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Numerical Methods

طرق التحليل العددي

MATH 331

جامعة بيرزيت

BIRZET UNIVERSITY

1161301 أيهم حشيش

* Chapter 2 *

قيمة تقريبية

How to Solve the equation $\boxed{f(x) = 0}$ numerically ?

zeros , roots , solution of $f(x)$

تكرار

* There are five numerical (iterative) method for solving $f(x) = 0$

1. Bisection Method
2. False - Position Method (Regula - Falsi Method)
3. Fixed Point iteration
4. Newton's Method (Newton - Raphson method)
5. The Secant Method

Sequence of ~~it~~ iteration
approximation

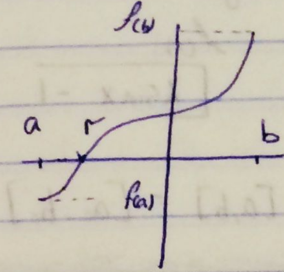
$P_0, P_1, P_2, P_3, \dots\dots$ converges r

Ex $1.5, 1.67, 1.77, 1.78, 1.87, 1.95, 1.97, 1.99, 1.999, 1.9999, \dots \rightarrow 2$

متتالية

ينصّف

* Bisection Method تنصف لبزانو



Recall if $f \in C[a,b]$ & $f(a)f(b) < 0$

متصل

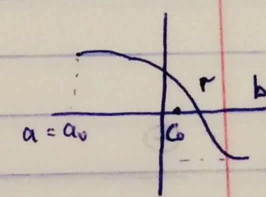
لأن أحدهم موجب والآخر سالب

⇐ الجذر بينهم

r : exact root $f(r) = 0$

$\Rightarrow \exists$ at least $r \in (a,b)$ s.t

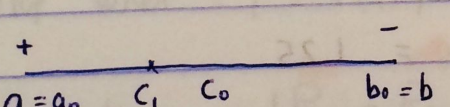
$f(r) = 0$ (r root of $f(x)$)



Bisection $f \in C[a,b]$; $[a,b] : [a_0, b_0]$

$f(a_0) f(b_0) < 0$

First iteration : $C_0 = \frac{a_0 + b_0}{2} \Rightarrow f(C_0)$



حسب الرسمة

if $f(C_0) < 0 \Rightarrow [a_1, b_1] = [a_0, C_0]$

$C_1 = \frac{a_1 + b_1}{2}$ نضع c على اليسار ⇐ $f(C_0) < 0$

if $f(C_0) > 0 \Rightarrow [a_1, b_1] = [C_0, b_0]$

على اليمين

$C_1 = \frac{a_1 + b_1}{2}$, $f(C_1)$?? $[a_2, b_2]$

$C_2 = \frac{a_2 + b_2}{2}$

Bisection iteration : $C_n = \frac{a_n + b_n}{2}$ bop

* Note Rad Mode : Mode Mode 2

Ex Solve the equation $x \sin x = 1$ in $[1,2]$ using the bisection method. Find three iteration

Solution: $\boxed{\underbrace{x \sin x - 1}^{f(x)}} = 0$ OR $1 - x \sin x = 0$ ✓ أحد الأطراف يجب أن يكون صفر

