Ex: Solve the equation $x = e^{-x} + 1$ with $P_0 = 1.5$ using newtons method. Find five iteration.

Solution $f(x) = x - e^{-x} - 1$, $P_0 = 1.5$

$x = e^{-x} + 1$

أو العكس

$f'(x) = 1 + e^{-x}$

$$P_{n+1} = P_n - \frac{P_n - e^{-P_n} - 1}{1 + e^{-P_n}}$$ *

$x = e^{-x} + 1$

1.5 - Ans- ...

$P_0 = 1.5$, $P_1 = 1.273638286 \Rightarrow 0.23$

$P_2 = 1.278462001 \Rightarrow 0.003$

$P_3 = 1.278464543 \Rightarrow 0.000002$

$P_4 = 1.278464543 \Rightarrow 0$

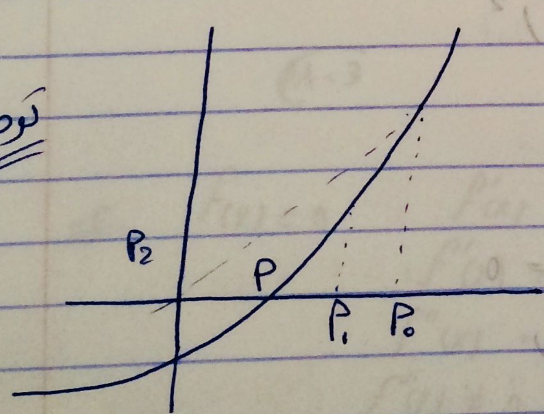
$P_5 = 1.278464543 \Rightarrow 0$

5 The Secant Method

$f(x) = 0$, P_0, P_1 شرطها

$$P_2 = P_1 - \frac{f(P_1)(P_1 - P_0)}{f(P_1) - f(P_0)}$$

توضيح



ما نبع على الكالكوليتر

2 condition ⇐

$$P_3 = P_2 - \frac{f(P_2)(P_2 - P_1)}{f(P_2) - f(P_1)}$$

$$P_{n+1} = P_n - \frac{f(P_n)(P_n - P_{n-1})}{f(P_n) - f(P_{n-1})}$$

EX Estimate $\sqrt[3]{20}$ using $P_0 = 2$, $P_1 = 3$ using a Secant method. Find two Iteration

أعطيتهم P_0, P_1

$P_2 = ??$

Solution $f(x) = x^3 - 20$, $P_2 = 3 - \frac{f(3)(3-2)}{f(3) - f(2)} = 3 - \frac{7}{7-12} = 2.63157$

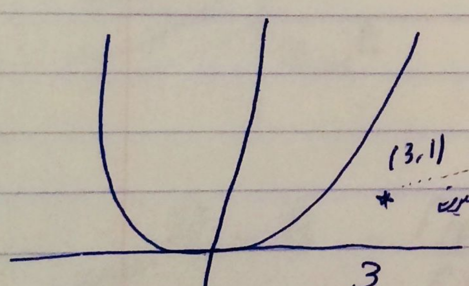
$$P_3 = \overset{P_2}{2.63157} - \frac{f(2.63157)(2.63157 - 3)}{f(2.63157) - f(3)} =$$

في هذه الطريقة ممكن أن يكون الجواب سالب

Q Find the Point on $y = x^2$ that is the Closet to Point $(3,1)$

في مثل هذه الأسئلة
استخدم نيوتن

x	y
3	1



$$\text{Distance} = \sqrt{(x - \hat{x})^2 + (y - \hat{y})^2}$$

$$(x - 3)^2 + (x^2 - 1)^2$$

$$\text{Distance} = \sqrt{(x - 3)^2 + (x^2 - 1)^2}$$

$$d^2 = (x-3)^2 + (x^2 - 1)$$

$$2dd' = \underbrace{\boxed{2(x-3) + 2(x-1)(2x) = 0}}_{f(x)}$$

مثلا $f(0) =$

$f(1) = \quad > 0$ $\qquad \frac{1+2}{2} = 1.5$

$f(2) = \quad < 0$

$P_0 = 1.5 \Rightarrow$ Newton

$f(x) = (x-2)^3 (x+1)^1$

root : $p = 2$

$p = -1$

* Multiplicity of roots

Def Let P be a root for $f(x)$

if $f(p) = f'(p) = f''(p) = \dots = f^{(M-1)}(p) = 0$

but $f^{(M)}(p) \neq 0$

then we say P has Multiplicity M.

