Student's Name : Ayham Hashesh

Instructor : Mahamoud Ghannam

Numerical Methods

طرق التحليل العددي

MATH 331

جامعة بيرزيت

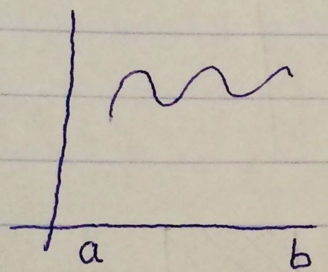
BIRZET UNIVERSITY

1161301 أيهم حشيش

* Chapter 1 :-

$\boxed{1.1}$ Review of calculus

* Continuity : f is Continuos on $[a,b]$

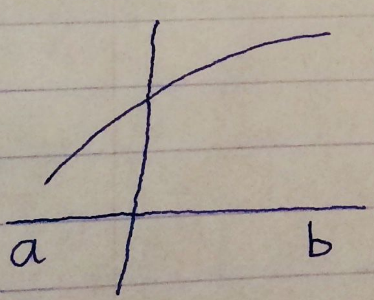


* $\forall$: for all

$f(x) \in C[a,b]$
$f(x) \in C'[a,b]$

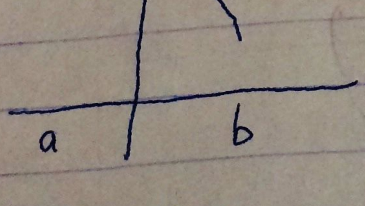
* f is increasing & Continuos on $[a,b]$

$f'(x) > 0$



* f is decreasing & Continuos on $[a,b]$

$f'(x) < 0$



* <u>Theorem</u> if f is continuos on $[a,b]$ then

f has an absolute Max (upper bound) $= M$

& f " " " min (lower bound) $= m$

$\Rightarrow \quad m \leq f(x) \leq M \quad \forall x \in [a,b]$

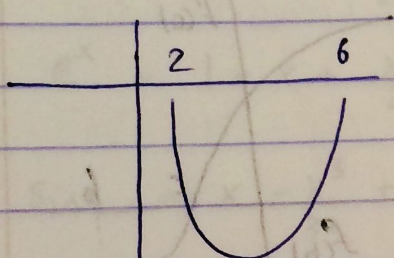
من الأطراف أو النقاط الحرجة

المشتقة الأولى

تعطي max

مش f(x)

<u>Ex</u> $f(x) = x^2 - 8x + 3$, $[2,6]$ find M, m



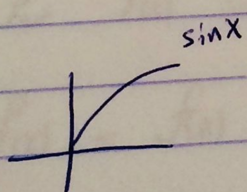
$f(2) = -9$] M
$f(6) = -9$

$4 \in [2,6]$

$f'(x) = 2x - 8 = 0$

$\boxed{x = 4}$ عوض بالاقتران $\Rightarrow f(4) = -13$ m

<u>Ex</u> $f(x) = \sin x$, $[0,1]$



$M = f(1) = \sin 1$

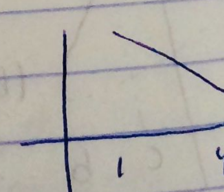
$90 \notin [0,1]$

$m = f(0) = \sin 0 = 0$

$0 \leq \underset{\sin x}{f(x)} \leq \sin 1$

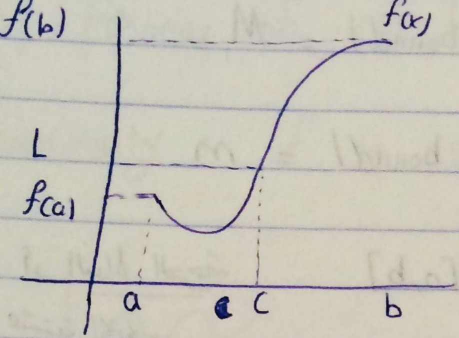
<u>Ex</u> $f(x) = \dfrac{2}{(x+1)}$, $1 \leq x \leq 4$

$\Rightarrow$ decreasing $\Rightarrow$ $M = f(1) = \dfrac{2}{64} = \dfrac{1}{32}$