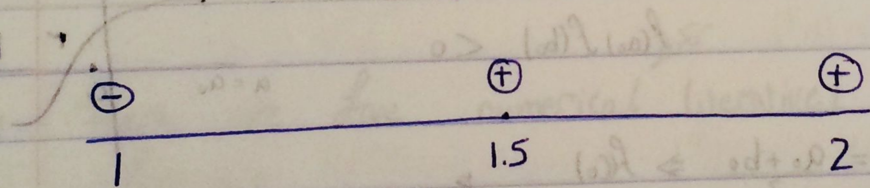
لأن هذه الطريقة فقط عندما $f(x) = 0$

$[a,b] = [a_0, b_0] = [1,2]$ $C_0 = \frac{a_0 + b_0}{2} = \frac{1+2}{2} = 1.5$

$f(1) = -0.15853$ * Rad Mode *

$f(2) = 0.81859$

$f(1.5) = 0.49624$ نأخذ حد موجب وحد سالب



$[a_1, b_1] = [1, 1.5]$ $C_1 = \frac{1 + 1.5}{2} = 1.25$

$f(1.25) = 0.18623$

$\Rightarrow [a_2, b_2] = [1, 1.25]$

$C_2 = \frac{1 + 1.25}{2} = 1.125$

* Stopping Methods

① the question determine

② <u>Stop</u> when $|C_n - C_{n-1}| < \epsilon$ [default]

قيمة مطلقة

given

ابسلون أو دلتا

*// ϵ : accuracy (error) & $|C_n - C_{n-1}| \approx E$

C_k	$\lvert C_k - C_{k-1}\rvert \approx E$	
C_0	/	
C_1	$\lvert C_1 - C_0\rvert$	$> \epsilon$
C_2	$\lvert C_2 - C_1\rvert$	$> \epsilon$
$\vdots$		
C_{n-1}		
C_n	$\lvert C_n - C_{n-1}\rvert$	$< \epsilon$

الفرق بيصير أقل من ابسلون ⇐ stop

$\boxed{r \approx C_n}$

③ Stop when $|f(C_n)| < \epsilon$

④ Stop when $\dfrac{|P_n - P_{n-1}|}{|P_n|} < \epsilon$ $\quad \dfrac{|C_n - C_{n-1}|}{|C_n|}$

$\boxed{\text{Note } P_n \equiv C_n}$

Bisection Method <u>Advantage</u> : always Converges

$$\{C_n\}_{n=0}^{\infty} = \{C_0, C_1, C_2, \ldots\}$$

$$** \lim_{n\to\infty} C_n = r$$

Disadvantage : too slow

Question Prove that $\lim_{n \to r} C_n = r$ for bisection method

Prof

باي سكشن => الطريقة بالتنصيف

$a_0 = a$ c_1 $b = b_0$

a_1 b_1

$$b_1 - a_1 = \frac{b-a}{2}$$

$$b_2 - a_2 = \frac{b_1 - a_1}{2} = \frac{b-a}{2^2}$$

$$b_3 - a_3 = \frac{b_2 - a_2}{2} = \frac{b-a}{2^3}$$

$$b_n - a_n = \frac{b-a}{2^n} \quad * \ ①$$

a_0 a_n c_n b_n b_0

$$0 < |r - c_n| < \frac{b_n - a_n}{2} \quad ②$$

القيمة الحقيقية

التقريب

الايرور

Error $|E|$ upper bound for error for bisection method

by Sandwich Thm $\lim_{n \to \infty} |r - c_n| = 0$

$$\lim_{n \to \infty} r - c_n = 0$$

$$\lim_{n \to \infty} r = \lim_{n \to \infty} c_n$$

$$r = \lim_{n \to \infty} c_n$$

Quiz ⇒ الخميس ⇒ ch1

20-2-2018

Extra 1 ⇒ HomeWork (الخميس)

$$\lim_{n\to\infty} c_n = r$$

$$\underbrace{|r - c_n|}_{|E|} < \frac{b-a}{2^{n+1}}$$

upper bound for E in bisection أكبر ايرور ممكن

$$\frac{b-a}{2^{n+1}} < \epsilon$$

لمعرفة عدد مرات التقريب في البايسكشن

$$\frac{2^{n+1}}{b-a} > \frac{1}{\epsilon}$$

$$2^{n+1} > \frac{b-a}{\epsilon} \Rightarrow \ln(2^{n+1}) > \ln\left(\frac{b-a}{\epsilon}\right)$$

$$(n+1)\ln 2 > \ln\left(\frac{b-a}{\epsilon}\right)$$

حفظ

$$n > \frac{\ln \frac{b-a}{\epsilon}}{\ln 2} - 1$$

used to find # of iteration (theoretically) for bisection to achieve accuracy ϵ

Ex find the # of iteration needed to solve $x\sin x = 1$ in $[1,2]$ using the bisection method with error less than 10^{-7} (with accuracy of 10^{-7})

