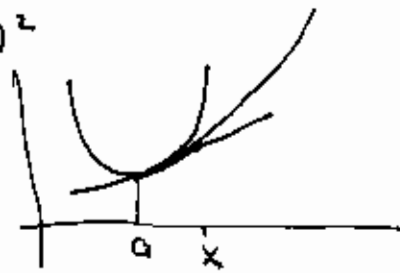


Taylor Theorem :-

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$



$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$$f(x) \approx f(a) + f'(a)(x-a) \quad \text{Linear estimation.}$$

$$\text{Error} = \frac{f''(a)}{2!}(x-a)^2 + \dots$$

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\text{Error} = \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

in general

$$f(x) \approx f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\text{Error} = \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \dots$$

(infinite Terms).

Taylor :-

$$\text{Error} = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1} \quad c \text{ between } a, x$$

$$|\text{Error}| \leq \max_{a \leq x \leq b} \frac{|f^{(n+1)}(x)|}{(n+1)!} (x-a)^{n+1}$$

لكن المشكلة كيف نجد Error حد ذاته

$$\Rightarrow f(x) \approx \underbrace{f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{P_n(x)} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{\text{Error } E_n(x)}$$

• $e^x, a=0$

$$e^x = f(0) + f'(0)(x-0) + \frac{f''(c)}{2!}(x-0)^2$$

$$e^x = 1 + x + \frac{e^c}{2!}x^2$$

$$e^x \approx 1+x \quad \text{with error } \frac{e^c}{2!}x^2$$

$$e^x = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(c)}{3!}(x-0)^3$$

$$e^x \approx 1 + x + \frac{x^2}{2}$$

$$\text{error} = \frac{e^c x^3}{6}$$

$$e^{0.1} \approx 1 + 0.1 + \frac{0.01}{2} \approx 1.105$$

$$\text{error} = \frac{e^c (0.001)}{6} < 1 \times 10^{-3}$$

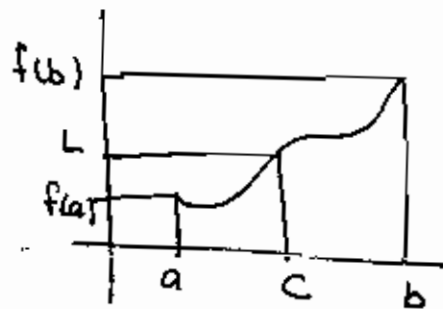
$$c \in [0, 0.1]$$



$$\text{upper bound for error } \frac{e^c (0.001)}{6} \leq \frac{e^1 (0.001)}{6} \leq 0.0005$$

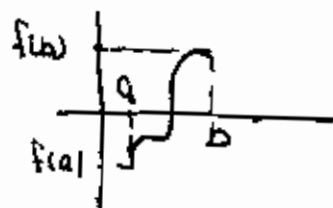
intermediate value theorem (IVT)

- $f(x)$ is continuous
- L between $f(a)$ and $f(b)$
- Then $\exists c \in (a, b)$ such that $f(c) = L$



bolzano

- $f(x)$ is continuous
- $f(a) = f(b) < 0$
- Then $\exists c \in (a, b)$ such that $f(c) = 0$



mean value theorem (MVT)

- $f(x)$ is continuous on $[a, b]$
- $f(x)$ is differentiable on (a, b)
- then $\exists c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$



Section 1.3

Error analysis

Def: suppose that p^n is an approximation to P

the error is $E_p = P - p^n$

the relative error $R_p = \frac{E_p}{P} = \frac{P - p^n}{P}$

ex:- 1. let $x = 3.141592$

$$x^n = 3.14$$

٣ منازل متأكدية من

$$E_x = 3.141592 - 3.14 = 0.001592$$

$$R_x = \frac{0.001592}{3.141592} = 0.000507$$

2. let $y = 1,000,000$

$$\hat{y} = 999,996$$

٥ منازل متأكدية
من الخلف في

$$E_y = 4$$

$$R_y = \frac{4}{1,000,000} = 4 \times 10^{-6}$$

3. let $z = 0.000,012$

$$\hat{z} = 0.000,009$$

٣ منازل متأكدية
ولا هناك منزلة

$$E_z = 0.000,003$$

$$R_z = 0.25$$

normalized decimal Form:-

$$\pm 0.d_1d_2d_3 \dots \times 10^n$$

$$d_1 \neq 0$$

$$\bullet x^2 = 2$$

$$x^2 = 2 = 0$$

$$\begin{array}{c} \cdot \quad \cdot \quad \cdot \\ \hline 1 \quad 1 \quad 1 \\ 1.25 \quad 2 \end{array}$$