Another Def Let P be a root for $f(x)$, we say that P has Multiplicity M if we can write $f(x)$ as:

$f(x) = (x-p)^M \cdot h(x)$; $h(p) \neq 0$ Mult. عدد مرات التكرار

قوة ↓

Ex: $f(x) = (x-2)^3 (x+1)$ $P = 2$, $P = -1$

Find Multiplicity for each root

Sol $P = 2$: $f(x) = (x-2)^3 (x+1)$, $h(x) = x+1$

$h(2) = 3 \neq 0$

$M = 3$

or $f(2) = 0$: $f'(x) = (x-2)^3 + 3(x-2)^2 (x+1)$

$f'(2) = 0$

$f''(x) = 3(x-2)^2 + 3(x-2)^2 + (x+1)\, 6(x-2)$

$f''(2) \neq 0$?

$f^3(x) =$

* $f^3(2) \neq 0 \Rightarrow M = 3$

$\underline{P=-1}$ $f(x) = (x+1)(x-2)^3$

$(x - -1)\underbrace{(x-2)^3}_{h(x)}$

$h(-1) \neq 0 \Rightarrow M=1$

<u>or</u> $f(-1) = 0$

$f'(-1) = 27 \neq 0$

لا تساوي صفر ✓ ←

$\boxed{P=1}$

<u>Note</u> $f(x-1)\ln x$ $h(1) = 0 \Rightarrow$ so differ. it

$df = \ln x + \frac{x-1}{x} + 1 - \frac{1}{x} \Rightarrow 0$

$\boxed{M=2}$ $ddf = \frac{1}{x} + \frac{1}{x^2} \neq 0 \Rightarrow \boxed{M=2}$

* <u>Notations</u> * P has Multiplicity $=1 \Rightarrow P$ is called simple root

* P has $M > 1 \Rightarrow P$ is called Multiple root

$M=2 \Rightarrow$ double root

$M=3 \Rightarrow$ Cubic root

<u>Ex</u> $f(x) = (x-1)\ln x$, find M for root

<u>Sol</u> $P=1$ root

$f'(x) = \frac{x-1}{x} + \ln x \Rightarrow f'(1) = 0$ $1 - \frac{1}{x} + \ln x$

$f''(x) = \frac{x-x+1}{x^2} + \frac{1}{x} \Rightarrow f''(1) = 2 \neq 0$ $+\frac{1}{x^2} + \ln x$

$\Rightarrow M=2$, $P=1$ is double root

<u>Ex</u> $f(x) = (x-1)^2 \ln x$, $P=1$ is cubic root $\underline{M=3}$

* $f(x) = x^3 - 70$

$f'(x) = 3x^2 \neq 0 \Rightarrow$ simple root

* **Order of Convergence**: is the number that Measures the speed of convergence of any numerical method (any sequence P_n)

$R > 0$ عدد موجب

* Notation sequance $\{P_n\}_{n=0}^{\infty} \longrightarrow P$ exact root

$|E_n| = |P - P_n|$ القيمة الحقيقية - التقريب

$|E_{n+1}| = |P - P_{n+1}|$

Def Let $\{P_n\}$ be a sequence converge to P then

if $\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^R} = A$, then we say the order of convergence is $R>0$, where A is called the asymptotic error constant $(A>0)$

Note Find Order of Convergence $\Rightarrow$ R && A

Def Sequence $[P_n]_{n=0}^{\infty} \longrightarrow P$ (1-3-2018)

$$|E_n| = |P - P_n|$$

$$\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^R} = A \quad , R, A > 0$$

تمثيل السرعة

we say P_n converge to P with order of convergence $= R$

* A : asymptotics error constant

~~for~~ for n large $\frac{|E_{n+1}|}{|E_n|^R} \approx A$

$$\Rightarrow \quad \boxed{|E_{n+1}| \approx A|E_n|^R}$$

constant

I claim : $R\uparrow$, convergence is faster.