⇐ نحسب مرة وحدة أدق

Sol.

$$n > \frac{\ln\left(\frac{2-1}{10^{-7}}\right)}{\ln 2} - 1 \Rightarrow n > 22.2$$

$\boxed{n = 23}$ or more (23 على الأقل)

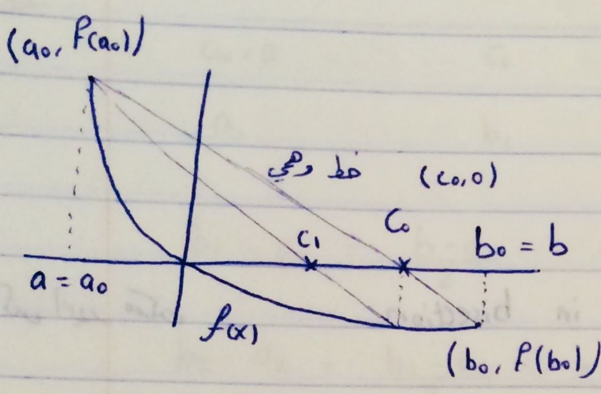
اجابة تقريبية

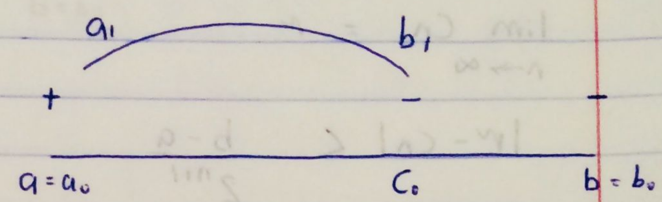
لو 10^{-2} (كبرنا) ⇐ تقل الاتيريشن

[2] False Position Method

نفس الطريقة السابقة
لكن C_n مختلفة فقط

$\Rightarrow$ need $f(x) = 0$, $[a, b] = [a_0, b_0]$





للإطلاع $\Leftarrow$

now what is C_0 ?

$$\text{Slope of } L = \frac{f(b_0) - f(a_0)}{b_0 - a_0} = \frac{0 - f(b_0)}{C_0 - b_0}$$

$$\frac{C_0 - b_0}{-f(b_0)} = \frac{b_0 - a_0}{f(b_0) - f(a_0)}$$

$$C_0 = b_0 - \frac{f(b_0)(b_0 - a_0)}{f(b_0) - f(a_0)}$$

<u>Similarily</u> : $C_1 = b_1 - \dfrac{f(b_1)(b_1 - a_1)}{f(b_1) - f(a_1)}$

$\vdots$

$$\boxed{C_n = b_n - \frac{f(b_n)(b_n - a_n)}{f(b_n) - f(a_n)}}$$

False - Position Formula

Ex $x \sin x = 1$ $[1,2]$ Solve by using Fals-Position

Ex Estimate $\sqrt[4]{12}$ in $[1,2]$ using the false-Position method
Find two iteration

Sol نحتاج فترة ⇐ جاهزة
نحتاج اقتران ⇐ ?

$$x = \sqrt[4]{12} \Rightarrow x^4 = 12 \Rightarrow \underbrace{x^4 - 12}_{f(x)} = 0 \quad , [a_0, b_0] = [1,2]$$

$-$ $+$

1 1.73333 2

$f(1) = -11$

$f(2) = 4$

نحن كله مع لعشر (الحد المطلوب) خانة ⇐ Rounding // chopping

$$C_0 = 2 - \frac{f(2)(2-1)}{f(2) - f(1)} = 2 - \frac{4}{4 - -11} = 1.73333$$

$f(c_0)$ ⇐ لخطوة نأخذ 1 أو 2 معه

$f(c_0) = -2.97331 \Rightarrow [a_1, b_1] = [1.73333, 2]$

نحسب

$$C_1 = 2 - \frac{f(2)(2 - 1.73333)}{f(2) - f(1.73333)} = 1.84703$$

3 Fixed-Point iteration (FPI Method)

$f(x) = 0 \Rightarrow x = P$ s.t $f(P) = 0$??

