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Numerical Methods

طرق التحليل العددي

MATH 331

جامعة بيرزيت

BIRZET UNIVERSITY

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Chapter 3

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How to Solve a square $n \times n$ system of equation

System → linear system $A_{n\times n} X_{n\times 1} = b_{n\times 1}$

System → nonlinear System
1. Newton Iteration
2. FPI [Jacobi Iteration]
3. Gauss-Seidel Iteration [Seidel Iteration]

2x2 System

one Point

$f_1(x, y) = 0$

$f_2(x, y) = 0$

$\Rightarrow$ Exact equation $(P, q) = \begin{pmatrix} P \\ q \end{pmatrix}$

Ex $\quad 2x^2 + \cos y = 1$

$\quad 3xy^2 + \ln(x+y) = x$

بنحولها للشكل العام

3x3 System

$f_1(x, y, z) = 0$

$f_2(x, y, z) = 0$

$f_3(x, y, z) = 0$

$\Rightarrow$ Exact Solution $(P, q, r) = \begin{pmatrix} P \\ q \\ r \end{pmatrix}$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = |A| = ad - bc$$

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\big(\quad\big)_{2\times 2}\big(\quad\big)_{2\times 1} = \big(\quad\big)_{2\times 1} \quad \checkmark$$

* Jacobian Matrix of 2x2 System

$$f_1(x, y) = 0$$
$$f_2(x, y) = 0$$

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

مشتقة f_1 بالنسبة لـ x ، $y \Leftarrow$ const

Ex ① find the Jacobian matrix of system

$$x^2y^2 + \ln(xy) = 3$$
$$xy^2 + y^2 = 4$$

② find $J^{-1}\big|_{(1,2)}$ عند

Solution ① $f_1 = x^2y^2 + \ln(xy) - 3$

$$f_2 = xy^2 + y^2 - 4$$

$$\Rightarrow \begin{pmatrix} 2xy^2 + \frac{1\cancel{y}}{x\cancel{y}} & 2yx^2 + \frac{1}{y} \\ y^2 & 2xy + 2y \end{pmatrix}$$

② $J|_{1,2} = \begin{pmatrix} 9 & 4,5 \\ 4 & 8 \end{pmatrix}$

$|J| = 72 - 18 = 54 \neq 0$

$J^{-1}|_{1,2} = \frac{1}{54}\begin{pmatrix} 8 & -4,5 \\ -4 & 9 \end{pmatrix} = \begin{pmatrix} 0.148 & -0.0833 \\ -0.0741 & 0.167 \end{pmatrix}$

<u>Methods</u> $\boxed{1}$ Newtons iteration for only 2x2 system

$f_1(x, y) = 0$

given (P_0, q_0)

$f_2(x, y) = 0$

$$\Rightarrow \begin{pmatrix} P_{n+1} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} P_n \\ q_n \end{pmatrix} - J^{-1}|_{(P_n, q_n)} \cdot \begin{pmatrix} f_1(P_n, q_n) \\ f_2(P_n, q_n) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} P_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} P_0 \\ q_0 \end{pmatrix} - J^{-1}|_{(P_0, q_0)} \begin{pmatrix} f_1(P_0, q_0) \\ f_2(P_0, q_0) \end{pmatrix}$$

%100 جاي
امتحان

Ex Solve the system below using newtons Iteration Starting with (1,2), find only one Iteration.

$f_1(x, y) \Rightarrow x^2y^2 + \ln(xy) - 3 = 0$

$f_2(x, y) \Rightarrow xy^2 + y^2 - 4 = 0$

$$\begin{pmatrix} P_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - J^{-1}_{1\,(1,2)} \cdot \begin{pmatrix} f_1(1,2) \\ f_2(1,2) \end{pmatrix}$$

$$f_1(1,2) = 4 + \ln 2 - 3 = 1.693$$

$$f_2(1,2) = 4$$

ملاحظات : الشكل العام
٣-٤ شارك

$$J^{-1}_{1\,1,2}$$

$$\begin{pmatrix} P_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0.148 & -0.0833 \\ -0.0741 & 0.167 \end{pmatrix} \begin{pmatrix} 1.693 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} -0.0826 \\ 0.543 \end{pmatrix} = \begin{pmatrix} 1.0826 \\ 1.457 \end{pmatrix} \begin{matrix} \rightsquigarrow P_1 \\ \rightsquigarrow q_1 \end{matrix}$$