صي الجزائفي

$$C_0 = \frac{1+2}{2} = 1.5$$

$$C_1 = \frac{1+1.5}{2} = 1.25$$

$$C_2 = \frac{1+1.25}{2} = 1.125 = 0.1125 \times 10^1$$

2 significant digits \Rightarrow Error $\leq 10^{-2}$

بعد اطب منزلة غير صفرية

Def: the number \hat{P} is said to approximate P to d significant digits if d is the largest positive integer for which

$$\frac{|P - \hat{P}|}{|P|} < \frac{10^{-d}}{2}$$

$$\text{i.e. } 2|R_e| \approx 10^{-d}$$

ex:-

1. $x = 3.141592$

$x^{\wedge} = 3.14$

$R_x = 3.141592 - 3.14 = 0.001592$

$R_x = \frac{0.001592}{3.141592} = 0.000507$

$2|R_x| = 0.001014 \approx 10^{-3}$
 $< 10^{-4}$

2. $2|R_y| = 8 \times 10^{-6} < 10^{-3}$

10^{-2}

10^{-3}

10^{-4}

10^{-5}

10^{-6}

3. $2|R_z| = 0.5 \not< 10^{-1}$

No significant bits.

• if $P = \pm 0.d_1d_2 \dots d_nd_{n+1} \dots \times 10^n$ is the normalized decimal form of the number P , $d_1 \neq 0$, then the k^{th} digit chopped floating point representation of P is

$$fl_{\text{chop}}(P) = \pm 0.d_1d_2 \dots d_k \times 10^n$$

the k^{th} digit round off floating point representation of P is

$$fl_{\text{round}}(P) = \pm 0.d_1d_2 \dots d_{k-1}r_k \times 10^n$$

where r_k is obtained by rounding $d_k, d_{k+1}, d_{k+2} \dots$

• $P = 0.1234|444445$

4 digits chopped

$fl_{\text{chop}}(P) = 0.1234$

$$f_L(p) = 0.1235$$

round

Final

- use 4 digits arithmetic (round) في منازل بعد اولى منزلة غير صفرية

$$\frac{\frac{3}{7} + \frac{5}{8} + (\frac{11}{15})}{21} = ?? \quad \text{or} \quad \frac{\frac{3}{7} + 0.5967 + \frac{11}{15}}{21} = ??$$

$$\frac{(0.4286 + 0.5967) + 0.7333}{21}$$

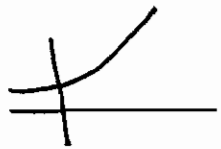
$$0.4286 + 0.5967 = 1.0253 \approx 1.025$$

$$1.025 + 0.7333 = 1.7583 \approx 1.758$$

$$\frac{1.758}{21} = 0.08371$$

- order of estimation

$$e^x \approx 1+x$$



$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^x \approx 1+x$$

$$e^h \approx 1+h$$

$$h \approx 0$$

order of approximation.

$$\text{Error} = \frac{h^2}{2!} \approx O(h^2)$$

$$e^{0.1} \approx 1 + 0.1 \approx 1.1$$

$$\text{error} = \downarrow \text{Const} \cdot h^2$$

$$e^{0.1} = 1.105170918$$

$$= C(0.1)^2$$

$$= C(0.01)$$

$$\leq 10^{-2}$$

$$e^h = 1 + h + \frac{h^2}{2!}$$

$$\text{Error} = Ch^3 = O(h^3)$$

$$e^{0.1} = 1 + 0.1 + \frac{0.01}{2}$$

$$= 1.105$$

$$\text{Error} \approx C(0.1)^3$$

$$\approx C(0.001) \leq 10^{-3}$$

$$\sin(0.1) \approx 0.1$$

$$- \sin h \approx h \quad \text{with error } O(h^3)$$

$$\sinh \approx h - \frac{h^3}{3!} \quad \text{with error } O(h^5)$$

suppose $e^h \approx 1+h$ Error = $O(h^2)$ (0.01)

