Theorem 1.8

Suppose a, b, c are integers. if a, b are even and c is odd then the equation $ax+by=c$ doesn't have an integer solution

Proof:- By contradiction

suppose that there are even integers x_0, y_0 and an odd integer c_0 such that $ax_0+by_0=c_0$

$\Rightarrow \exists\, k_0, l_0, t_0 \in \mathbb{Z}$ such that

$x_0 = 2k_0, \quad y_0 = 2l_0, \quad c_0 = 2t_0+1$

$\Rightarrow a(2k_0) + b(2l_0) = 2t_0+1$

$\Rightarrow 2ak_0 + 2bl_0 = 2t_0+1$

$\Rightarrow 2(ak_0 + bl_0 - t_0) = 1$

$\Rightarrow 2s_0 = 1$ where $s_0 = ak_0 + bl_0 - t_0 \in \mathbb{Z}$.

$\Rightarrow 1$ is even a contradiction $\times$.

Definition:- A real number r is rational iff there are integers s, t, such that $r = \frac{s}{t}$. $t \neq 0$

and r is irrational iff r is not rational.

Theorem 1.9: if r is a real number such that $r^2 = 2$ then r is irrational.

proof:- by contradiction

suppose that $r \in \mathbb{R}$ and $r^2 = 2$ and r is rational

$\Rightarrow \exists\ s, t \in \mathbb{Z},\ t \neq 0$ such that $r = \frac{s}{t}$

where s, t have no common divisor grater than 1

$\Rightarrow r^2 = 2 = \frac{s^2}{t^2}$

$\Rightarrow s^2 = 2t^2$, so s^2 is even

$\Rightarrow s$ is even...(1)

s^2 is even $\Rightarrow s$ is even

by contrapositive

suppose s is odd $\Rightarrow s = 2s+1$

$\Rightarrow s^2 = 4s^2+4s+1$

$= 2(s^2+2s)+1$

$\Rightarrow s^2$ is odd $\times$.

$\Rightarrow s = 2l,\ l \in \mathbb{Z}$.

$\Rightarrow s^2 = 4l^2 = 2t^2$

$\Rightarrow t^2 = 2l^2$ is even

$\Rightarrow t$ is even ...(2)

$\Rightarrow$ from (1), (2) s, t are both divisible by 2

i.e 2 is a common divisor of s, t $\times$.

a contradiction

To proof $p \Rightarrow q$

Direct Method

Assume p

⋮

Logical sequence of steps

⋮

Conclude q

Contrapositive Method

Assume $\sim q$

⋮

logical sequence of steps

⋮

conclude $\sim p$.

contradiction method.

Assume p and $\sim q$

⋮

Logical sequence of steps

⋮

Reach a contradiction

Rolls theorem.

if f is continous on $[a,b]$, differentiable on (a,b) and $f(a) = f(b)$ then there is $c \in (a,b)$ such that $f'(c) = 0$

Theorem 1.10: Let $x, y \in \mathbb{R}$

if $x \neq y$ then $e^x \neq e^y$.

proof:- Direct method

suppose $x \neq y$ then either $x > y$ or $y > x$

So we can assume $x > y \Rightarrow r = x - y > 0$

$\Rightarrow \exists\, r > 0$ such that $x = y + r$

$\Rightarrow e^x = e^{y+r} = e^y . e^r$

$\Rightarrow e^r > 1$ since $e > 1$

$\Rightarrow e^x = e^y . e^r > e^y \Rightarrow e^x \neq e^y$.

Contrapositive Method.

Suppos $e^x = e^y \Rightarrow \ln e^x = \ln e^y$
$\Rightarrow x = y$.

Contradiction Method

suppose $e^x = e^y$ and $x \neq y$.
$\Rightarrow \ln e^x = \ln e^y$
$\Rightarrow x = y$ X. a contradiction

Theorem 1.11 Let $a, b, c \in \mathbb{R}$ then $a^2 + b^2 + c^2 \geq ab + bc + ca$

$\Rightarrow (a-b)^2 + (a-c)^2 + (b-c)^2 \geq 0$

$\Rightarrow a^2 - 2ab + b^2 + a^2 - 2ac + c^2 + b^2 - 2bc + c^2 \geq 0$

$\Rightarrow 2a^2 + 2b^2 + 2c^2 \geq 2ab + 2ac + 2bc$

$\Rightarrow a^2 + b^2 + c^2 \geq ab + ac + bc$.

Theorem 1.13 Let $x \in \mathbb{R}$ then

$x = 1$ iff $x^3 - 3x^2 + 4x - 2 = 0$

$\boxed{\Rightarrow}$ Direct suppose $x = 1 \Rightarrow x^3 - 3x^2 + 4x - 2 = 1^3 - 3(1)^2 + 4(1) - 2 = 0$

$\boxed{\Leftarrow}$ Suppose $x^3 - 3x^2 + 4x - 2 = 0 \Rightarrow (x-1)(x^2 - 2x + 2) = 0$
$\Rightarrow x = 1$ or $x = 1 \mp i$
$\Rightarrow x = 1$.

Theorem 1.14 :-

Prove that there do not exist prime numbers a, b, c such that $a^3 + b^3 = c^3$.

Proof:

This is equivalen to.

If a, b, c are primes then $a^3 + b^3 \neq c^3$

By Contradiction

Suppose that a, b, c are primes and $a^3 + b^3 = c^3$.

either a, b are both odd

$\Rightarrow a^3, b^3$ are odd

$\Rightarrow a^3 + b^3$ is even

$\Rightarrow c^3$ is even

$\Rightarrow c$ is even

$\Rightarrow c = 2$ $\times$.

If at least one of a, b is even say $b = 2$

$\Rightarrow b^3 = c^3 - a^3$

$\Rightarrow 8 = (c-a)(c^2 + ca + a^2)$

$\Rightarrow c^2 + ca + a^2 \geq 12$. $\times$.

Theorem 1.16 :-

If $x \in \mathbb{R}$ such that $x^2 - x - 2 < 0$ then $-1 < x < 2$

Proof Suppose $x^2 - x - 2 < 0$

$\Rightarrow (x-2)(x+1) < 0$

$\Rightarrow x-2 < 0$ and $x+1 > 0$ Or $x-2 > 0$ and $x+1 < 0$

$\Rightarrow x < 2$ and $x > -1$ Or $x > 2$ and $x < -1$

$\Rightarrow -1 < x < 2$ Or a contradiction

$\Rightarrow -1 < x < 2$.

Theorem 1.17:-

prove that there do not exist natuaral numbers m and n such that $\frac{7}{17} = \frac{1}{m} + \frac{1}{n}$.

proof: By contradiction

suppose that there are natural numbers m, n such that $\frac{1}{m} + \frac{1}{n} = \frac{7}{17}$.

Case1 $m = 2 \Rightarrow \frac{1}{2} + \frac{1}{n} = \frac{7}{17}$ (but $\frac{1}{n}$ is positive)

a contradiction

Case2:- $n = 2 \Rightarrow \frac{1}{m} + \frac{1}{2} = \frac{7}{17}$ a contradiction

Case3:- $m \geq 5, n \geq 5 \Rightarrow \frac{1}{m} + \frac{1}{n} \leq \frac{1}{5} + \frac{1}{5} < \frac{7}{17}$

a contradiction

Case4:- One of m and n must be 3 or 4

$\Rightarrow \frac{7}{17} - \frac{1}{3} = \frac{4}{51} \neq \frac{1}{k}$

also $\frac{7}{17} - \frac{1}{4} = \frac{11}{68} \neq \frac{1}{k}$

all cases implie a contradiction