$m = f(4) = \dfrac{2}{7^3}$

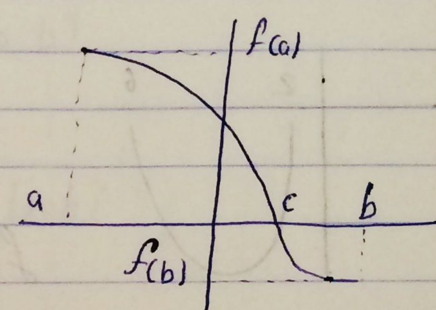
* Intermediate Value Theorem [IVT]



$f \in C[a,b]$ & L between $f(a), f(b)$

$\Rightarrow \exists\ c$ in $[a,b]$ s.t $f(c) = L$

* Belzano Theorem صورة أحدهم موجب والآخر سالب



$f \in C[a,b]$ & $f(a)f(b) < 0$

$\Rightarrow \exists\ c \in (a,b)$ s.t $f(c) = 0$

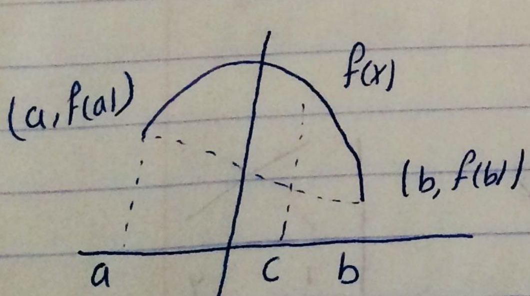
$x = c$ is a root (zero) for $f(x)$

$x = c$ " " Solution for the equation $\boxed{f(x) = 0}$

* Mean Value Theorem [MVT]

شرط 1 $f \in C[a,b]$ & شرط 2 $f' \in C[a,b]$

$\Rightarrow \exists\ c \in (a,b)$ S.T :



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$|f(b) - f(a)| = |f'(c)|\,|(b-a)|$$

Recall : $|x \mp y| \leq |x| + |y|$

* Rolle's Theorem

F cont on $[a,b]$ & defferentable on $[a,b]$ & $f(a) = f(b)$

$\Rightarrow C \in [a,b]$ S.T $f'(c) = 0$

* Taylor Series [Taylor expansion]

[1] $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

$e^x = 1 + x + \frac{x^2}{2}$, for x close to $\boxed{\text{zero}}$ center

[2] $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

[3] $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

given $f(x)$ & a center $x = a$, The Taylor expansion (series) for $f(x)$ about $x = a$ is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!}$$

$\Rightarrow$ used to approximate $f(x)$ for x close to a

* Taylor Theorem

Taylor expansion for $f(x)$ about $x = a$ &

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(x)(x-a)^2}{2!} + \dots + f^{n}(a)\frac{(x-a)^n}{n!}$$

the error of this equation is

$$E(x) = \frac{f^{n+1}(c)(x-a)^{n+1}}{(n+1)!}$$, where c is a constant between x and a

الحد التالي مش +1

$f(x)$ $\qquad\qquad\qquad$ $P_3(x)$

Ex $\quad e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ $\qquad$ $\boxed{1}$ Find Error

$$E = \frac{f^{(4)}(c)\, x^4}{24}$$

$$f(x) = e^x = f'(x) = f^{(n)}(x)$$

$$E = \frac{e^c x^4}{24}, \quad c \text{ between } x \text{ \& } \underline{\underline{0}}$$