$\sin h = h - \frac{h^3}{3!}$ Error = $O(h^5)$ (0.00001)

$e^h + \sin h \approx 1+h + \frac{h^3}{3!}$ with Error $O(h^2) + O(h^5)$
 $\approx 1+2h + O(h^2)$

لا توتر
 الاصغر هي التي توتر

def: order of approximation

assum that $f(h)$ is approximated by $p(h)$ and there exists a real constant $M \geq 0$ and a positive integer n so that

$$\frac{|f(h) - p(h)|}{|h^n|} \leq M \text{ for small } h$$

we say $p(h)$ approximate $f(h)$ with order of approximation $O(h^n)$ and we write $f(h) = p(h) + O(h^n)$

$$|f(h) - p(h)| \leq M|h^n|$$

$$f(h) - p(h) \approx Ch^n$$

Ex: show that $p(h) = 1+h$ estimate of $f(h) = e^h$ with order $O(h^2)$

or show that $e^h = 1+h + O(h^2)$

$$e^h = 1+h + \frac{h^2}{2!} + \frac{h^3}{3!} + \dots$$

$$\frac{|e^h - (1+h)|}{|h^2|} = \frac{\frac{h^2}{2} + \frac{h^3}{3!} + \frac{h^4}{4!} + \dots}{h^2} = \frac{1}{2} + \frac{h}{3!} + \frac{h^2}{4!} + \frac{h^3}{5!} + \dots$$

↓ harmonic series $(\sum \frac{1}{n})$ divergens

$< \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$

$< \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots \rightarrow$

geometric series = $\frac{1/2}{1-1/2} = 1$

$$e^h = 1+h + O(h^2)$$

Exercise

Show that

$$1 - \sin h = h - \frac{h^3}{3!} + O(h^5)$$

$$2 - f(h) = \sum_{k=0}^n f^{(k)}(h) \frac{h^k}{k!} + O(h^{n+1})$$

Theory:- \bar{a}

$$\text{assume that } f(h) = P(h) + O(h^n) \\ g(h) = Q(h) + O(h^m)$$

$$\text{and } r = \min[m, n]$$

then

$$f(h) + g(h) = P(h) + Q(h) + O(h^r)$$

$$f(h) - g(h) = P(h) - Q(h) + O(h^r)$$

$$\frac{f(h)}{g(h)} = \frac{P(h)}{Q(h)} + O(h^r) \quad Q(h), g(h) \neq 0.$$

Ex:-

$$f(h) = P(h) + O(h^3)$$

$$g(h) = Q(h) + O(h^2)$$

$$\frac{f(h)}{g(h)} = \frac{P(h)}{Q(h)} + O(h^2)$$

Ex:- (loss of significant)

$$f(x) = x(\sqrt{x+1} - \sqrt{x})$$

$$g(x) = \frac{x}{\sqrt{x+1} + \sqrt{x}}$$

use 6 digits arithmetic and round to find $f(500)$, $g(500)$

$$\begin{aligned} f(500) &= 500(\sqrt{501} - \sqrt{500}) \\ &= 500(22.3830 - 22.3607) \\ &= 500(0.022300) = 11.1500. \end{aligned}$$

$$g(500) = \frac{500}{\sqrt{501} + \sqrt{500}} = \frac{500}{22.3830 + 22.3607} = 11.1748$$

exact answer = 11.174755...

الطريقة الثانية أسهل لأن في العلة الأولة
عملية الجمع ضربنا
significant digits

$$\frac{3}{17} = 0.176470588 + \epsilon$$

Notes:-

$$P = \tilde{P} + \epsilon_P$$

$$Q = \tilde{Q} + \epsilon_Q$$

$$P + Q = \tilde{P} + \tilde{Q} + \epsilon_P + \epsilon_Q$$

$$= \tilde{P} + \tilde{Q} + \epsilon_{P+Q}$$

$$P \cdot Q = (\tilde{P} + \epsilon_P)(\tilde{Q} + \epsilon_Q)$$

$$= \tilde{P}\tilde{Q} + \tilde{P}\epsilon_Q + \tilde{Q}\epsilon_P + \epsilon_P\epsilon_Q$$

$$= \tilde{P}\tilde{Q} + \epsilon_{PQ}$$

$$\bullet P = 9.8 \times 10^6 + 35 \times 10^{-9}$$

$$\tilde{Q} = 3.6 \times 10^7 + 2.4 \times 10^{-9}$$

Chapter 2

Section 2.2

$$f(x) = 0$$

suppose $\exists c \in (a, b)$ such that $f(c) = 0$

estimate c ??