Method 1	Method 2
فرضاً $R = 1$	$R = 2$

R أكبر ← أسرع

* Suppose $|E_n| \approx 0.1$

$|E_{n+1}| \approx A|E_n|^1 \approx A(0.1)$ $\qquad$ $|E_{n+1}| \approx A|E_n|^2 = A(0.1)^2 = 0.01$

الايرور تقل ، الكونفيرجنس تزيد ، السرعة تزيد

R && A

Ex find the order of convergence of the sequence $= \frac{1}{6^n}$

Solution $\{\frac{1}{6}, \frac{1}{36}, \frac{1}{216}, \ldots\}$, $P_n = \frac{1}{6^n}$

$P = 0$ اللمت تروح عليه

$$\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^R} = \lim_{n\to\infty} \frac{|P_{n+1} - P|}{|P_n - P|^R}$$

$$= \lim_{n\to\infty} \frac{\left|\frac{1}{6^{n+1}} - 0\right|}{\left|\frac{1}{6^n} - 0\right|^R}$$

$\cancel{\times 6^{R} 6}$

$$\Rightarrow \lim_{n\to\infty} \frac{\frac{1}{6^{n+1}}}{\frac{1}{6^{nR}}} = \lim_{n\to\infty} \frac{6^{nR}}{6^{n+1}} = \lim_{n\to\infty} 6^{nR-n-1}$$

$\underline{\underline{1^R = 1}}$

$$\Rightarrow \lim_{n\to\infty} 6^{-1} * 6^{n(R-1)} = \boxed{\frac{1}{6}} \lim_{n\to\infty} 6^{n(R-1)}$$ ٣ حالات

$$= \frac{1}{6} \begin{cases} \infty , & R > 1 \quad \times \\ 0 , & R < 1 \quad \times \\ \frac{1}{6} , & R = 1 \quad \checkmark \end{cases}$$

لأن a عدد صحيح

$$\Rightarrow R = 1 , \quad A = \frac{1}{6}$$

* <u>Notes</u> $R = 1 \Rightarrow$ Convergence is linear

ممكن يكون كسر

$R = 2 \Rightarrow$ " " quadratic أسرع

$R = 3 \Rightarrow$ " " cubic

<u>نظرية مهمة</u>

Newton Method , $P_{n+1} = P_n - \dfrac{f(P_n)}{f'(P_n)}$ for $f(x)$

assume $\{P_n\} \to P$

$\boxed{1}$ if P is simple root (M = 1) , <u>then</u> $R = 2$

مليلسي

$$A = \left| \frac{f''(P)}{2 f'(P)} \right|$$

تعني

(i.e : $\lim \dfrac{|E_{n+1}|}{|E_n|^2} = \left|\dfrac{f''(P)}{2f'(P)}\right|$

$\boxed{2}$ if P is a multiple root $\boxed{(M>1)}$

then $R = 1$, $A = \dfrac{M-1}{M}$

simple ⇒ سريعة

multiple ⇒ بطيئة

(i.e $\lim\limits_{n\to\infty} \dfrac{|E_{n+1}|}{|E_n|} = \dfrac{M-1}{M}$

* <u>Secant Method</u> : $P_{n+1} = P_n - \dfrac{f(P_n)(P_n - P_{n-1})}{f(P_n) - f(P_{n-1})}$

1 if P simple root

$R = 1.618$, $A = \left|\dfrac{f''(P)}{2f'(P)}\right|^{\underline{\underline{0.618}}}$

