10.1 Relative Maxima and Minima: C

$10.1 = 8 + 18 + 20 + 29 + 36 + 50 + 53$

8 $y = x^3 - 3x^2 + 6x + 1$

$y' = 3x^2 - 6x + 6$

$y' = 0$ or y' ~~und.~~

$y' = 0 \Rightarrow$ البسط = صفر

y' is undefined $\Rightarrow$ المقام = صفر

$3x^2 - 6x + 6 = 0$

$x^2 - 2x + 2 = 0$

~~$(x \quad 1)(x \quad 2)$~~

$\text{Discriminant} = (-2)^2 - 4(1)(2)$

$= 4 - 8 = -4$

No solution

So No critical values.

$\longleftarrow + + \ + \ + \ + \longrightarrow$ sign diagram of y'

y is increasing on $(-\infty, \infty)$

No Rel. Maximum

No Rel. Min.

18 $y = \frac{x^4}{4} - \frac{x^3}{3} - 2$

$$\boxed{\begin{aligned} \bar{y} &= 4\frac{x^3}{4} - 3\cdot\frac{x^2}{3} \\ \bar{y} &= x^3 - x^2 \end{aligned}}$$

$\bar{y} = x^3 - x^2$

$\bar{y} = 0$ or $\bar{y}$ ~~undefined~~

$x^3 - x^2 = 0$

$x^2(x-1) = 0$

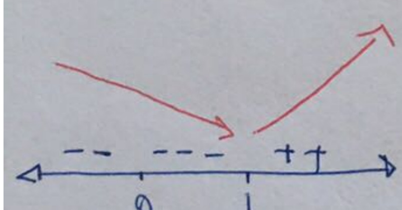
$x = 0,\ x = 1$

How many Critical values? 2.

what are the critical values? $x = 0,\ x = 1$

what are the critical points? $(0, f(0)) = (0, -2)$

$(1, f(1)) = (1, \frac{1}{4} - \frac{1}{3} - 2)$



$\bar{y}$ [sign diagram of $\bar{y}$]

y is increasing on $[1, \infty)$

y is decreasing on $(-\infty, 1)$

at $x = 1$, we have Relative Minimum.

at $x = 0$, Neither maximum nor minima

$x = 0$ Horizontal point of Inflection

$y = -(x-3)^{\frac{2}{3}}$

$y' = -\frac{2}{3}(x-3)^{-\frac{1}{3}}$

$$\boxed{y' = \frac{-2}{3\sqrt[3]{x-3}}}$$

$\cancel{y'=0}$ or y' is undefined

$3\sqrt[3]{x-3} = 0$

$\left(\sqrt[3]{x-3}\right)^3 = (0)^3$

$x-3=0$

$x=3$

How many critical values? 1

what is the critical value? $x=3$

what is the critical point? $(3, f(3)) = (3,0)$

$f(3) = y\big|_{x=3} = -(3-3)^{2/3} = 0$

$$\begin{array}{ccc} + + + & & - - - \\ \hline & 3 & \end{array} \quad y'$$

y is increasing on $(-\infty, 3)$

y is decreasing on $(3, \infty)$

$(3, 0)$ Relative maximum point.

$\boxed{29}$ $y = 3x^5 - 5x^3 + 1$

$y' = 15x^4 - 15x^2$

$y' = 0$ or $\cancel{y' \text{ und.}}$

$15x^4 - 15x^2 = 0$

$15x^2 [x^2 - 1] = 0$

$15x^2 = 0$ or $x^2 - 1 = 0$

$x = 0$ $\quad$ $x = \pm 1$

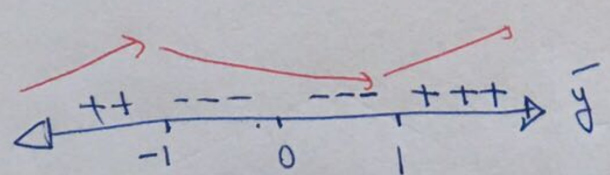
How many critical values? 3

What are the critical values? $x = 0$, $x = 1$, $x = -1$

What are the critical points? $(0, f(0)) = (0, 1)$

$(1, f(1)) = (1, -1)$

$(-1, f(-1)) = (-1, 3)$



y is increasing on $(-\infty, -1) \cup (1, \infty)$

y is decreasing on $(-1, 1)$

at $(-1, 3)$ Relative maximum point

$(1, -1)$ Relative Minimum

$(0, 1)$ Horizontal point of Inflection (No max. No min.