section 2.2

- we estimate c by bisection method
- we assume $f(a) \cdot f(b) < 0$, f is continuous

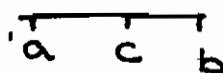
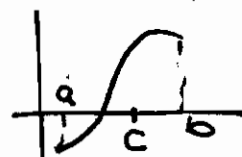
$$c = \frac{a+b}{2}$$

• Find $f(c)$

- if $f(c) = 0 \Rightarrow$ is done

- if $f(a) \cdot f(c) < 0 \Rightarrow r \in [a, c]$

Else $r \in [c, b]$

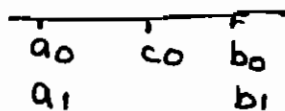


Bisection

let $[a_0, b_0] = [a, b]$

$$c_0 = \frac{a_0 + b_0}{2}$$

find $f(c_0)$



if $f(c_0) = 0$ Done

else if $f(a_0) \cdot f(c_0) < 0$ then $[a_1, b_1] = [a_0, c_0]$

else $[a_1, b_1] = [c_0, b_0]$

...

nth step

n+1 step $f(c_n)$

if $f(c_n) = 0$ Done

else if $f(a_n) \cdot f(c_n) < 0$ then $[a_{n+1}, b_{n+1}] = [a_n, c_n]$

Else $[a_{n+1}, b_{n+1}] = [c_n, b_n]$

stop if $|c_{n+1} - c_n| < 10^{-10}$

or stop if $|b_n - a_n| < 10^{-10}$

or stop if $|f(c_n)| < 10^{-10}$

notes:-

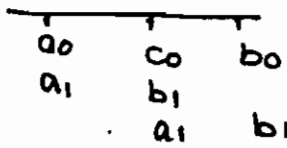
$$b_1 - a_1 = \frac{b_0 - a_0}{2}$$

$$b_2 - a_2 = \frac{1}{2} (b_1 - a_1) = \frac{1}{4} (b_0 - a_0)$$

$$\bullet b_n - a_n = \frac{1}{2^n} (b_0 - a_0)$$

$$\bullet a_0 \leq a_1 \leq a_2 \leq \dots \leq a_n \leq \dots \leq r \leq \dots \leq b_n \leq b_{n-1} \leq \dots \leq b_2 \leq b_1$$

تکبر a's
تضعف b's



- a₁ = a₀

$$[a_1, b_1] = [a_0, b_0]$$

or

$$a_1 = c_0 = \frac{a_0 + b_0}{2} > \frac{a_0 + a_0}{2} = \frac{2a_0}{2} = a_0$$

$a_1 > a_0$ $b_0 > a_0$

$$[a_1, b_1] = [c_0, b_0]$$

- b₁ = b₀

or

$$b_1 = c_0 = \frac{a_0 + b_0}{2} < \frac{b_0 + b_0}{2} = b_0$$

$a_0 < b_0$
 $b_1 < b_0$

$$[a_1, b_1] = [a_0, b_0]$$

$$[a_1, b_1] = [c_0, b_0]$$

a_n ↑ r
b_n ↓ r

Theory Bisection theorem:-

Assume that $f \in C[a, b]$ and that there exists a number $r \in [a, b]$ such that $f(r) = 0$, if $f(a) \cdot f(b) < 0$ and $[C_n]_{n=0}^{\infty}$ represents the sequence of midpoints generated by the bisection process then

$$|r - C_n| \leq \frac{b-a}{2^{n+1}}, \quad n = 0, 1, 2, \dots$$

and

$$\lim_{n \rightarrow \infty} C_n = r$$

$$\text{Error} \leq 10^{-5}$$

$$\frac{b-a}{2^{n+1}} < 10^{-5} = \epsilon$$

$$\frac{b-a}{\epsilon} < 2^{n+1}$$

$$\ln \frac{b-a}{\epsilon} < (n+1) \ln 2$$

$$(n+1) \geq \frac{\ln \left(\frac{b-a}{\epsilon} \right)}{\ln 2}$$

$$n = \text{int} \left[\frac{\ln \left(\frac{b-a}{\epsilon} \right)}{\ln 2} \right]$$