$f(x) = 0 \Rightarrow x - g(x) = 0$ $\qquad f(x) = x - g(x)$

يوجد عدد لا نهائي منهم

$\Rightarrow x = g(x)$

$x = P \Rightarrow f(P) = 0 \Rightarrow P - g(P) = 0 \Rightarrow P = g(P)$

$x = P$ is a fixed Point for $g(x)$

قيمة ثابتة

Def: $x = p$ is called a fixed point for $g(x)$ if $g(p) = p$

مش root الا اذا $0 = P \Rightarrow$ fixed p & root

Ex find the fixed pts of $g(x) = x^2 - 2$

Sol $x = x^2 - 2 \Rightarrow x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$\boxed{x = 2}$ $\boxed{x = -1}$

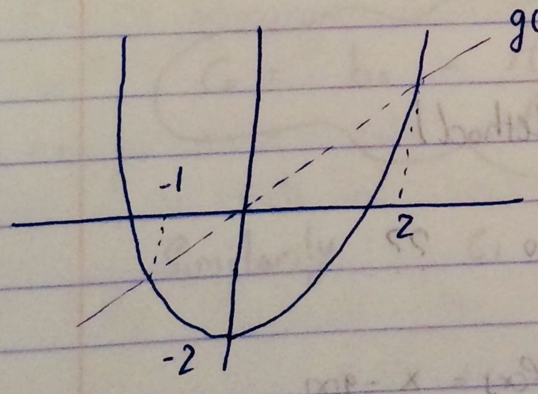
Check: $g(2) = 2^2 - 2 = 2$ ✓

$g(-1) = (-1)^2 - 2 = -1$ ✓

Ex $g(x) = x$ identity function كل نقطة الها نفس صورتها

$\Rightarrow x = x \Rightarrow$ كل النقاط f.p

* Geometrically speak, the fixed pts of $g(x)$ are the intersection point between $y = g(x)$ and $y = x$



HW find fixed pt of $g(x) = x\ln x - \ln x + 1$

Sol: $x\ln x - \ln x + 1 = x$

$x\ln x - \ln x + 1 - x = 0$

$\ln x(x-1) + (1-x) = 0$

$\Rightarrow (x-1)(\ln x - 1) = 0$

$x = 1$, $\ln x = 1 \Rightarrow x = e$

* <u>Fixed Point Iteration</u> كل الشغل على g مش f

$P_{n+1} = g(P_n)$, given P_0

معطى في الامتحان

Initial (guess) Value

استخدم بلزانو

$f(x) =$

$f(1) =$ -

$f(2) =$ +

$\Rightarrow 1.5 = P_0$

$P_1 = g(P_0)$

$P_2 = g(P_1)$

$P_3 = g(P_2)$

⋮

$P_n \Big]_{n=0}^{\infty}$

if $\lim_{n\to\infty} P_n = P \Rightarrow P$ fixed pt for $g(x)$

$\Rightarrow P$ Not for $f(x)$

Ex Consider the following FPI :