$\boxed{2}$ FPI 2X2 , 3X3

* $\boxed{2X2}$

$$f_1(x,y) = 0 \Rightarrow x = g_1(x,y) \Rightarrow P_{n+1} = g_1(P_n, q_n)$$
$$f_2(x,y) = 0 \Rightarrow y = g_2(x,y) \Rightarrow q_{n+1} = g_2(P_n, q_n)$$

$P_0, q_0 \Rightarrow$ given

* $\boxed{3X3}$

$$f_1(x,y,z) = 0 \Rightarrow x = g_1(x,y,z) \Rightarrow P_{n+1} = g_1(P_n, q_n, r_n)$$
$$f_2(x,y,z) = 0 \Rightarrow y = g_2(x,y,z) \Rightarrow q_{n+1} = g_2(P_n, q_n, r_n)$$
$$f_3(x,y,z) = 0 \Rightarrow z = g_3(x,y,z) \Rightarrow r_{n+1} = g_3(P_n, q_n, r_n)$$

In $\boxed{2X2}$

$$P_1 = g_1(P_0, q_0) \qquad P_2 = g_1(P_1, q_1)$$
$$q_1 = g_2(P_0, q_0) \qquad q_2 = g_2(P_1, q_1)$$

In $\boxed{3X3}$ the same

Ex Solve the system using FPI with $(P_0, q_0) = (1, 0.5)$

Find (P_2, q_2)

$$x = x^2 + \sin(\pi y) \quad g_1(x, y)$$

$$y = \cos(\pi x) - y \quad g_2(x, y)$$

$$P_1 = g_1(1, 0.5) = 1 + 1 = 2$$

$$q_1 = g_2(1, 0.5) = -1.5$$

$$P_1, q_1 = (2, -1.5)$$

$$P_2 = g_1(2, -1.5) = 5$$

$$q_2 = g_2(2, -1.5) = 2.5$$

$$(P_2, q_2) = (5, 2.5)$$

$\boxed{3}$ Gauss - Seidal Iteration

Is an improvement of FPI

$\boxed{2\times2}$

$$x = g_1(x, y)$$

$$y = g_2(x, y)$$

given (P_0, q_0)

$$P_1 = g_1(P_0, q_0)$$

$$q_1 = g_2(P_1, q_0)$$

(P_1, q_1)

$p_2 = g_1(p_1, q_1)$
$q_2 = g_2(p_2, q_1)$

$p_{n+1} = g_1(p_n, q_n)$

$q_{n+1} = g_2(p_{n+1}, q_n)$

$\boxed{3\text{X}3}$
$X = g_1(X, Y, Z)$
$Y = g_2(X, Y, Z)$ (p_0, q_0, r_0)
$Z = g_3(X, Y, Z)$

$p_1 = g_1(p_0, q_0, r_0)$
$q_1 = g_2(p_1, q_0, r_0)$
$r_1 = g_3(p_1, q_1, r_0)$

Ex Solve the system below using Seidel iteration using (1, 1, 1) Find (p_2, q_2, r_2)

$g_1(X, Y, Z) = -(X-1)^2 - (y-1)^2 - (Z-1)^2$
$g_2(X, Y, Z) = 2X^2 + 3y^2 - 5Z^2$
$g_3(X, Y, Z) = e^X - \cos(\pi Y)$

$p_1 = g_1(1, 1, 1) = 0$

$q_1 = g_2(0, 1, 1) = -2$ p_1, q_1, r_1
$(0, -2, 0)$

$r_1 = g_3(0, -2, 1) = 0$

$p_2 = g_1(0, -2, 0) = -11$
$q_2 = g_2(-11, -2, 0) = 254$
$r_2 = g_3(-11, 254, 0) = -0.999$

$$3x_1 - x_2 + x_3 = 7$$
$$x_1 + 5x_2 - 3x_3 = 8$$
$$4x_1 - x_2 + x_3 = -2$$

$$Ax = b$$

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 1 & 5 & -3 \\ 4 & -1 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad b = \begin{pmatrix} 7 \\ 8 \\ -2 \end{pmatrix}$$