36) $f(x) = x - 3x^{2/3}$, $\quad f'(x) = \dfrac{x^{\frac{1}{3}} - 2}{x^{\frac{1}{3}}}$

$f' = 0$ or f' undefined

$x^{\frac{1}{3}} - 2 = 0$ or $x^{\frac{1}{3}} = 0$

$(x^{\frac{1}{3}})^3 = (2)^3$ $\qquad x = 0$

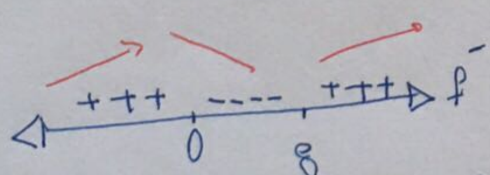
$x = 8$

How many critical values? 2

what are the critical values? $x = 8, x = 0$

what are the critical points? $(0, f(0)) = (0, 0)$

$(8, f(8)) = (8, -4)$



f is increasing on $(-\infty, 0) \cup (8, \infty)$

f is decreasing on $(0, 8)$

$(0, 0)$ Relative maximum point

$(8, f(8)) = (8, -4)$ Relative Minimum point

$y = \frac{90}{\sqrt{P+5}}$, $P > 10$

$y = 90(P+5)^{-\frac{1}{2}}$

y = sales volume

P = price.

$y' = 90(-\frac{1}{2})(P+5)^{-\frac{3}{2}} \cdot 1$

$\boxed{y' = \frac{-45}{(P+5)^{\frac{3}{2}}}}$

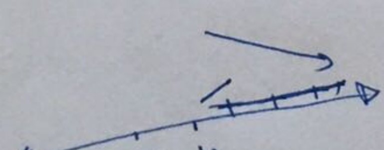
$y' \cancel{=} 0$ or y' undefined

$(P+5)^{3/2} = 0$

$\left((P+5)^{\frac{3}{2}}\right)^{\frac{2}{3}} = 0$

$P+5=0$

$\boxed{P=-5}$ refused



No critical values

~~y~~ (sales volume ~~incre~~ decreases as P increase.)

10.2 Concavity & Points of Inflection

1+17+22+37

① $f(x) = x^3 - 3x^2 + 1$

at a) $x = -2$

b) $x = 3$

$f'(x) = 3x^2 - 6x$

$f''(x) = 6x - 6$

a) $f''(-2) = 6(-2) - 6$

$= -12 - 6 = -18$ ⊖

$f(x)$ is concave down.

