Chapter 1 : The Logic and Language of Proofs.

1.1 Propositions

Definition:- * Aproposition or astatement is a sentence that is either true or false but not both.

*propositions are denoted by Capital letters P, q, r, ...

* The truth value of aproposition is whether it is true or false-

Examples:-

P: $1+2=5$ is a proposition and it is false -

q: $5>2$ is aproposition and it is true -

r: math 243 is an easy course is not aproposition

s: number of BZUniversity students is 15251 is aproposition

t: Batool was born in November 15, 2001 is a proposition.

Logical Connectives: $\sim$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$

Definition:- from simple propositions say p, r, s, --- we can define other propositions using logical Connictives

Definition: if p, q are propositions then $\sim p$ (not p), $p \wedge q$ (p and q), $p \vee q$ (p or q) $\rightarrow$ (if ... then ...) and $p \leftrightarrow q$ (p if and only if) are defined as follows.

p	q	$\sim p$	$p \wedge q$	$p \vee q$	$p \to q$	$p \leftrightarrow q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Example:- which of the following compound statements is true

1- if $1+2=5$ then $5>2$

2- $1+2=5$ and $5>2$

3- $1+2=5$ or $5>2$

4- if $5>2$ then $1+2=5$

5- $1+2=5$ iff $5>2$.

6- $1+2=5$ iff $2>5$

Solution suppose $p: 1+2=5$, $q: 5>2$. $r: 2>5$ then p is False, q is true, r is false.

then 1. if $1+2=5$ then $5>2$
$F \Rightarrow T$ is true statement.

2. $1+2=5$ and $5>2$ the statement is false
$F \wedge T$

3. $1+2=5$ or $5>2$ is true statement.
$F \vee T$

4. if $5>2$ then $1+2=5$
$T \rightarrow F$ is false statement.

5. $1+2=5$ iff $5>2$
F iff T is false statement.

6. $1+2=5$ iff $2>5$ is true statement.
F iff F

Defn:- a) the converse of $p \rightarrow q$ is $q \rightarrow p$.

b) the contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$.

Example:- write converse and contrapositive of: if Omar is in math 243 then Omar is smart

solution:- Let p: Omar is in math 243.
q: Omar is smart.

then the given statement is $p \rightarrow q$.

* So Converse of $P \to q$ is $q \to P$ that is
if Omar is smart then Omer is in math 243

* and the contrapositive of $P \to q$ is $\sim q \to \sim P$
that is if Omer is not smart then Omar is not
in math 243

1.2 Expressions and Tautologies

Definition :- Two compound statements are equivalent
iff they have same truth value

Example:- $\sim(P \wedge q)$ is equivalent to $(\sim P) \vee (\sim q)$
and we write $\sim(P \wedge q) \equiv (\sim P) \vee (\sim q)$.

Since Columns * and ** are identical

P	q	$P \wedge q$	$\sim P$	$\sim q$	$\sim(P \wedge q)$ *	$(\sim P) \vee (\sim q)$ **
T	T	T	F	F	F	F
T	F	F	F	T	T	T
F	T	F	T	F	T	T
F	F	F	T	T	T	T

and we write $\sim(P \wedge q) \equiv (\sim P) \vee (\sim q)$

Defn: * A tautology is a compound statement that is always true

* A contradiction is a compound statement that is always false.

Example: which of the following is a tautology and which is a contradiction

a) $(P \to q) \leftrightarrow (\sim P) \vee q)$

b) $(P \wedge q) \wedge \sim P$.

answer

P	q	$P \to q$	$\sim P$	$(\sim P) \vee q$	$(P \to q) \leftrightarrow (\sim P) \vee q$	$P \wedge q$	$(P \wedge q) \wedge \sim P$
T	T	T	F	T	T	T	F
T	F	F	F	F	T	F	F
F	T	T	T	T	T	F	F
F	F	T	T	T	T	F	F

$(P \to q) \leftrightarrow (\sim p \vee q)$ is a tautology since its truth value is always true.