* <u>Cost</u> : Number of operation $(+, -, \div, *)$

$$\frac{2+3*5-1}{4} \Rightarrow \text{Cost} = 4$$

$$\text{Cost of } 2^2 = 2*2 \Rightarrow \text{Cost} = 1$$

$$\text{Cost of } 2^3 = 2*2*2 \Rightarrow \text{Cost} = 2$$

$$\underset{2\times 2}{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{Coast of } |A| = a*d - b*c$$
$$\Rightarrow \text{Coast} = 3$$

Coast of A^2

$$A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$$

$$\text{Coast} = 4(\overset{*}{2} + \overset{+}{1}) = 12$$

عدد الضربات

عدد العمليات على مربع واحد

مراجعة

Coast of A^3 = $A * A * A$ ⇒ Cost of 24

Coast of $4A^2 + |A|\,A^3$

أي رقم * مصفوفة

$\begin{pmatrix} & \\ & \end{pmatrix}_{2\times 2}$ ⇐ 4 مرات

$4+12$

$\Rightarrow 16 + 24 + 3 + 4 + 4 = 51$

$|A|$

حاصل ضرب الرقم الناتج عن المحدد بالمصفوفة الناتجة عن A^3

حاصل جمع المصفوفة الناتجة عن $4A^2$ مع المصفوفة الناتجة عن $|A|A^3$

* $A_{n\times n}$ Cost of A^2

$$\begin{pmatrix} - & - & - \\ - & - & - \\ - & - & - \end{pmatrix} \begin{pmatrix} | & | & | \\ | & | & | \end{pmatrix} = \begin{pmatrix} \square & \square & \\ & & \end{pmatrix}_{n\times n}$$

Cost

عدد المربعات، مهمة، ضرب، جمع

$$= n^{2}\,[\,n + n - 1\,]$$

$$= 2n^3 - n^2$$

Ex $A_{n\times n}$, $b_{n\times 1}$. find Cost of Ab

$$\begin{pmatrix} & \\ & \end{pmatrix}_{n\times n} \begin{pmatrix} \\ \\ \end{pmatrix}_{n\times 1} = \begin{pmatrix} \square \\ \vdots \\ \square \end{pmatrix}_{n\times 1}$$

Cost $= n(n+n-1)$

$= 2n^2 - n$

* Recall *

20-3-2018

[1] $\sum_{K=1}^{n} K = 1+2+3+4+\dots+n = \frac{n(n+1)}{2}$

[2] $\sum_{K=1}^{n} K^2 = 1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$

[3] $\sum_{K=1}^{n-1} K = \frac{n(n-1)}{2}$

[4] $\sum_{K=1}^{n-1} K^2 = \frac{n(n-1)(2n-1)}{6}$

[5] $\sum_{K=1}^{n} c = nc$

[6] $\sum_{K=1}^{n} a_K + b_K = \sum_{K=1}^{n} a_K \pm \sum_{K=1}^{n} b_K$

[7] $\sum_{K=1}^{n} c a_K = c \sum_{K=1}^{n} a_K$

* upper triangular System : $n \times n$

الأصفار تحت
الأرقام فوق

$$a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + \dots + a_{1n}X_n = b_1$$

$$a_{22}X_2 + a_{23}X_3 + \dots + a_{2n}X_n = b_2$$

$$a_{33}X_3 + \dots + a_{3n}X_n = b_3$$

$$\vdots$$

$$a_{(n-1)(n-1)}X_{(n-1)} + a_{n-1\,n}X_n = b_{n-1}$$

$$a_{nn}X_n = b_n$$

Ex $2x_1 + x_2 - x_3 = 3$

$6x_2 + x_3 = 6$ 3x3 upper Δ system

$5x_3 = 0$

* upper Δ solved by back substitution [B.S]

