3.1 Proof by Induction الاستقراء الرياضي

Definition A set S of natural numbers is said to be **inductive** provided that if $n \in S$, then $n+1 \in S$.

Ex $S=\{2,4,6,8,\dots\}$ not inductive

$2 \in S$ but $2+1=3 \notin S$.

$T=\{5,6,7,8,9,\dots\}$, T is inductive

$S=\{1,2,3,4,\dots\}$

T is inductive, $5 \in T$

$T=\{1,2,3,4,\dots$
$=\{2,3,4,\dots\}$
$\{3,4,5,\dots\}$
$\{4,5,6,\dots\}$
$\{5,6,\dots\}$

$\{5,6,\dots\} \subseteq T$

The Principle of Mathematical Induction

If S is an inductive set and 1 belongs to S, then S is the set of all natural numbers.

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$\{5,6,\dots\} \subseteq S$

prove that $\forall n \in \mathbb{N}$, $1+2+3+\dots+n=\frac{n(n+1)}{2}$.

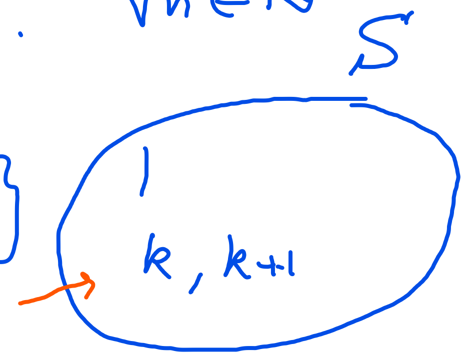
$$1+2+3+\cdots+n=n(n+1)/2$$

Now, having arrived at a conjecture, how do we prove or disprove it? We could attempt to make an infinite list of problems, one for each natural number, and then prove each statement separately. Is this feasible? The list would be as follows:

$n=1$: $1=1(1+1)/2$ or $1=1$
$n=2$: $1+2=2(2+1)/2$ or $1+2=3$
$n=3$: $1+2+3=3(3+1)/2$ or $1+2+3=6$
$\vdots$
n: $1+2+3+\cdots+n=n(n+1)/2$
$n+1$: $1+2+3+\cdots+(n+1)=(n+1)[(n+1)+1]/2$
$\vdots$

Prove using mathematical induction that

$P(n)$: $1+2+3+4+\cdots+n = \frac{n(n+1)}{2}$. $\forall n \in \mathbb{N}$.



Define $S = \{n \in \mathbb{N} \mid P(n)\}$

$= \{n \in \mathbb{N} \mid 1+2+\cdots+n = \frac{n(n+1)}{2}\}$

✓ $1 \in S$ Yes.

✓ Is S inductive

Suppose $k \in S$ i.e ✓ $1+2+3+\cdots+k = \frac{k(k+1)}{2}$ ✓ ... ①

need to prove $k+1 \in S$ i.e $1+2+3+\cdots+k+k+1 \stackrel{?}{=} \frac{(k+1)(k+2)}{2}$??

① adding to equation ① $k+1$ to both sides

$1+2+\cdots+k+k+1 = \frac{k(k+1)}{2} + \frac{k+1}{1}$

$= \frac{k(k+1)+2(k+1)}{2}$

$= \frac{(k+1)(k+2)}{2}$ ✓

So $S = \mathbb{N}$.

② $1+2+\cdots+k+k+1 = (1+2+\cdots+k)+k+1$

$= \frac{k(k+1)}{2} + k+1$

$= \frac{(k+1)(k+2)}{2}$ ✓

First principle of Mathematical Induction

Question use induction to prove that

$1+2+\cdots+k = \frac{k(k+1)}{2}$ $\forall n \in \mathbb{N}$

prove 1) the statement is true for $n=1$ since $1 = \frac{1(1+1)}{2}$

2) Suppose the statement is true for $n=k$, i.e

suppose $1+2+\cdots+k = \frac{k(k+1)}{2}$ — ①

and we prove the statement is true for $n=k+1$ i.e

EXAMPLE 1

Prove by induction that for any natural number n, 6 divides $n^3 - n$. ✓

Let $S = \{n \in \mathbb{N}: 6 \text{ divides } n^3 - n\}$. Then $S \subseteq \mathbb{N}$. In order to use the Principle of Mathematical Induction to prove that $S = \mathbb{N}$, we must prove two things:

we need to prove $1+2+\cdots+k+k+1 = \dfrac{(k+1)(k+2)}{2}$ — ②

Now adding $k+1$ to both sides of eqn ① above

implies $(1+2+\cdots+k)+(k+1) = \dfrac{k(k+1)}{2} + k+1$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

prove:

① the statement is true for $n=1$ since $6 \mid 1^3 - 1$ or $6/0$ ✓

② suppose it is true for $n=k$ i.e suppose $6 \mid k^3 - k$ and need to prove it is true for $n=k+1$ that is Q.E.D.

need to prove $6 \mid (k+1)^3 - (k+1)$

$6 \nmid 2$

$6 \nmid 5$

Now :- given $k^3 - k = 6m$, $m \in \mathbb{Z}$ for some.

need to prove $(k+1)^3 - (k+1) = 6n$??

$(k+1)^3 = (k+1)(k+1)^2$

$= (k+1)(k^2+2k+1)$

$= k^3 + 2k^2 + 1 + k^2 + 2k + 1$

$= k^3 + 3k^2 + 3k + 1$

Now $(k^3 + 3k^2 + 3k + \cancel{1}) - (k + \cancel{1})$

$= (k^3 - k) + (3k^2 + 3k)$

$= (k^3 - k) + 3(k)(k+1)$

$= 6m + 3 \cdot 2t$

$= 6m + 6t = 6(m+t) = 6n$, $n = m+t \in \mathbb{Z}$

The Extended Principle of Mathematical Induction

1st

Let $k \in \mathbb{N}$ and let S be a subset of $\mathbb{N}$ such that

a) $k \in S$, and

b) If $n \geqslant k$ and $n \in S$, then $n+1 \in S$.

Then $\{n \in \mathbb{N} : n \geqslant k\} \subseteq S$.

prove that $\forall n \geq 5$, $P(n)$

prove (1) the stat is true when $n=5$

(2) suppose it is true when $n=k$
and prove it is true for $n=k+1$

The Second Principle of Mathematical Induction

Let S be a set of natural numbers with the following properties:

a) $1 \in S$.

b) For each $n \in \mathbb{N}$, if $\{1, 2, 3, \ldots, n\} \subseteq S$, then $n + 1 \in S$.

Then $S = \mathbb{N}$.

The Extended Second Principle of Mathematical Induction

Let $k \in \mathbb{N}$ and let S be a subset of $\mathbb{N}$ such that

a) $k \in S$, and

b) If $n \geqslant k$ and $\{k, k+1, \ldots, n\} \subseteq S$, then $n + 1 \in S$.

Then $\{n \in \mathbb{N} : n \geqslant k\} \subseteq S$.

Proposition 3.2

Every natural number greater than 1 is either a prime number or the product of prime numbers.