But $(P \wedge q) \wedge \sim P$ is a contradiction since its truth value is always false.

propositions

Each of the following are equivalence.

a) $\sim(\sim P) \equiv P$.

b) $P \to q \equiv \sim P \vee q$.

c) $\sim(P \to q) \equiv P \wedge (\sim q)$

d) $\sim(P \wedge q) \equiv (\sim P) \vee (\sim q)$ } Demorgans lows.

e) $\sim(P \vee q) \equiv (\sim P) \wedge (\sim q)$

f) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

g) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ } distributive lows.

h) $p \vee (q \vee r) \equiv (p \vee q) \vee r$

i) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$ } Associative

Proposition

Each of the following is a toutology.

a) $p \rightarrow p$

b) $(p \leftrightarrow q) \rightarrow (q \leftrightarrow p)$

c) $(p \leftrightarrow q) \leftrightarrow (\sim p \leftrightarrow \sim q)$

d) $[(p \rightarrow q) \wedge (q \rightarrow p)] \leftrightarrow (p \leftrightarrow q)$

e) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

f) $[(p \leftrightarrow q) \wedge (q \leftrightarrow r)] \leftrightarrow (p \leftrightarrow r)$

g) $\sim (p \vee q) \leftrightarrow (\sim p) \wedge (\sim q)$

h) $\sim (p \wedge q) \leftrightarrow (\sim p) \vee (\sim q)$

i) $[p \wedge (q \vee r)] \leftrightarrow [(p \wedge q) \vee (p \wedge r)]$

j) $[p \vee (q \wedge r)] \leftrightarrow [(p \vee q) \wedge (p \vee r)]$

k) $[p \vee (q \vee r)] \leftrightarrow [(p \vee q) \vee r]$

l) $[p \wedge (q \wedge r)] \leftrightarrow [(p \wedge q) \wedge r]$

proposition: Each of the following is a tautology.

a) $(p \rightarrow q) \wedge [p \wedge (\sim q)]$

b) $[(p \vee q) \wedge \sim p] \wedge (\sim q)$

c) $(p \wedge q) \wedge (\sim p)$

The abbreviation "iff" is sometimes used for "if and only if." An example of a biconditional proposition is "The sun is shining if and only if Wayne is playing golf." In order for this biconditional proposition to be true, the proposition "If the sun is shining, then Wayne is playing golf" and its converse must be true.

A biconditional proposition "P if and only if Q" involves two conditional propositions, "If P, then Q" and "If Q, then P." Note that "P if and only if Q" is true whenever "If P, then Q" and "If Q, then P" are both true; and "P if and only if Q" is false whenever either "If P, then Q" or "If Q, then P" is false.

EXERCISES 1.1

1. Which of the following are propositions?

a) $2^2 + 3^2 = 17$

b) $8x^3 + 6x^2 - 4x + 2$

c) If n is a positive integer, then the sum of the first n positive integers is given by $n(n+1)/2$.

d) Will you marry me?

e) No, I am already married to Leslie!

2. Use the definition of $\rightarrow$ to determine the conditions under which each of the following compound propositions is true.

a) If $8 > 5$, then $3 < 1$.

b) If $a = 2$, then $1 < 3$.

c) If $5 > 8$, then $3 < 1$.

d) If $1 < 3$, then $a = 2$.

e) If $1 > 3$, then $a = 2$.

3. Identify the hypothesis and the conclusion in each of the following compound propositions.

a) If Mary is 24 years old, then I am a monkey's uncle.

b) n^2 is odd whenever n is an odd integer.

c) That r is a rational number implies that r^2 is rational.

d) When a is irrational, $a^2 + a$ is irrational.

e) When a is rational and b is irrational, $a + b$ is irrational.

f) If a is irrational, then a^2 and $2a$ are irrational.

g) In order to pass the driver's test, the candidate must be able to parallel park.

? **h)** In order to pass the vision test, it is sufficient for the candidate to read the line Q S Z P W M 4.

4. Assume that "Mary is a girl" is a true statement and that "Mary is ten years old" is a true statement. Which of the following are true?

a) If Mary is ten years old, then Mary is a girl.

b) Mary is ten years old if and only if Mary is a girl.

5. Assume that "Joe is a girl" is a false statement and that "Mary is ten years old" is a true statement. Which of the following are true?

   a) If Mary is ten years old, then Joe is a girl.

   b) If Joe is a girl, then Mary is ten years old.

6. Assume that "Joe is a girl" is a false statement and that "Joe is ten years old" is a false statement. Which of the following are true?

   a) Joe is ten years old or Joe is a girl.

   b) If Joe is ten years old, then Joe is a girl.

   c) Joe is ten years old if and only if Joe is a girl.

   d) Joe is not a ten year-old girl.