Solution $x_3 = \frac{0}{5} = 0$ Step 1

$x_2 = \frac{6 - (1)(x_3)}{6} = \frac{6 - 1(0)}{6} = 1$ Step 2

$\Rightarrow$ 3 Iteration

$x_3 = \frac{3 + (1)(x_3) - (1)(x_2)}{2} = 1$ Step 3

Cost $= 9$, $n \times n \Rightarrow \boxed{n^2}$ = Cost if upper Δ system

now Cost of $n \times n$ upper Δ system using B.S

Step	+, −	*, ÷
1	0	1
2	1	2
3	2	3
⋮	⋮	⋮
K	K-1	K
⋮		
n		

+, − : $\sum_{K=1}^{n} K-1$

*, ÷ : $\sum_{K=1}^{n} K$

Total Cost of B.s $= \sum_{K=1}^{n} K-1 + \sum_{K=1}^{n} K$

$$\Rightarrow \sum_{K=1}^{n} (2K-1)$$

$$= 2\sum_{K=1}^{n} K - \sum_{K=1}^{n} 1$$

$$= 2\frac{(n)(n+1)}{2} - n \quad = n^2 + n - n = n^2$$

To solve any $n\times n$ linear System : $A_{n\times n} x_{n\times 1} = b_{n\times 1}$

Augmented matrix : $[A|b]_{n\times n+1}$

$\boxed{1}$ Gaussian Elimination G.E

$$[A|B] \longrightarrow [\underset{\text{upper } \Delta}{U}|c] + BS$$

$\boxed{2}$ LU Factorization : $Ax = b$

ⓐ $A = LU \Rightarrow L\underbrace{\boxed{Ux}}_{Y} = b$

ⓑ $LY = b$ solve for Y [Forward Substitution]

(c) $UX = Y$ solve for x [B.S]

$\boxed{3}$ Cremers Rule [$Ax = b$]

$$X_1 = \frac{|A_i|}{|A|} \qquad i = 1, \dots, n$$

Ex 2×2 linear System

$$Ax = b : \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Find Total Cost

$$X_1 = \frac{|A_1|}{|A|} , \quad X_2 = \frac{|A_2|}{|A|}$$

$$\text{Cost} = \overset{\text{Cost}}{|A|} + \overset{\text{Cost}}{|A_1|} + \overset{\text{Cost}}{|A_2|} + \overset{\div}{2}$$

نحسبها مرة

$$= 3 + 3 + 3 + 2 = 11$$

<u>Ex</u> 3X3

<u>Solution</u> $X_1 = \frac{|A_1|}{|A|}$, $X_2 = \frac{|A_2|}{|A|}$, $X_3 = \frac{|A_3|}{|A|}$

$$\text{Total Cost} = 4(\text{Cost of } |A|) + \overset{\div}{3}$$

مش تنساهم

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \Rightarrow |A| = a*(e*i - h*f) - b*(d*i - f*g) + c*(d*h - e*g)$$

$$\text{Cost of } |A| = 14$$

$$\text{Total Cost} = 4(14) + 3 = 59$$

<u>HW</u> Find total cost of Cramer Rule of 4x4 Solution 3..

$\boxed{4}$ Gauss-Jordan Reduction

$[A|b] \longrightarrow [I|X]$, I : Identity

Cost = Cost التحويل

$\boxed{5}$ Inverse Method

$Ax = b$ ⓐ find A^{-1} $[A|I] \longrightarrow [I|A^{-1}]$

ⓑ $X = \underset{n\times n}{A^{-1}} b_{n\times 1}$

الشرح بالتفصيل

G.E $[A|b] \longrightarrow [u|c]$ + B.S

Recall : Row operations $\boxed{1}$ Switch two rows

$\boxed{2}$ Multiply a row by a nonzero constant

$\boxed{3}$ Replace a row by adding/subtracting it to a multiple of another row.