b) $f''(3) = 6(3) - 6 = 12$ ⊕

$f(x)$ is concave up

⑰ $y = x^4 - 16x^2$

$y' = 4x^3 - 32x$

$y'' = 12x^2 - 32$

$y' = 0$ or y' is undefined

X

$4x^3 - 32x = 0$

$4x[x^2 - 8] = 0 \Longrightarrow x = 0$

$x^2 - 8 = 0 \to x^2 = 8$

$x = \pm\sqrt{8}$

How many critical values? 3

What are critical values! $x = 0, x = \sqrt{8}, x = -\sqrt{8}$

$y = x^4 - 16x^2$ / $y' = 4x^3 - 32x = 4x[x^2 - 8]$

What are the critical points $(0, f(0)) = (0, 0)$

$(\sqrt{8}, -64)$

$(-\sqrt{8}, -64)$

$f(\sqrt{8}) = (\sqrt{8})^4 - 16(\sqrt{8})^2$

$= 64 - 16 * 8$

$= 16[4 - 8] = 16[-4] = -64$

$-- \quad +++ \quad --- \quad +++ \quad f'$

$-\sqrt{8} \quad 0 \quad \sqrt{8}$

f is increasing on $(-\sqrt{8}, 0) \cup (\sqrt{8}, \infty)$

f decreasing on $(-\infty, -\sqrt{8}) \cup (0, \sqrt{8})$

$(-\sqrt{8}, -64)$, $(\sqrt{8}, -64)$ are relative minimum points

$(0, 0)$ Relative maximum point

$f''(x) = 6x - 6$

$f''(x) = 0$ or f'' is ~~undefined~~

$6x - 6 \leq 0$

How many inflection points?
at most 1 $[\ \leq 1\ ]$

→ $6x = 6$

$x = 1$

$\cap \quad \cup$

$--- \quad +++ \quad f''$

1

f concave up on $(1, \infty)$

f concave down on $(-\infty, 1)$

$(1, f(1)) = (1, -15)$ is an inflection point

$\boxed{22}$ $y = x^{\frac{4}{3}}(x-7)$

$y' = \dfrac{7x^{\frac{1}{3}}(x-4)}{3}$

$y'' = \dfrac{28(x-1)}{9x^{2/3}}$

$y' = 0$ or y' is undefined

$7x^{\frac{1}{3}}(x-4) = 0$

$7x^{\frac{1}{3}} = 0$ or $x - 4 = 0$

$\boxed{x = 0}$ $\boxed{x = 4}$

$\Downarrow$

$7x^{\frac{1}{3}} = 0$

$\left(x^{\frac{1}{3}}\right)^3 = (0)^3 = 0$

$x = 0$

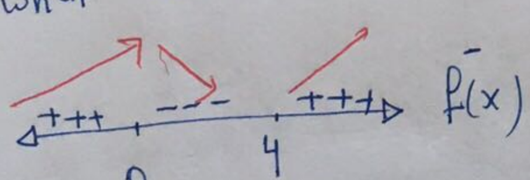
How many critical values? 2

What are the critical values? $x = 0$, $x = 4$

What are the critical points? $(0, f(0)) = (0,0)$

$(4, f(4))$

what are the



f is increasing on $(-\infty, 0) \cup (4, \infty)$

f is decreasing on $(0, 4)$

$(0, f(0)) = (0,0)$ Relative maximum.

$(4, f(4))$ Relative Minimum

$\cancel{f = x^{\frac{4}{3}}}$

$9x^{2/3} = 9\left(x^{\frac{1}{3}}\right)^2$

$f'' = \dfrac{28(x-1)}{9x^{2/3}}$

$f'' = 0$ or f'' und.

$28(x-1) = 0$

$x - 1 = 0$

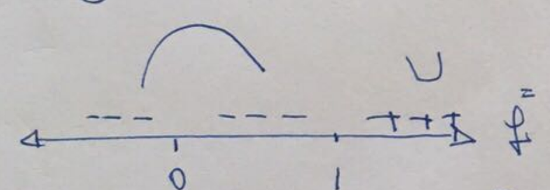
$x = 1$

$9x^{2/3} = 0$

$\left(x^{2/3}\right)^{3/2} = (0)^{3/2} = 0$

$x = 0$

How many inflection points? ≤ 2 [at most 2]



f concave down on $(-\infty, 1)$

f concave up on $(1, \infty)$

we have only one inflection point. which is $(1, f(1))$

$(1, -6)$

$\boxed{0}$. we don't have inflection points

10.3: Optimizations in Business and Economics

4 + 8 + 10 + 18 + 28 + 36 + 39 + 41

4 Find the absolute maximum and minimum of

$f(x) = x^3 - x^2 - x, \quad [-0.5, 2]$

Solution:

$f'(x) = 3x^2 - 2x - 1$

$f'(x) = 0$ or $f'(x)$ undefined

X

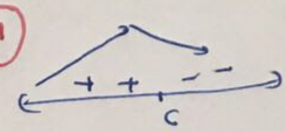
$3x^2 - 2x - 1 = 0$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-\,-2 \pm \sqrt{(-2)^2 - 4(3)(-1)}}{2(3)}$

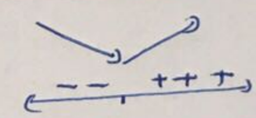
$= \dfrac{2 \pm \sqrt{4+12}}{6} =$

$= \dfrac{2 \pm 4}{6}$ → $\dfrac{6}{6} = 1$; $\dfrac{2-4}{6} = -\dfrac{1}{3}$

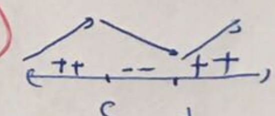
Remember

1


abs max. at $x = c$

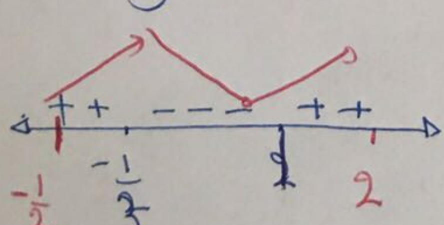
2


abs. min at $x = c$

3


can't be decided

How many critical values? $\underline{\underline{2}}$



at $x = -\frac{1}{2}$ → Rel. minimum point

$\left(-\frac{1}{2}, f\left(-\frac{1}{2}\right)\right) = \left(-\frac{1}{2}, 0.125\right)$

at $x = -\frac{1}{3}$, Rel. maximum point which is $\left(-\frac{1}{3}, f\left(-\frac{1}{3}\right)\right) = \left(-\frac{1}{3}, 0.185\right)$

at $x = 1$, Rel. min. point $(1, f(1)) = (1, -1)$

at $x = 2$, Rel. max. point $(2, f(2)) = (2, 2)$

el. maximum points are $\left(\frac{-1}{3}, 0.185\right)$

$(2, 2) \Rightarrow$ abs. maximum

Rel. Minimum points are $(1, -1) \Rightarrow$ abs. minimum

$\left(\frac{-1}{2}, 0.125\right)$

(8) $R(x) = 2800x - 8x^2 - x^3$

Find max. Revenue from sales

Solution:-

$R'(x) = 2800 - 16x - 3x^2$

$R'(x) = 0$ or $R'(x)$ is ~~undefined~~

$2800 - 16x - 3x^2 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4(a)(c)}}{2a} = \frac{-\,-16 \pm \sqrt{(-16)^2 - 4(-3)(2800)}}{2(-6)}$$

$$= \frac{16 \pm \sqrt{256 + 33600}}{-12} = \frac{16 \pm 184}{-12} = \begin{cases} \cancel{-16.667} \\ 14 \end{cases}$$

$$\underset{0}{|}\; + + + + \;\underset{14}{|}\; - - - - - \rightarrow R'(x)$$

at $x = 14$ we have maximum Revenue, which is $R(14)$

$R(14) = 2800(14) - 16(14) - 3(14)^2 = 34888$ \$

---

or $R'' = \cancel{\ldots}$ $R'' = -16 - 6X$

$R''(14) = -16 - 6(14) = -16 - 84 = -100$ so maxim

Second Derivative Test for extreme Val... ③

Remember: second Derivative Test

① c: critical value.

② $\boxed{f''(c): +}$ Rel. minimum

$\boxed{f''(c): -}$: Rel. maximum

$\boxed{f''(c) = 0}$ test Fails

True or False.

① If $f''(10) = 4$, then at $x = 10$ we have Rel. minimum, ~~at~~

② If $f''(10) = 4$ and $f'(10) = 0$. So we have Rel. minimum at $x = 10$

An agency charges \$100 per person for a trip to a concert if 70 people travel in a group. But for each person above the 70, the charge will be reduced by \$1. How many people will maximize the total Revenue for the agency. if the trip is Limited to at most 90 people?