7. Write the converse and the contrapositive of each of the following propositions.

   a) If $\sqrt{2} < \sqrt{5}$, then $2 < 5$. b) If $2 \geqslant 5$, then $\sqrt{2} \geqslant \sqrt{5}$.

8. On a certain island (Manhattan), the inhabitants are divided into two types, those who always tell the truth and those who always lie. One day a visitor stops three inhabitants of the island to ask directions to a well-known museum (The Guggenheim). "All three of us are liars," warns the first inhabitant, "Not so; only two of us are liars," says the second. "Not so," says the third, "the other two guys are lying." Which, if any, of the three islanders can the visitor trust to give honest directions?

9. A visitor to an island whose inhabitants *always* tell the truth or *always* lie encounters an islander to makes the following two statements.

   1. I love Bertha.

   2. If I love Bertha, then I love Sally Lou.

   Does the islander love Bertha? Does he love Sally Lou? Argue persuasively in a sentence or two that your assessment is correct.

10. Jimmy the Greek has been asked to give odds on two basketball games to be played by an obscure college we will call "Tech." Tech is to play State U and Southern U. Although Jimmy is quite knowledgeable about the Boston Celtics, the truth is that he has not scouted the Tech team in years. It is suggested that Jimmy should consult the oracles, which he does. The oracle at Delphi says, "Tech will beat State U or Tech will beat Southern U." Jimmy is quite pleased with this information until someone mentions to him that the oracles are known for their equivocal sayings and that he should seek a second opinion just to be on the safe side. He does. The oracle at Xanthi says, "If Tech doesn't beat State U, then Tech will beat Southern U." Now Jimmy is distraught. He has the vague feeling that one of the oracles is saying more than the other, but he's not sure. Straighten Jimmy out. [*Authors' remark*: We hope that the word ORacle reminds you of the word OR.]

11. William Sessions, Director of the F.B.I., addressed the National Press Club in September 1988 and was asked a final question, which he promised to answer: "As a Texan, would you like to see a Texan President or a Texan Vice-President?" Bearing in mind that the Republicans were running a Texan for President (George Bush) and the Democrats were running a Texan for Vice-President (Lloyd Bentsen) and bearing in mind that William Sessions is an intelligent man, guess correctly the answer Sessions gave.

12. Explain why the connective $\oplus$ defined by

$\square$	$\diamond$	$\oplus$
T	T	F
T	F	T
F	T	T
F	F	F

is sometimes called *exclusive or*.

13. a) Write the converse of the contrapositive of the following statement: If $3 > 1$, then $5 > 1$.

b) Write the contrapositive of the converse of the following statement: If $3 > 1$, then $5 > 1$.

14. Do there exist propositions p and q such that both $p \to q$ and its converse are true? Do there exist propositions p and q such that both $p \to q$ and its converse are false. Justify each answer with an example or argument.

1.2 Expressions and Tautologies

In the preceding section, we considered several fundamental ways in which to combine propositions to form compound propositions and we used boxes (squares and diamonds) as placeholders in writing down the associated truth tables. There is an obvious disadvantage to our placeholder notation. What other figures can we use? We could use pentagons, hexagons, and so on, but if we are considering a compound proposition made up of many simple propositions, it is annoying to count up sides, and although the human eye can readily distinguish between a square and a pentagon, the distinction between a small nonagon and a small decagon is not so easily apparent. We therefore introduce the idea of a **propositional expression**. We are given an alphabet of **variables**, and we have $\vee$, $\wedge$, $\neg$, and parentheses. We add the signs $\to$ and $\leftrightarrow$. Using variables, these signs, and parentheses, we form expressions. We rely on the reader's intuition rather than attempt a formal definition of a propositional expression. Each variable is a propositional expression, and if the preceding symbols are put together in a way that appears meaningful, the result is an expression. For example, $\neg(X \vee Y) \wedge (\neg X \to (Y \leftrightarrow Z))$ is an expression, as is $(X \vee Y) \leftrightarrow (X \wedge Y)$, but we hope that you do not believe that $XX \to)) \wedge (Y \leftrightarrow$ is also an expression. Note that it makes no sense to ask if the expression $(X \vee Y) \leftrightarrow (X \wedge Y)$ is true. As it is written, the expression has no meaning whatsoever, but when we replace the variables X and Y with propositions P and Q, we obtain a proposition $P \vee Q \leftrightarrow P \wedge Q$, which like any other proposition is either true or false.