$\Rightarrow g(x) = \sqrt{5x-6}$

$P_{n+1} = \sqrt{5P_n - 6}$ with $P_0 = 2.5$

Find the first 20 iteration

Sol Calculation : $2.5 = \sqrt{(5 * Ans - 6)}$

$P_1 = 2.5495097$

$P_2 = 2.597604$ $P_{40} = 2.9993957$

$P_3 = 2.64386$

⋮

$P_{15} = 2.9449836$ Ans

$\lim_{n\to\infty} P_n = 3$

Check $g(3) = \sqrt{5(3)-6} = 3$

3 fixed Pt of g

$f(3) = 9 - 5(3) + 6 = 0$ $\Rightarrow$ 3 root of $f(x)$

if $P_0 = 6.5$ $\Rightarrow$ تبرح على 3

بس مش شرط

if $P_0 = 2$ $\Rightarrow$ 2 بترجع على 2

* **FPI** : $g(x)$, P_0

22-2-2018

$P_{n+1} = g(P_n)$ * $g(x)$ is continuos

$\{P_0, P_1, P_2, \ldots\} = P_n]_{n=0}^{\infty}$

$\lim_{n\to\infty} P_n = P$ إذا $P = \lim$ is $P \Leftarrow$ f.P

$\Rightarrow P$ is a fixed Point of $g(x)$

Proof: need to show $g(P) = P$

$P_{n+1} = g(P_n)$

$\lim_{n\to\infty} P_{n+1} = \lim_{n\to\infty} g(P_n)$

$P = g(\lim_{n\to\infty} P_n)$ cont g لأن

$P = g(P)$

Theorem: given a function $g(x)$ & $[a,b]$

if ① $g(x)$ is continuos in $[a,b]$ D $[2,3]$ R $[2.2, 2.7]$

② $g(x) \in [a,b]$ $\forall x \in [a,b]$ (Range Subset (S) Domain)

then at least a fixed pt for $g(x)$ in $[a,b]$ ممكن أكثر من وحدة

إضافة لذلك Moreover, if ③ $|g'(x)| \leq K < 1$ $\forall x \in [a,b]$

then ⓐ $\exists!$ fixed Point P for $g(x)$ in $[a,b]$

there exist a unique

ⓑ FPI of $g(x)$ will converge to P for any $P_0 \in [a,b]$

at least 1 fixed : شرطين

only one fixed : ٣ شرط

لتحقق

Note K : upper bound for $|g'(x)|$ on $[a,b]$ $= \max\limits_{a \le x \le b} |g'(x)|$

Note if $|g'(x)| > 1$ $\forall x \in [a,b]$ then FbI

تحو

will not converge to P, Provided that $P_0 \neq P$

اثبات Unique fixed Point : 1 2 3

اثبات fixed Point : 1 2

Proof: Suppose $\boxed{1}$ & $\boxed{2}$

حالة 1 ~ if $g(a) = a$, we are done, a is fixed Point

حالة 2 ~ if $g(b) = b$, " " "

حالة 3 ~ if $g(a) \neq a$ & $g(b) \neq b$

a g(a) (g(b)) b

بالداخل

$\Rightarrow g(a) > a$ & $g(b) < b$

$\Rightarrow g(a) - a > 0$ & $g(b) - b < 0$

now define $h(x) = g(x) - x$

Apply Belzano on $h(x)$ in $[a,b]$

$h(x)$ is Continuos on $[a,b]$ لمح شرطين

$h(a)\,h(b) < 0$

بلزانو

$\Rightarrow \exists$ at least $P \in (a,b)$ Such that $h(P) = 0$

$g(P) - P = 0$ $\Rightarrow g(P) = P$, P fixed Point for $g(x)$

Uniqeness لكن لم نفحص f.p في الإثبات السابق ، هنا وجود

<u>Now</u> suppoc 3 is also satisfies

To prove uniqeness $\Rightarrow$ الطريقة (assume ther are two f.p p & q for $g(x)$ in $[a,b]$) من هنا

$g(p) = p$, $g(q) = q$ [Assume We have 2 f.p]

Apply M.V.T on $g(x)$ on $\{p, q\} \subseteq [a,b]$

g Cont on $[p, q]$
g' exist on $[p, q]$ لازم أقل من واحد

$$\Rightarrow \exists\ c \in (p, q) \text{ s.t } g'(c) = \frac{g(q) - g(p)}{q - p}$$

$$g'(c) = 1 \Rightarrow |g'(c)| = 1$$

فرضنا انهم أقل من 1 Contradiction $\Rightarrow$ implies uniqeness

<u>Now</u> we will show that FPI of $g(x)$: $p_{n+1} = g(p_n)$ will converge to p for any $p_0 \in [a,b]$