2 if P multiple root

$R = 1$, A depend on M

* <u>Bisection</u>

$R = 1$, $A = 0.5$ always

بطيئة

* <u>False - Position</u>

لا يعتمد على الرووت

ع آخر

$R = 1$, A depend on $f(x)$

* <u>FPT</u> :

depends on $g(x)$

Ex P(Known) في هذا المثال

$f(x) = (x+2)(x-1)^2$

$P = -2 \Rightarrow M = 1$

$P = 1 \Rightarrow M = 2$

1. Newton find R, A theoretically for each root

$\boxed{P=-2}$, $R = 2$, $A = \left|\frac{f''(-2)}{2f'(-2)}\right|$

$f(x) = (x+2)(x^2-2x+1) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3$

$f''(x) = 6x$

يعني $\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^2} = \frac{2}{3} = A$

$\Rightarrow A = \left|\frac{(-12)}{2(9)}\right| = \frac{12}{18} = \frac{2}{3} = A$

$\boxed{P=1}$: $R = 1$, $A = \frac{M-1}{M} = \frac{2-1}{2} = \frac{1}{2}$

$\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|} = \frac{1}{2} = 0.5$

2. R, A, Secant

$\boxed{P=-2}$: $R = 1.618$, $A = \left|\frac{f''(-2)}{2f'(-2)}\right|^{0.618} =$

$\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^{1.618}} = 0.77835$

$P=1$, $R=1$, A نظرياً لا نعرف

3 Bisection , P, A $\Rightarrow$ $P=-2$, $P=1$

$R=1$, $A=0.5$

$$\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|} = 0.5$$

4 False-Position : $R=1$, $A=?$

Ex $f(x) = x^3 - 27$ (6-3-2018)

Solution $P = 3$ (Known)

P_0 مجهولة لأنّ

① find R, A if we used Newton method (theoretically)
② Start with 3.5 using three iteration of Newton
Find R, A numerically

① $f'(x) = 3x^2$

$f'(3) = 27 \neq 0 \Rightarrow \boxed{M=1}$ simple root

$\boxed{R=2}$, $A = \left|\frac{f''(3)}{2f'(3)}\right| = \left|\frac{18}{2(27)}\right| = \frac{18}{54} = \frac{1}{3} = 0.333$

* this means $\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^2} = \frac{1}{3}$

② $P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$

P is Known

$\frac{P - P_{n+1}}{(P - P_n)^R}$

$0.068027211 / 0.5^2$

P_k	$\lvert E_k \rvert = \lvert 3 - P_k \rvert$	$\frac{\lvert E_{k+1} \rvert}{\lvert E_k \rvert^{2}} = A$
3.5	0.5	0.272108
3.068027211	0.068027211	0.323533
3.001497216	0.001497216	0.332355
3.000000747	0.000000747	$\downarrow \frac{1}{3}$

تناقص بسرعة

Ex $f(x) = (x-2)^2 (x+1)$

Solution Root : $P = 2$, $M = 2$; $P = -1$, $M = 1$

$R = 1$
$\Rightarrow A = \frac{1}{2}$

$$f(x) = x^3 - 3x^2 + 4$$
$$f'(x) = 3x^2 - 6x$$
$$f''(x) = 6x - 6$$

Newton $P = 2$	$E_k = \lvert P - P_k \rvert$	Newton $\boxed{P = -1}$
$P_0 = 2.5$	0.3	$P_0 = -1.5$
$P_1 = 2.26667$	0.26662	$P_1 = -1.111$
$P_2 = 2.13856$	0.13056	$P_2 = -1.00740$
$P_3 = 2.070777$		$P_3 = -1.000036311$
$P_4 = 2.03579$		$P_4 = -1.000000001$
$P_5 = 2.0180008$		$P_5 = -1$
$\vdots$		