Remark A proposition is either true or false. An expression has no meaning until its variables are replaced by propositions; thus an expression cannot

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Proposition 1.4

> Each of the following expressions is a contradiction.
>
> **a)** $(P \to Q) \wedge (P \wedge \neg Q)$
>
> **b)** $[(P \vee Q) \wedge \neg P] \wedge (\neg Q)$
>
> **c)** $(P \wedge Q) \wedge (\neg P)$

PROOF We prove part (a) and leave the proofs of parts (b) and (c) as exercises.

By Proposition 1.1(b), $P \wedge \neg Q \leftrightarrow \neg(P \to Q)$. By definition, $\neg(P \to Q) \wedge (P \to Q)$ is a contradiction. Therefore, $(P \to Q) \wedge (P \wedge \neg Q)$ is a contradiction by Proposition 1.3(a). (Of course, it can also be proved by constructing the appropriate table.) □

As another example of a proof of a logical proposition that does not rely on the construction of a table, we deduce Proposition 1.1(c) from Propositions 1.1(a), 1.2(c), 1.1(b), and 1.2(f).

By Proposition 1.1(a), $P \vee Q \leftrightarrow \neg P \to Q$. Therefore, by Proposition 1.2(c), $\neg(P \vee Q) \leftrightarrow \neg(\neg P \to Q)$. By Proposition 1.1(b), $\neg(\neg P \to Q) \leftrightarrow \neg P \wedge \neg Q$. Hence, by Proposition 1.2(f), $\neg(P \vee Q) \leftrightarrow \neg P \wedge \neg Q$.

Exercises 1.2

15. Prepare a truth table for each of the following expressions.

a) $P \wedge Q \to P \vee Q$ **b)** $P \to (Q \to R)$ **c)** $(P \to Q) \to R$

16. Which of the following pairs of expressions are equivalent?

a) $\neg(P \vee Q), \neg P \vee \neg Q$ **b)** $\neg P \wedge \neg Q, \neg(P \wedge \neg Q)$ **c)** $\neg(P \leftrightarrow Q), \neg P \leftrightarrow Q$

17. a) Using only negation and disjunction, find an expression in terms of variables P and Q that is equivalent to $P \to Q$.

b) Verify your answer to part (a) using an appropriate table.

18. Construct a truth table for each of the following expressions.

a) $\neg(P \wedge Q)$ **b)** $\neg P \wedge \neg Q$

c) $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$ **d)** $P \to (Q \to R)$

e) $(P \to Q) \to R$ **f)** $\neg(P \vee Q) \to R$

g) $\neg((P \vee Q) \to R)$

19. Let P be the proposition $2 = 5$. If possible, find a proposition Q such that $P \vee Q \leftrightarrow P \wedge Q$ is true. If possible, find a proposition Q such that $P \vee Q \leftrightarrow P \wedge Q$ is false.

20. a) Write the truth tables for the expressions $(P \leftrightarrow Q) \to R$ and $P \leftrightarrow (Q \to R)$.

b) Find propositions P, Q, and R such that one of the two compound propositions $(P \leftrightarrow Q) \to R$ and $P \leftrightarrow (Q \to R)$ is true and the other is false.

21. Under the assumption that $P \to Q$ is false, give truth values for $Q \to P$, $P \leftrightarrow Q$, $P \vee Q$, and $\neg P \wedge \neg Q$.

22. Which of the following statements are true?
   a) $(P \leftrightarrow Q) \Leftrightarrow (Q \leftrightarrow P)$
   b) $\neg(P \leftrightarrow Q) \Leftrightarrow (\neg P \to \neg Q)$
   c) $(\neg P \leftrightarrow Q) \Leftrightarrow (P \leftrightarrow \neg Q)$
   d) $(\neg P \to Q) \Leftrightarrow (\neg P \wedge \neg Q)$

23. Let P, Q, and R be variables. Prove that each of the following expressions is a tautology.
   a) $P \to P$
   b) $\neg(\neg P) \leftrightarrow P$
   c) $P \wedge (Q \vee R) \leftrightarrow (P \wedge Q) \vee (P \wedge R)$
   d) $\neg(P \vee Q) \leftrightarrow \neg P \wedge \neg Q$

24. Prove that each of the following propositional expressions is a contradiction.
   a) $((P \vee Q) \wedge \neg P) \wedge \neg Q$
   b) $(P \wedge Q) \wedge \neg P$