Apply M.V.T on $g(x)$ on $[p_0, p]$

$\Rightarrow \exists\ c \in (p_0, p)$ s.t

$$g'(c) = \frac{g(p) - g(p_0)}{p - p_0} \Rightarrow \frac{g'(c)}{1} = \frac{p - p_1}{p - p_0}$$

$$|p - p_1| = |g'(c)|\ |p - p_0|$$

$$\boxed{|p - p_1| \leq k\,|p - p_0|}$$

$|P-P_1| < |P-P_0|$: P_1 is closer to P from P_0

Similarily, Apply M.V.T on $g(x)$ on $[P_1, P]$

$\Rightarrow |P-P_2| \leq K|P-P_1| \leq K^2|P-P_0|$

$\Rightarrow |P-P_2| < |P-P_1|$ FPI will converge to p

$|P-P_3| \leq K^3|P-P_0|$

$$\boxed{\underbrace{|P-P_n|}_{|E|} \leq K^n|P-P_0|}$$

upper bound for $|E|$ for FPI of $g(x)$

* Theorem for FPI of $g(x)$ ① $|P-P_n| \leq K^n|P-P_0|$

$$|P-P_n| \leq \frac{K^n|P_1-P_0|}{1-K}$$ upper bound

Used to find number of iteration theoretically for FPI

$$\frac{K^n|P_1-P_0|}{1-K} < \epsilon$$

$$K^n < \frac{\epsilon(1-K)}{|P_1-P_0|} \Rightarrow n \ln K < \ln\left(\frac{\epsilon(1-K)}{|P_1-P_0|}\right)$$

الشي

$$\Rightarrow n > \frac{\ln\left(\frac{\epsilon(1-K)}{|P_1-P_0|}\right)}{\ln K}$$

مهم جداً

Ex Consider $g(x) = \sqrt{2+\ln x}$ in $[1,2]$

ⓐ show that $g(x)$ has a unique fixed pt in $[1,2]$

ⓑ Show that FPI of $g(x)$ will converge

ⓒ Estimate the fixed pt of $g(x)$ with error less than 10^{-2}, $P_0 = 1.5$

ⓓ Find # of iteration needed to estimate f.p with accuracy of $5 * 10^{-4}$, $P_0 = 1.5$

Solution ⓐ (1) g is clearly continuous

(2) $g'(x) = \dfrac{\frac{1}{x}}{2\sqrt{2+\ln x}} = \dfrac{1}{2x\sqrt{2+\ln x}}$ مشتقة ما تحت الجذر $\dfrac{}{2\sqrt{\ }}$

على الفضاء x +

g is increasing in $[1,2]$ → $\sqrt{2+\ln x}$ + الجذر دائماً +

$g(1) \le g(x) \le g(2)$

$\sqrt{2} \le g(x) \le \sqrt{2+\ln 2}$

$1 < 1.414 \le g(x) \le 1.64 < 2 \Rightarrow \exists$ fixed Point   Range ⊆ Domain

(3) $|g'(x)| = \left|\dfrac{1}{2x\sqrt{2+\ln x}}\right| = \dfrac{1}{2x\sqrt{2+\ln x}}$ which is decreasing [مقلوب]

$\le \dfrac{1}{2(1)\sqrt{2+\ln 1}} = \underbrace{\boxed{0.35355}}_{K} < 1$

عوضنا الواحد فقط لأننا نريد upper

$\Rightarrow$ (a), (b)

(c) $P_{n+1} = g(P_n) = \sqrt{2+\ln P_n}$, $P_0 = 1.5$

P_K	$\lvert P_K - P_{K-1} \rvert$
1.5	—
1.550956	0.05 > 0.01
1.56169	0.011 > 0.01
1.56389	0.002 < 0.01

