1.6 More proofs

Theorem 1.26 if x is any real number such that $x^2 - x - 2 < 0$ then $-1 < x < 2$

proof:- Direct:- Suppose $x^2 - x - 2 < 0$

$\Rightarrow (x-2)(x+1) < 0$

$\Rightarrow x-2 < 0$ and $x+1 > 0$ or $x-2 > 0$ and $x+1 < 0$

$\Rightarrow x < 2$ and $x > -1$ or $x > 2$ and $x < -1$

$\Rightarrow x < 2$ and $x > -1$ or a contradiction

$\Rightarrow -1 < x < 2$.

Th 1.17 proof that there do not exist natural number m and n such that $\frac{1}{m} + \frac{1}{n} = \frac{7}{17}$

proof:- Direct (by cases)

Case 1) Neither m nor n can be 2 since $\frac{1}{2} + \frac{1}{n} > \frac{7}{17}$

$\frac{1}{2} + \frac{1}{m} > \frac{7}{17}$.

Case 2:- both $m, n > 5$ is impossible since

$\frac{1}{m} + \frac{1}{n} < \frac{1}{5} + \frac{1}{5} < \frac{7}{17}$.

Case 3:- So either m or n must be 3 or 4

say $n = 3 \Rightarrow \frac{1}{3} + \frac{1}{m} = \frac{7}{17} \Rightarrow \frac{1}{m} = \frac{4}{51}$ $\times$.

and $\frac{1}{4} + \frac{1}{m} = \frac{7}{17} \Rightarrow \frac{1}{m} = \frac{11}{68}$ $\times$.

So $\nexists\ m, n \in \mathbb{N}$; $\frac{1}{m} + \frac{1}{n} = \frac{7}{17}$.

92/39 Prove that if $n \in \mathbb{N}$ then n^2+n is even

Proof:- suppose $n \in \mathbb{N} \Rightarrow n$ is even or n is odd

$\Rightarrow n = 2k, k \in \mathbb{Z}$ or $n = 2k+1, k \in \mathbb{Z}$.

$\Rightarrow n^2+n = 4k^2+2k = 2(2k^2+k) = 2k'$ Or $n^2+n = 4k^2+4k+1+2k+1 = 2(2k^2+2k+k+2) = 2k'$

$\Rightarrow n^2+n$ is even.

Other proof:- Direct suppose $n \in \mathbb{N}$.

$\Rightarrow n^2+n = n(n+1)$ is even since it is the product of two consecutive numbers so one of them must be even.

Try:- proof by contradiction (Excercise).

suppose $n \in \mathbb{N}$ and n^2+n is odd

case1:- n is even $\Rightarrow n^2+n$ is even (see above) X.

Case2:- n is odd $\Rightarrow n^2+n$ is even (see above) X.

(94/39) Prove that if $a, b \in \mathbb{Q}$ with $a < b$ then there is a rational number r; $a < r < b$.

Direct proof:- Let $a = \frac{m}{n}$, $b = \frac{s}{t} \in \mathbb{Q}$

$\Rightarrow \exists r \in \mathbb{Q}$, take $r = \frac{a+b}{2} = \frac{mt + ns}{2\cancel{m}t}$

then $r \in \mathbb{Q}$ since $mt + ns$, $nt \in \mathbb{Z}$.

and $a < r$ ~~since $r - a = \frac{mt+ns}{2nt} - \frac{m}{n} = \frac{mt-ns-2mt}{2nt}$~~

since $r - a = \frac{a+b}{2} - a = \frac{b-a}{2} > 0 \Rightarrow a < r$

also $b - r = b - \frac{b+a}{2} = \frac{b-a}{2} > 0 \Rightarrow r < b$.

(96) (a) Find $x, y \in \mathbb{R}^+$ such that $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$.

answer. $x = 25, y = 9 \Rightarrow \sqrt{25+9} \neq 5 + 3$

$\Rightarrow \sqrt{34} \neq 8$.

(b) if $x, y \in \mathbb{R}^+$ then $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$.

proof:- By contradiction

suppose $x, y \in \mathbb{R}^2$ and $\sqrt{x+y} \not\leq \sqrt{x} + \sqrt{y}$

$\Rightarrow \sqrt{x} + \sqrt{y} < \sqrt{x+y}$

$\Rightarrow x + y + 2\sqrt{xy} < x + y$

$\Rightarrow \sqrt{xy} < 0$ X.

(101) prove that if $x \in \mathbb{R}$, $x<2$ then $\exists y \in \mathbb{R}$ $y<0$ such that $x = \frac{2y}{1+y}$.

Solution:- Let $x \in \mathbb{R}$, $x<2$, Take $y = \frac{-x}{x-2}$

then $y<0$ and

$$\frac{2y}{1+y} = \frac{2\left(\frac{-x}{x-2}\right)}{1+\frac{-x}{x-2}} = \cancel{\frac{2x-4-x}{x-2}} \cancel{\frac{x-2}{x-2}}$$

$$= \frac{-2x}{x-2} \Big/ \frac{-2}{x-2}$$

$$= x \checkmark$$

(102) prove that if $p \in \mathbb{Z}$, p is odd then $x^2+x-p=0$ has no integer solution

Solution:- suppose that $\exists p \in \mathbb{Z}$, p is odd ~~prime~~ and $x^2+x-p=0$ has an integer solution

$\Rightarrow x = \frac{-1 \mp \sqrt{1+4p}}{2} \in \mathbb{Z}$.

$\Rightarrow \cancel{\mp}\sqrt{1+4p} = 2k$.

$\Rightarrow$ either $\sqrt{1+4p} = 1+2k$ or $\sqrt{1+4p} = -1-2k$.

$\Rightarrow p = k^2+k = k(k+1)$ is even $\times$

Chapter 2 — sets.

Defⁿ :- A set is a Collection of Well defined objects.
Sets are denoted by Capital letters $A, B, \dots$ and objects by small letters. and between brackets of the form $\{\ \}$.

Examples :

① $A = \{1, 2, 5\}$.

② $B = \{2, 4, 6, \cdots, 100\}$, $C = \{1, 3, 5, 7, \cdots\}$

③ $D = \{x \mid x \text{ is a Bir Zeit University student}\}$

④ $E = \{x \in \mathbb{R} \mid 1 < x < 100\}$

Defⁿ :- subsets of $\mathbb{R}$

$[1,2] = \{x \in \mathbb{R} \mid 1 < x < 2\}$

$(1,5] = \{x \in \mathbb{R} \mid 1 < x \leq 5\}$

$[2,\infty) = \{x \in \mathbb{R} \mid 2 \leq x\}$

$(-\infty,1] = \{x \in \mathbb{R} \mid x \leq 1\}$

$\mathbb{R} = (-\infty, \infty)$.

Def^n:- if A is a set and a is an object in A then we write $a \in A$, and if b is not in A we write $b \notin A$.

Def^n:- 1) if A, B are sets then A is a subset of B written $A \subseteq B$ iff $x \in A \Rightarrow x \in B$

that is every element in A is in B.

notice that $A \nsubseteq B$ iff $\exists\, x \in A$ such that $x \notin B$.

2) Two sets A, B are equall iff A and B contain same elements.

notice that $A \subseteq B$ and $B \subseteq A$.