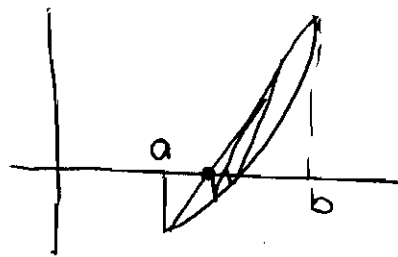
if $n+1 \geq 16.7$
 $n = 16$

in example

$$n = \text{int} \left[\frac{\ln \frac{2}{10^{-3}}}{\ln 2} \right] = 10$$

- False position method

- $f(a) \cdot f(b) < 0$
- f is continuous

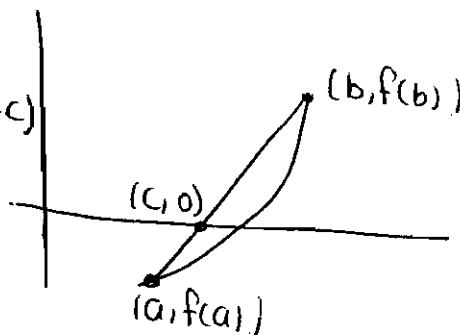


$$\text{slope} = \frac{f(b) - 0}{b - c} = \frac{f(b) - f(a)}{b - a}$$

$$f(b)(b-a) = (f(b) - f(a))(b-c)$$

$$b-c = \frac{f(b)(b-a)}{f(b) - f(a)}$$

$$c = b - \frac{f(b)(b-a)}{f(b) - f(a)}$$



Section 2.1

Fixed point iteration

To solve $F(x)=0$ we solve $x=g(x)$ [where $F(x)=x-g(x)$]
↓
[Fixed point]

i.e. to Find the roots of $F \rightarrow$ we Find the Fixed point of $g(x)$.

Def: p is a fixed point of g iff $g(p)=p$.

1. $g(x) = \frac{1}{x}$.

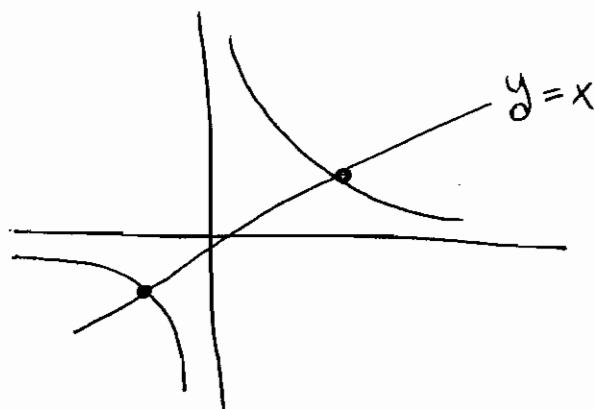
Fixed points $1, -1$.

$g(p) = p$.

$\frac{1}{p} = p \Rightarrow p^2 = 1 \Rightarrow p = \pm 1$.

2. $g(x) = x+1$. No Fixed points.

3. $g(x) = x$ all points are Fixed points.



Def: Fixed point iteration:-

start with P_0 , $P_{n+1} = g(P_n)$, $n=0, 1, 2, 3, \dots$

$P_1 = g(P_0)$

$P_2 = g(P_1)$

⋮

Theorem:

if the Fixed point iteration converges to P , then P is the Fixed point of g .

algorithm

$$[a_0, b_0] = [a, b]$$

$$c_0 = b_0 - \frac{f(b_0)(b_0 - a_0)}{f(b_0) - f(a_0)}$$

$f(c_0)$

if $f(c_0) = 0$ done.

else if $f(c_0) \cdot f(a_0) < 0 \Rightarrow [a_1, b_1] = [a_0, c_0]$

else $[a_1, b_1] = [c_0, b_0]$

$$c_n = b_n - \frac{f(b_n)(b_n - a_n)}{f(b_n) - f(a_n)}$$

• example

Solve $x \sin x = 1$.