Row

$R_K \longrightarrow R_K - M_{Kp} R_p$

Note M_{Kp} : Multiplier

Ex $Ax = b$ $4\times 4 \Rightarrow \boxed{n=4}$

Solve it using G.E & Find Cost

$$[A|b] = \left[\begin{array}{cccc|c} 2 & 6 & 4 & 3 & 22 \\ 1 & 2 & 1 & 4 & 13 \\ 4 & 2 & 2 & 1 & 20 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right]$$

Step 1 : **Multipliers** $M_{21} = \frac{1}{2}$ 0.5

M المستخدم لتصفير الصف الثاني العمود الأول

$M_{31} = \frac{4}{2} = 2$

$M_{41} = \frac{-3}{2} = -1.5$

$$\Rightarrow \begin{array}{l} \\ R_2 - \frac{1}{2}R_1 \\ R_3 - 2R_1 \\ R_4 + 1.5R_1 \end{array} \left[\begin{array}{cccc|c} 2 & 6 & 4 & 3 & 22 \\ 0 & -1 & -1 & 2.5 & 2 \\ 0 & -10 & -6 & -5 & -24 \\ 0 & 10 & 9 & 6.5 & 39 \end{array}\right]$$

4, 4 Iteration

4, 4

4, 4

Step	+, -	*, ÷
1	3(4)	3(4), 3

Multipliers
قسمة

لكل صف اربعة طرح

اربعة ضرب

* 3 لانه اول شي معمول يتصفر

عدد الصفوف الي اشتغلنا عليهم

Step 2 : M $\Rightarrow$ $M_{32} = \frac{-10}{-1} = +10$, $M_{42} = \frac{10}{-1} = -10$

$$\begin{array}{l} \\ \\ R_3 - 10R_2 \\ R_4 \end{array} \left[\begin{array}{cccc|c} 2 & 6 & 4 & 3 & 22 \\ 0 & -1 & -1 & 2.5 & 2 \\ 0 & 0 & 4 & -30 & -44 \\ 0 & 0 & -1 & 31.5 & 59 \end{array}\right]$$

Step	+, -	*, ÷
2	2(3)	2(3), 2

Step 3 : $M_{43} = -\frac{1}{4} = -0.25$

$$\left[\begin{array}{cccc|c} 2 & 6 & 4 & 3 & 22 \\ 0 & -1 & -1 & 2.5 & 2 \\ 0 & 0 & 4 & -30 & -44 \\ 0 & 0 & 0 & 24 & 45 \end{array}\right]$$

Step	+, -	*, ÷
3	1(2)	1(2), 1

Step	+, -	*, ÷	
1	3(4)	3(4)	3
2	2(3)	2(3)	2
3	1( ) 2	1(2)	1

$20 + 20 + 6 \Rightarrow 46$

Cost of $[A|b] \longrightarrow [U|c] = 46$

* Cost of B.S $= 4^2 = 16$

Total Cost of G.E $= 46 + 16 = \boxed{62}$

$\underline{\underline{n-1}} \longleftarrow K \Leftarrow$ step عدد

$$\left.\begin{array}{l} \div \quad n-K \\ \\ n-K \quad (n-K+1) \end{array}\right] \; n \times n \text{ system}$$

Newton for Jacobian $\Rightarrow$ Quiz

G.E 4x4

22-3-2018

$[A|b]$ → $[U|c]$ + B_s
aug → upper

Step	+, -	*, ÷
1	3 (4)	3(4), 3
2	2 (3)	2(3), 2
3	1 (2)	1(2), 1

Now G.E for nxn

Step	+, -	*, ÷
1	$(n-1)(n)$	$(n-1)n$, $n-1$
2	$(n-2)(n-1)$	$(n-2)(n-1)$, $n-2$
3	$(n-3)(n-2)$	$(n-3)(n-2)$, $n-3$
⋮	⋮	⋮
K	$(n-k)(n-k+1)$	$(n-k)(n-k+1)$, $n-k$
⋮	⋮	⋮
$\boxed{n-1}$		

عدد steps الخطوات

Cost of $[A|b]$ → $[U|c]$

$$= \sum_{k=1}^{\boxed{n-1}} (n-k)(n-k+1) + \sum_{k=1}^{n-1} (n-k)(n-k+1) + \sum_{k=1}^{n-1} n-k$$

