X) = 30 - 6X$

- $P'(X) = 0$

$30 - 6X = 0$

$-30 \qquad -30$

$\frac{-6X}{-6} = \frac{-30}{-6} \qquad \therefore X = 5$



$\therefore$ max. profit at

$X = 5$

$p = 30 - 5 = 25$

$\longrightarrow CS = \int_0^5 (30 - X)\,dX - 5(25)$

$= 30X - \frac{X^2}{2} \Big|_0^5 - 125$

$= 150 - \frac{25}{2} - 125 = 12.5$