كل وحدة ناقصة الي قبلها

(d) $n > \dfrac{\ln\left(\dfrac{\epsilon(1-K)}{|P_1 - P_0|}\right)}{\ln K} \Rightarrow n > \dfrac{\ln\left(\dfrac{5*10^{-4}(1-0.35355)}{|1.550956 - 1.5|}\right)}{\ln(0.35355)}$

واذا غير P_0 منحسبها من

$n > 4.867$

$\boxed{n = 5}$

$\pi = 3.14\ldots$

$\Rightarrow g(p) = p$

27-2-2018

* **Theorem** if P is a fixed pt for $g(x)$
then [1] $|g'(p)| < 1$ then P is called attractive
[2] $|g'(P)| > 1$ = = = = repulsive
تروح لقيمة غير f.p تنتج ∞ او another f.p

if $|g'(p)| = 1 \Rightarrow$ depend on P

EX Find the fixed pts of $g(x) = \frac{x^2}{2} + \frac{1}{x} - \frac{1}{2}$

Then classify them to attractive or repulsive

Solution $x = g(x) \Rightarrow x = \frac{x^2}{2} + \frac{1}{x} - \frac{1}{2}$

$$2x^2 = x^3 + 2 - x$$
$$x^3 - 2x^2 - x + 2 = 0$$
$$x^2(x-2) - (x-2) = 0$$
$$(x-2)(x^2-1) = 0$$
$$x = 2,\ x = 1,\ x = -1$$

$g'(x) = x - \frac{1}{x^2} \Rightarrow |g'(2)| = |2 - \frac{1}{4}| = \frac{7}{4} > 1$

$\Rightarrow P = 2$ is repulsive

$|g'(1)| = 0 < 1 \Rightarrow P = 1$ is attractive

$|g'(-1)| = -1 - 1 = -2 \Rightarrow |-2| = 2 > 1 \Rightarrow P = -1$ is repulsiva

repulsive may go to $\boxed{P=1}$ because it's a attractive

توضيح $P_0 = -1.5$ $P_0 = 0.5$
FPI $\rightarrow \infty$ FP2 $\rightarrow 1$

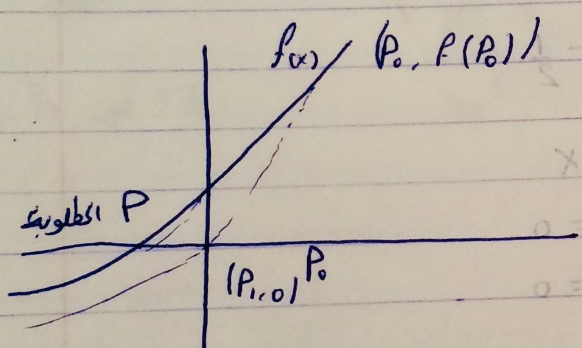
FPI $P_{n+1} = g(P_n)$

$$= \frac{P_n^2}{2} + \frac{1}{P_n} - \frac{1}{2}$$

equal $^-0.5$ $\quad ^-0.5 = \frac{Ans^2}{2} + \frac{1}{Ans} - \frac{1}{2}$

$\boxed{4}$ Newton's Method

$\underline{f_{(x)}} = 0$, $\underline{P_0}$ تخمين



تقاطع المماس
مع محور السينات

تقريب $P_1 = P_0 - \frac{f(P_0)}{f'(P_0)}$

$$P_2 = P_1 - \frac{f(P_1)}{f'(P_1)}$$

$g(x) = x - \frac{f(x)}{f'(x)}$

$P_{n+1} = g(P_n) = P_n - \frac{f(P_n)}{f'(P_n)}$

$\vdots$

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$$

Note Newton's method can be consider an example of FPI with $g(x) = x - \frac{f(x)}{f'(x)}$