Solution:

# of people	money
70	100
70+1	100−1
70+2	100−2
⋮	
70+x	100−x

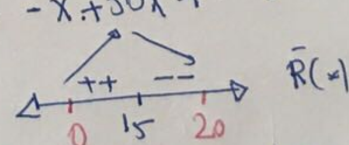
$R(x) = (70+x)(100-x) = 7000 + 100x - 70x - x^2 = -x^2 + 30x + 7000$

$R(x) = -x^2 + 30x + 7000$

$R'(x) = -2x + 30$

$R'(x) = 0$ or $R'(x)$ is ~~und~~.

$-2x + 30 = 0 \rightarrow \boxed{x = 15}$ we have to check →



at $x = 15$, we have Rel. max.

~~R(15)~~.

So $70 + 15 = 85$ will maximize the profit

(18) $C(x) = 250 + 6x + 0.1x^2$, $x \geq 0$

* Producing how many units will minimize the average cost?
* what is the minimum average cost

$\bar{C}(x) = \text{average cost} = \frac{C(x)}{x} = \frac{250 + 6x + 0.1x^2}{x} = \frac{250}{x} + \frac{6x}{x} + \frac{0.1x^2}{x}$

$$\boxed{\bar{C}(x) = \frac{250}{x} + 6 + 0.1x, \; x > 0}$$

$\bar{C}(x) = 250x^{-1} + 6 + 0.1x$

$\bar{C}'(x) = 250(-1)x^{-2} + 0.1$

$= \frac{-250}{x^2} + 0.1$

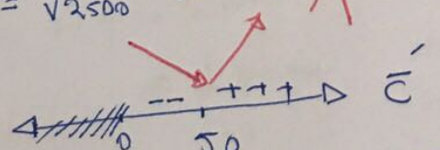
$\bar{C}'(x) = \frac{-250 + 0.1x^2}{x^2}$

$\bar{C}'(x) = 0$ or $\bar{C}'(x)$ is undefined

$-250 + 0.1x^2 = 0$ or $x^2 = 0 \rightarrow \cancel{x = 0}$

$x^2 = \frac{250}{0.1} = 2500$

$x = \sqrt{2500} = 50, \cancel{-50}$



* at $x = 50$, minimum average cost
* minimum average cost $= \bar{C}(50) = \frac{250}{50} + 6 + 0.1(50) = 5 + 6 + 5 = 16$

To check that at $x = 50$, we have minimum, we can use the second derivative test.

$\bar{C}''(x) = -250(-2)x^{-3} = \frac{500}{x^3}$

$\bar{C}''(50) = \frac{500}{(50)^3}$: $\oplus$ minimum

28 $C(x) = 800 + 100x^2 + x^3$

$R(x) = 60\,000x - 50x^2$

Find the level of production that will maximize the profit and find the max. profit.

Solution:

$P(x) = R(x) - C(x)$

$= 60000x - 50x^2 - (800 + 100x^2 + x^3)$

$= 60000x - 50x^2 - 800 - 100x^2 - x^3$

$\boxed{P(x) = -x^3 - 150x^2 + 60\,000x - 800}$

$P'(x) = -3x^2 - 300x + 60000$

$P'(x) = 0$ or $P'(x)$ is ~~undefine~~

$-3x^2 - 300x + 60000 = 0$

$\rightarrow x^2 + 100x - 20\,000 = 0$

$(x + 200)(x - 100) = 0$

$\rightarrow x = \cancel{-200}$ $x = 100$

* $P''(x) = -6x - 300$

* $P''(100) = -6(100) - 300 = -900$.

we have max. at $P(x)$, what is the max profit

T. $P(100) = -(100)^3 - 150(100)^2 + 60000(100) - 800$

$= 3499200$ \$

36 $\bar{C} = \frac{100}{x} + 30 + \frac{x}{10}$

price is 46\$ per unit

production is Limited to 150 units

Find level of production that yields max. profit.