25. Write, in symbols, the converse, the contrapositive, and the negation of each of the following propositional expressions.
   a) $P \to (Q \vee R)$
   b) $P \to (Q \wedge R)$
   c) $(P \vee Q) \to R$
   d) $(P \wedge Q) \to R$

26. Which of the following expressions is logically equivalent to $P \to Q$?
   a) $P \to Q$
   b) $\neg P \to Q$
   c) $\neg Q \to P$
   d) $\neg P \to \neg Q$
   e) $\neg Q \to \neg P$
   f) $P \vee \neg Q$
   g) $P \wedge \neg Q$
   h) $\neg P \vee Q$
   i) $\neg P \wedge Q$
   j) $P \vee Q$
   k) $P \wedge Q$
   l) $Q \to P$

27. Suppose each of the following three statements is true.

   John is smart.

   John or Mary is ten years old.

   If Mary is ten years old, then John is not smart.

   Which of the following statements are true?
   a) Mary is ten years old.
   b) John is ten years old.
   c) Either John or Mary is not ten years old.

28. By constructing truth tables, prove each of the following.
   a) Proposition 1.1(a)
   b) Proposition 1.1(b)
   c) Proposition 1.1(c)

29. Assuming Propositions 1.1(c) and (e) and Proposition 1.2(c), deduce Proposition 1.1(d) without constructing a table.

30. Prove each of the following.
   a) Proposition 1.2(d)
   b) Proposition 1.2(e)

31. Use the results of Exercise 30 to show that if P, Q, and R are propositions and $P \to Q$, $Q \to R$, and $R \to P$ are true, then either P, Q, and R are all true or P, Q, and R are all false. *Do not construct a truth table.*

32. Prove each of the following parts of Proposition 1.2.
   a) Part (a)
   b) Part (b)
   c) Part (c)
   d) Part (f)
   e) Part (g)
   f) Part (h)
   g) Part (i)
   h) Part (j)
   i) Part (k)

33. Prove Proposition 1.3.

34. Prove each of the following.
    a) Proposition 1.4(b)
    b) Proposition 1.4(c)
35. Write a useful negation of each of the following propositions.
    a) If $2 \neq 4$, then $f(2) \neq f(4)$.
    b) If $4 > 2$, then $f(4) > f(2)$.
    c) If $a < b$, then $a^2 < b^2$.
    d) If $ab = 0$, then $a = 0$ or $b = 0$.
36. Write the contrapositive of each of the following propositions.
    a) If $ab \neq 0$, then $a = 0$ or $b = 0$.
    b) If $a \neq 0$ or $b \neq 0$, then $ab \neq 0$.
37. Write the converse of each of the propositions in Exercise 36.
38. Write the negation of the propositions in Exercise 36.
39. Write the negation and the contrapositive of the following statement: If x is an even integer or $x > 17$, then x is a multiple of 4 and $x \geqslant 5$.

*40. Give an example of two equivalent expressions that do not have the same truth table.

1.3 Quantifiers

Since our primary purpose is to teach you how to prove theorems, we have discussed propositional logic with this in mind. We now wish to discuss another topic in logic—namely, quantifiers. Be aware that our description of this topic is brief; we do not give the in-depth approach required for a rigorous presentation such as would be found in a course on logic. We hope, however, that enough is said here to help you write the negation of propositions and to prepare you to use quantifiers in the study of sets.

By a **sign**, we mean a mark that can be recognized, but we are interested only in mathematical signs. In this section, we discuss two types of signs, **constants** and **variables**. We are given a universe from which all constants are drawn, and so it is easy to say what a constant is; it is just a member of our given universe. A sentence such as "$x < 4$" is not a proposition. It contains a **variable**, which we could replace with a specific real number to make the sentence a proposition. Such a sentence is called a **propositional function** or **open sentence**. The set of objects that can replace the variable (in this case, the set of real numbers) is called the **universe** or **set of meanings**, and the word, or sign, in a propositional function to be replaced is called a **variable**. Thus, for example, in the open sentence "$x < 4$," the sign x is being used as a variable. Any object in the set of meanings must be a **constant**; that is, it must belong to the given universe. The collection of objects in the set of meanings that can be substituted to make a propositional function a true proposition is called the **truth set** of the propositional function. Thus the truth set of "$x < 4$" , the set of all real numbers that are less than 4.

*An exercise requiring the use of noncore material or one that is particularly difficult is marked with an asterisk.