مهمة

$$= \sum_{k=1}^{n-1} [2(n-k)(n-k+1) + n-k]$$

let $t = n-k$

الحدود الجديدة

$K = 1 \Rightarrow t = n-1$

$K = n-1 \Rightarrow t = 1$

$$\sum_{t=n-1}^{1} \left[2t(t+1) + t \right]$$

$$= 2\sum_{t=1}^{n-1} t^2 + 3\sum_{t=1}^{n-1} t$$

$$2\,\frac{(n)(n-1)(2n-1)}{6} + 3\,\frac{n(n-1)}{2}$$

$$\Rightarrow \frac{4n^3 - 6n^2 + 2n + 9n^2 - 9n}{6}$$

$$\Rightarrow \frac{4n^3 + 3n^2 - 7n}{6}$$

التحويل

دون B.s

B.s

$$\text{Total Cost of G.E} = \frac{4n^3 + 3n^2 - 7n}{6} + n^2$$

$$= \frac{4n^3 + 9n^2 - 7n}{6}$$

$$\approx \frac{2n^3}{3} \text{ for } n \text{ Large}$$

[أكبر قوى]

HW Consider the data $(1,1), (2,3), (3,5)$

ⓐ Let $P(x) = ax^2 + bx + c$

Find a, b, c such that $P(x_k) = y_k$ for the above data using G.E

ⓑ find Cost of Solving a 3X3 System using G.E

ⓒ " " " " an nxn System using G.E

(b)

Step	+, -	*, ÷
1	2(3)	2(3) , 2
2	1(2)	1(2) , 1

$19 + 9 = 28$

(a)

$P(1) = 1 \Rightarrow a + b + c = 1$

$P(2) = 2 \Rightarrow 4a + 2b + c = 3$

$P(3) = 5 \Rightarrow 9a + 3b + c = 5$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 3 \\ 9 & 3 & 1 & 5 \end{array}\right)$$

* LU Factorization :

$Ax = b$

(1) $A = LU$ L : Lower Δ

U : upper Δ

تغير $[A] \longrightarrow [U]$

b

$L\boxed{UX} = b$ (Y)

(2) Solve $LY = b$ by Forward Subtitution

$\Rightarrow$ Cost $n^2 - n$

(3) Solve $UX = Y$ by B.S $\Rightarrow$ Cost $= n^2$

Ex $A = \begin{bmatrix} 2 & 6 & 4 & 3 \\ 1 & 2 & 1 & 4 \\ 4 & 2 & 2 & 1 \\ -3 & 1 & 3 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 22 \\ 13 \\ 20 \\ 6 \end{bmatrix}$

Solve Using LU Factorization

Solution

$$\begin{bmatrix} 2 & 6 & 4 & 3 \\ 1 & 2 & 1 & 4 \\ 4 & 2 & 2 & 1 \\ -3 & 1 & 3 & 2 \end{bmatrix}$$

b بدون

حالة

Step 1 Multi. $M_{21} = \frac{1}{2}$

$M_{31} = \frac{4}{2}$

$M_{41} = \frac{-3}{2}$

$$\begin{array}{l} \\ R_2 - \frac{1}{2}R_1 \\ R_3 - 2R_1 \\ R_4 + \frac{3}{2}R_1 \end{array} \begin{bmatrix} 2 & 6 & 4 & 3 \\ 0 & -1 & -1 & 2.5 \\ 0 & -10 & -6 & -5 \\ 0 & 10 & 9 & 6.5 \end{bmatrix}$$

Step	+, -	*, ÷
1	3(3)	3(3), 3

Step 2 Mult $M_{32} = \frac{-10}{-1} = 10$

$M_{42} = \frac{10}{-1} = -10$

$$\begin{array}{l} \\ \\ R_3 - 10R_2 \\ R_4 + 10R_2 \end{array} \begin{bmatrix} 2 & 6 & 4 & 3 \\ 0 & -1 & -1 & 2.5 \\ 0 & 0 & 4 & -30 \\ 0 & 0 & -1 & 31.5 \end{bmatrix}$$

Step	+, -	*, ÷
2	2(2)	2(2), 2