$$P_{25} = 2.0000000547$$

* Accelerated Newton Iteration

Newton for Multiple roots is slow $R = 1$

P root $M > 1$

$$P_{n+1} = P_n - \frac{M f(P_n)}{f'(P_n)}$$ Accelerated Newton

$\underline{R = 2}$

<u>Ex</u> For previous example $P = 2$, $R = 1$, $\boxed{M = 2}$

use accelerated Newton method

$$P_{n+1} = P_n - \frac{2(P_n^3 - 3P_n^2 + 4)}{3P_n^2 - 6P_n}$$

$P_0 = 2.5$ في هذه الطريقة $R = 2$

$P_1 = 2.0333$

$P_2 = 2.000182$ $\boxed{R = 2}$

$P_3 = 2.0000005$ $\therefore$

find A theo. ① فرق الاريور

$\frac{1}{1} \quad \frac{1}{1^2}$ خرقش

②

* FPI , R, A

Theorem let P be a fixed pt for $g(x)$

If $g'(P) = g''(P) = \dots = g^{(n-1)}(P) = 0$

but $g^{(n)}(P) \neq 0$ Multiplicity لا علاقة له بـ $\Rightarrow$ root فش

then FPI will converge to P with $R = n$, $A = \left| \frac{g^{(n)}(P)}{n!} \right|$

$\Rightarrow$ * **Theoretically** $\Leftarrow$

Ex $g(x) = \frac{x}{2} + \frac{2}{x}$, $P = 2$ fixed Point

① find R, A, Theoritically if we used FPI

② Prove your claim in (a) using $P_0 = 2.5$ & three Iteration of FPI

$x = \frac{x}{2} + \frac{2}{x} \Rightarrow$ f.p =

Solution $g'(x) = \frac{1}{2} - \frac{2}{x^2}$

$g'(2) = \frac{1}{2} - \frac{1}{2} = 0$

$$g''(x) = \frac{4}{x^3}$$

$$P_{k+1} = \frac{Ans}{2} + \frac{2}{Ans}$$

$$g''(2) \neq 0 = \frac{4}{2^3} = \frac{1}{2}$$

$R = n$

FPI $\boxed{R = 2}$, $A = \left| \frac{g''(2)}{2!} \right| = 0.25$

أهم شي هذه الأشكال

اسهل

P_k	$P_k - P_{k-1}$	$\lvert E_k \rvert$	$\frac{\lvert E_{k+1} \rvert}{\lvert E_k \rvert^2}$
2.5		0.5	0.20
2.05		0.05	0.2439
2.000609756		0.000609756	0.25013
2.000000093		0.000000093	

$\downarrow$ $0.25 = \frac{1}{4}$

Proof need to show $\lim_{k \to \infty} \frac{|E_{k+1}|}{|E_k|^n} = \left| \frac{g^{(n)}(P)}{n!} \right|$

$P_{n+1} = g(P_n)$

Apply Taylor's Exp. for $g(x)$ about P

$$g(x) = g(P) + \cancel{g'(P)(x-P)} + \cancel{\frac{g''(P)(x-P)^2}{2}} + \dots + \cancel{\frac{g^{(n-1)}(P)(x-P)^{n-1}}{(n-1)!}} + \frac{g^{(n)}(c)(x-P)^n}{n!}$$

كل المشتقات صفر حسب تعريف الملتبلستي

$$g(x) = P + \frac{g^{(n)}(c)(x-P)^n}{n!}$$

$\boxed{x = P_k}$: $g(P_k) = P + \frac{g^{(n)}(c)(P_k - P)^n}{n!}$

$$P_{k+1} - P = \frac{g^{(n)}(c)(P_k - P)^n}{n!} \quad . \quad c \text{ between } P \text{ \& } P_k$$

$$\left| \frac{P_{k+1} - P}{(P_k - P)^n} \right| = \left| \frac{g^{(n)}(c)}{n!} \right|$$

$$\Rightarrow \lim_{n \to \infty} \frac{|E_{k+1}|}{|E_k|^n} = \left| \frac{g^{(n)}(P)}{n!} \right|$$

$P_k \approx P$

$c \approx P$

$P_k \to P$

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$$

Q Prove that if P is a simple root for $f(x)$, then Newton iteration will converge to P with $R = 2$, $A = \left|\frac{f''(P)}{2f'(P)}\right|$