$\boxed{2}$ Find an upper bound for |Error| when approximately $e^{0.5}$

now $\quad e^{0.5} \approx 1 + 0.5 + \frac{0.5^2}{2} + \frac{0.5^3}{6}$

$$E(x) = \frac{e^c x^4}{24} \qquad c \text{ between } 0.5 \text{ \& } 0$$

$$0 < c < 0.5$$

$$1 = e^0 < e^c < e^{0.5}$$

$$|E(0.5)| = \frac{e^c (0.5)^4}{24} \le \frac{e^{0.5} (0.5)^4}{24}$$

في الإمتحان الحل جواب رقم

1.3 Error

13-2-2018

If P is approximated by $\hat{P}$ then

① absolute error $|E| = |P - \hat{P}|$

القيمة الحقيقية

القيمة التقريبية

② Relative error $= |R| = \dfrac{|P - \hat{P}|}{|P|}$

Ex $P = 1.2351$, $\hat{P} = 1.2$

Find $|E|$, $|R|$

Solution $|E| = |1.2351 - 1.2|$

$= 0.0351$

$|R| = \dfrac{0.0351}{1.2351} = 0.0284$

Ex $999\,995 = P$, $1\,000\,000 = \hat{P}$

Solution $|E| = 5$ $|R| = \dfrac{5}{999\,995} = 0.000\,00\,50000\,25$

abs $|E|$ أفضل لكن سنستخدم Relative

Recall $e^x \approx \underbrace{1 + x + \frac{x^2}{2} + \frac{x^3}{6}}_{}$ $P_3(x)$

$|E| = \left| e^x - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3}\right) \right|$

* Source of Error

1. Round-off error : a number is approximated by a number

2. Truncation error : a formula is estimated by a formula

Round-off Error

* Finite digit Arithmatic

⇒ Computer has a Finite numbers of digit

* Floating Point Representation

$0.02341 = 0.2341 * 10^{-1}$
$= 2.341 * 10^{-2}$
$= 2341.0 * 10^{-5}$

* Normalized Floating Point

any number is represented by $\underbrace{0.d_1 d_2 d_3 \ldots d_k}_{\text{significant digit}} * 10^n$

where $d_1 \neq 0$

$0.000000012 \Rightarrow \underbrace{0.12}_{\text{significant digit}} * 10^{-7}$ [Normalized]

or [ناخد أول عنصر مش صفر ثم نعد]

Ex $0.0100234 * 10^{-4} \Rightarrow$ 6 significant digit

$1.200 \Rightarrow$ 4 significant digit

* Rule : Start Counting from left from the first nonzero digit
خانة غير صفرية

0.000000012 : الآلة الحاسبة تعد مثلا
10 خانات من 1

* Two Types of Round-off : [1] Chopping : $0.d_1d_2 \dots d_k d_{k+1} d_{k+2} \dots$

$\approx 0.d_1d_2 \dots d_k$

[2] Rounding : $0.d_1d_2 \dots d_k d_{k+1} d_{k+2} \dots$

$$\approx 0.d_1d_2 \dots r_k \begin{cases} d_k & \text{if } d_{k+1} < 5 \\ d_k + 1 & \text{if } d_{k+1} > 5 \end{cases}$$

* We Assume that Both [chopping & Rounding] are Finite digit Arithmatic K [significant digit]

Ex Using 5 digit arithmatic estimate

[1] 2.0147985 ⇒ [a] chopping ⇒ 2.0147
[b] Rounding ⇒ 2.0148

[2] 0.000314295 ⇒ [a] Chopping ⇒ 0.00031429
[b] Rounding ⇒ 0.00031430

Ex Estimate $\frac{1}{7} + \frac{1}{8} + \frac{1}{9}$, using 2-digit rounding

Solution 0.14 + 0.13 + 0.11

مقربة وجاهزة 0.27 + 0.11 = 0.38

تقريب $\frac{1}{7}$ ، $\frac{1}{8}$ ، $\frac{1}{9}$

تقريب مجموع [$\frac{1}{7}$ ، $\frac{1}{8}$]

تقريب المجموع السابق مع $\frac{1}{9}$

تقريب المجموع النهائي

* Order of Operations :

[1] Brackets
[2] Powers
[3] ÷, * [From left to Right]
[4] +, - [= = = =]

Ex Estimate $$\frac{\frac{1}{6} \div \frac{1}{3} + (0.23)^3}{\cos e^2 * \sqrt{0.01348}}$$ Using 3-digit rounding