Find max. profit.

Solution:

$$\bar{C}(x) = \frac{C(x)}{x} \Rightarrow C(x) = x \cdot \bar{C}(x)$$

$$C(x) = x\left[\frac{100}{x} + 30 + \frac{x}{10}\right] = 100 + 30x + \frac{x^2}{10}.$$

$$\boxed{C(x) = 100 + 30x + \frac{x^2}{10}}$$

Revenue: $R(x) = P \cdot x$

$$\boxed{R(x) = 46 \cdot x}$$

$$P(x) = R(x) - C(x)$$

$$= 46x - \left(100 + 30x + \frac{x^2}{10}\right)$$

$$P(x) = 46x - 100 - 30x - \frac{x^2}{10}$$

$$\boxed{P(x) = -\frac{x^2}{10} + 16x - 100}$$

$$P'(x) = -\frac{2x}{10} + 16$$

$$P'(x) = -\frac{x}{5} + 16$$

$P'(x) = 0$ or $P'(x)$ is undefined

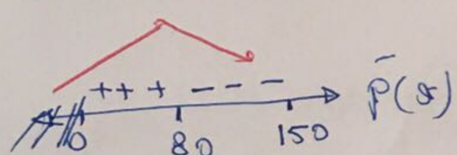
$-\frac{x}{5} + 16 = 0$

$\rightarrow -\frac{x}{5} = -16$

$\rightarrow -x = -80$

$\boxed{x = 80}$

at $x = 80$ we might have a relative max



at $x = 80$ units, profit is max.

max profit is $P(80) = -\frac{80^2}{10} + 16(80) - 100$

$= 540\$$

$\boxed{39}$ $\cdot P = 1960 - \frac{1}{3}x^2$ [Demand]

$\bar{C} = 1000 + 2x + x^2$

production is Limited to 1000 units and x is in hundreds of units

a) Find the quantity that will maximize profit

b) Find max. profit

Solution:

$R(x) = P \cdot x = \left(1960 - \frac{1}{3}x^2\right) x = 1960x - \frac{1}{3}x^3$

$$\boxed{R(x) = 1960x - \frac{1}{3}x^3}$$

$C(x) = x \cdot \bar{C}(x) = x\left[1000 + 2x + x^2\right]$

$= 1000x + 2x^2 + x^3$

$$\boxed{C(x) = 1000x + 2x^2 + x^3}$$

Profit $= P(x) = 1960x - \frac{1}{3}x^3 - \left[1000x + 2x^2 + x^3\right]$

$= 1960x - \frac{1}{3}x^3 - 1000x - 2x^2 - x^3$

$$\boxed{P(x) = -\frac{4}{3}x^3 + 960x - 2x^2}$$

$P'(x) = -4x^2 + 960 - 4x$

$\bar{P}'(x) = 0$ or $\bar{P}'(x)$ is ~~undefin~~

$-4x^2 - 4x + 960 = 0$

$x^2 + x - 240 = 0$

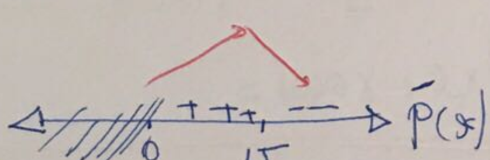
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1 - 4(1)(-240)}}{2}$$

$$= \frac{-1 \pm \sqrt{961}}{2} = \frac{-1 \pm 31}{2}$$

$\rightarrow \frac{-1+31}{2} = \frac{+30}{2} = +15$

$\rightarrow \frac{-1-31}{2} = \frac{-32}{2} = \cancel{-16}$



at $x = 15$, ~~Relative~~ abs. max. profit which is

$$P(15) = -\frac{4}{3}(15)^3 + 960(15) - 2(15)^2$$

$$= 9450 \text{ \$}$$

instead of this we can use the 2nd derivative test for max. min.

$P''(x) = -8x - 4$

$P''(15) = -8(15) - 4 = -124$ so max.