$M = 1 \Rightarrow f'(P) \neq 0$

سريع $g'(P) = 0$

Proofs Apply Taylor Exp. for $f(x)$ about P_n $\Rightarrow$ Center

$$f(x) = f(P_n) + f'(P_n)(X - P_n) + \frac{f''(c)(X - P_n)^2}{2}$$

Let $x = P$ $\quad f(P) = f(P_n) + f'(P_n)(P - P_n) + \frac{f''(c)}{2}(P - P_n)^2$

Divide by $f'(P_n)$ (حتى تصبح شكل المطلوب)

root P لأن

$f(P) = 0$

$f(P_n) \to 0$

$$0 = \frac{f(P_n)}{f'(P_n)} + P - P_n + \frac{f''(c)}{2f'(P_n)}(P - P_n)^2$$

$$P - \left(P_n - \frac{f(P_n)}{f'(P_n)}\right) = \frac{-f''(c)}{2f'(P_n)}(P - P_n)^2$$

$$|P - P_{n+1}| = \left|\frac{-f''(c)}{2f'(P_n)}\right| \left|(P - P_n)^2\right|$$

$$\frac{|E_{n+1}|}{|E_n|^2} = \left|\frac{f''(c)}{2f'(P_n)}\right|$$

$$\lim_{n\to\infty} \frac{|E_{n+1}|}{E_n^2} = \left|\frac{f''(P)}{2f'(P)}\right|$$

Since when $n \to \infty$ $\quad c \simeq P_n \simeq P$

c بين P_n, P

P

$$\frac{g''(P)}{2}$$

Q Prove that if P is a multiple root for $f(x)$ $(m > 1)$ then Newton has $R = 1$, $A = \frac{M-1}{m}$

Hint Apply Taylor about $x = P$ for $f(x)$

center $\Rightarrow$ $\underline{a}$ not $\underline{x}$

$$f(x) = \cancel{f(p)}^{0} + \cancel{f'(p)(x-p)}^{0} + \cdots + \cancel{\frac{f^{M-1}(p)}{(M-1)!}}^{0}(x-p)^{M-1} + \frac{f^{M}(c)}{M!}(x-p)^{M}$$

$$f(x) = \frac{f^{M}(c)(x-p)^{M}}{M!} \Rightarrow f'(x) = \frac{\cancel{M}\, f^{M}(c)(x-p)^{M-1}}{\cancel{M}(M-1)!}$$

المشتق $$f'(x) = \frac{\overset{\text{constant}}{f^{M}(c)}\,(x-p)^{M-1}}{(M-1)!}, \quad M! = M(M-1)!$$

نقسم $$\frac{f(x)}{f'(x)} = \frac{x-p}{M} \Rightarrow \frac{f(p_n)}{f'(p_n)} = \frac{p_n - p}{M} \rightarrow$$

$$\Downarrow$$

$$\frac{\dfrac{f^{\cancel{M}}(c)(x-p)^{M}}{M!}}{\dfrac{f^{M}(c)\,(x-p)^{M-1}}{(M-1)!}} = \frac{(x-p)^{M-(M-1)}}{M} = \frac{f(x)}{f'(x)}$$

let $x = p_n \Rightarrow \dfrac{f(p_n)}{f'(p_n)} = \dfrac{p_n - p}{M}$

للطرفين $$\underline{\underline{p_n}} - \frac{f(p_n)}{f'(p_n)} = p_n - \frac{p_n - p}{M}$$

$$\underset{-p}{} \quad p_{n+1} = \underset{-p}{p_n} - \frac{p_n - p}{M}$$

$$\frac{p - p_{n+1}}{p - p_n} = 1 - \frac{1}{M} = \frac{M-1}{M}$$

$$\left.\frac{|E_{n+1}|}{|E_n|} = \frac{M-1}{M}\right] \text{ Constant}$$

$$\lim \frac{|E_{n+1}|}{|E_n|} = \underbrace{\frac{M-1}{M}}_{\text{Constant}}$$