Note default in Calculator [D] ⇒ Degree mode, we use Rad mode always

Mode Mode ⇒ 2 ⇒ Rad

or shift Mode ⇒ 2 ⇒ Rad

Solution $\dfrac{0.167 \div 0.333 + (0.23)^3}{(\cos(2.72)^2) * \sqrt{0.0135}}$

$\Rightarrow \dfrac{0.502 + 0.0122}{\cos(7.40) * 0.116}$

$\Rightarrow \dfrac{0.514}{0.0509} = 10.1$

Notes ① $e = 2.72$

② $0.23 \Rightarrow$ لو أكبر من ٣ خانات يجب التقريب

③ $(0.23)^3 = (0.23)(0.23)(0.23)$ تكعيب مباشرة

④ لو كانت الجمع قبل القسمة ⇐ نقسم ، الأولوية للقسمة

* Loss of significant :- $P = 0.123479$ (6 sig)

$q = 0.123478$

$P - q = 0.000001$ [1 sig]

Loss ∈ عند طرح رقمين قريبين من بعض

Ex $\ln(x) - \ln(x+1) \Rightarrow$ **Sol** $\ln\left(\frac{x}{x+1}\right)$ [L.o.S] للتخلص من

* Propagation of error :- error may increase due to operation

ممكن تقل

Ex $P = \underbrace{\hat{P}}_{\text{التقريب}} + E_p$ $\quad P + q = \hat{P} + \hat{q} + E_p + E_q$

$q = \hat{q} + E_q$

* Order of Approximation :- Notation or Method used to express the truncation error for some estimation

Note When I use h instead of x, this means h is a variable close to $0 \Rightarrow |h| < 1$

$-1 \quad 0 \quad 1$

ex $\underset{f(h)}{e^h} = \underset{P(h)}{1 + h + \frac{h^2}{2}} \Rightarrow E = f(c)^3 \frac{h^3}{3} = e^c \frac{h^3}{6}$

C is constant between 0 & h [truncation Error], $E \approx ch^3$

We can express this error by $O(h^3) \longrightarrow$ order of approximation

Def : Order of approximation if $f(h) = P(h) + O(h^n)$ then $f(h) \approx P(h)$ with $E_{trunc} \approx C * h^n$

Ex $\sin h \approx h - \frac{h^3}{3!} + \frac{h^5}{5!} \Rightarrow$ Order of approximation : $O(h^7)$

15-2-2018

* Order of approximation :

$$f(h) = p(h) + O(h^n)$$

$$f(h) \approx p(h)$$

$$E \approx C \cdot h^n$$

$f(h) = p(h) + O(h^5)$

$f(h) = \bar{p}(h) + O(h^6)$

$\bar{p}(h)$ is a better approximation for $f(h)$

Ex $\sin h \approx h - \frac{h^3}{3!} + \frac{h^5}{5!}$, what is the order of approx ?

$$E = \frac{f^{(7)}(c)\, h^7}{7!} = \frac{\cos c \; h^7}{7!} = C * h^7$$

order of approx : $O(h^7)$

Theorem : $f(h) = P(h) + O(h^n)$

$g(h) = q(h) + O(h^n)$ then

$f(h) \overset{*}{\underset{\div}{\mp}} g(h) = p(h) \overset{*}{\underset{\div}{\mp}} q(h) + O(h^r)$ where

① $r = \min\{m, n\}$

$1 \approx 1 + 0.1$ لكن

② $h^k + O(h^r) = O(h^r)$ if $K \geq r$

h قوة أكبر أو تساوي r ⇐ يلغي

Ex $f(h) = 1 - 2h + h^2 + O(h^3)$

$g(h) = h^2 + 2h^4 + O(h^5)$

Find ① $f(h) + g(h)$

① $\Rightarrow 1 - 2h + h^2 + h^2 + 2h^4 + O(h^3)$

$\Rightarrow 1 - 2h + 2h^2 + O(h^3)$

② $f(h).g(h) = (1-2h+h^2)(h^2+2h^4) + o(h^3)$

$= h^2 + o(h^3)$

③ estimate $f(0.1)\,g(0.1)$ and find error

$f(0.1)\,g(0.1) \approx (0.1)^2 = 0.01$

$E \approx C \propto h^3$

$E = C(0.1)^3 = C\alpha\ 0.001$