$$f(x) = x \sin x - 1$$

$$f(0) = -1$$

$$f(2) = 0.81859485$$

$$c_0 = b_0 - \frac{f(b_0)(b_0 - a_0)}{f(b_0) - f(a_0)}$$

$$= 2 - \frac{0.81859485(2 - 0)}{0.81859485 - (-1)} = 1.09975017$$

$$f(c_0) = 1.09975017 \sin(1.09975017) - 1$$

$$= -0.02001912$$

$$[a_1, b_1] = [1.09975017, 2]$$

$$c_1 = b_1 - \frac{f(b_1)(b_1 - a_1)}{f(b_1) - f(a_1)} = 2 - \frac{0.81859485(2 - 1.09975017)}{0.81859485 - (-0.02001912)}$$

$$= 1.12124074$$

$$f(c_1) = 0.00983461$$

$$[a_2, b_2] = [1.09975017, 1.12124074]$$

$$c_2 = 1.11416120$$

$$c_3 = 1.11415714$$

بیتا 0 فزاید

proof:-

$$\text{if } \lim_{n \rightarrow \infty} P_n = P \Rightarrow \lim_{n \rightarrow \infty} P_{n+1} = \lim_{n \rightarrow \infty} g(P_n) = g(\lim_{n \rightarrow \infty} P_n) = g(P)$$

$$\text{since } P_{n+1} = g(P_n) \quad \downarrow = P.$$

example:-

$$\text{Solve } x^2 - 2x - 3 = 0 \Rightarrow f(x) = 0.$$

$$(x-3)(x+1) = 0.$$

$$x = 3.$$

$$x = -1.$$

$$x^2 = 2x + 3.$$

$$x = \sqrt{2x+3} = g(x).$$

$$\text{if } P_0 = 4.$$

$$P_1 = g(4) = g(P_0) = \sqrt{11} = 3.31662.$$

$$P_2 = g(P_1) = g(3.31662) = \sqrt{9.63325} = 3.10375$$

$$P_3 = 3.03439$$

$$P_4 = 3.01184$$

$$P_n \rightarrow 3$$

• Note that 3 is a fixed point of

$$g(x) = \sqrt{2x+3} \text{ because } g(3) = 3$$

way 2:- x ^{is} ^{divergence.}

$$2x = x^2 - 3.$$

$$x = \frac{x^2 - 3}{2} = g(x).$$

$$P_0 = 4$$

$$P_1 = g(4) = 6.5.$$

$$P_2 = g(6.5) = 19.625.$$

$$P_3 = 191.07.$$

way 3:-

$$x(x-2) = 3 \Rightarrow x = \frac{3}{x-2} = g(x).$$

$$P_0 = 4$$

$$P_1 = g(4) = \frac{3}{2} = 1.5.$$

$$P_2 = -6$$

$$P_3 = -0.375$$

$$P_4 = -1.26315.$$

$$P_5 = -0.919355$$

$$P_6 = -1.02762$$

$$P_7 = -0.990876$$

Theorem:- (Fixed point Theorem I)

Assume $g \in C[a,b]$ if $g(x) \in [a,b]$ for all $x \in [a,b]$ then g has a fixed point in $[a,b]$. Furthermore if $|g'(x)| \leq k < 1$ for all $x \in (a,b)$ then g has a unique Fixed point.

Proof:-

if $g(a) = a$ or $g(b) = b$ Done.

if not $g(a) > a$ and $g(b) < b$

let $h(x) = g(x) - x$ $\therefore h$ continuous.

$$h(a) = g(a) - a > 0$$

$$h(b) = g(b) - b < 0$$

by Bolzano $\exists c \in \mathbb{C}$ such that $h(c) = 0$.

$$\begin{aligned} g(c) - c &= 0 \\ \boxed{g(c) = c} \end{aligned}$$

Uniqueness

Suppose $\exists P_1, P_2$ such that $g(P_1) = P_1, g(P_2) = P_2$.

Using mean value theorem on (P_1, P_2) .

$$\exists c \in (P_1, P_2) \text{ such that } \left| \frac{g(P_2) - g(P_1)}{P_2 - P_1} \right| = |g'(c)| < 1$$

$$\frac{P_2 - P_1}{P_2 - P_1} = 1 \Rightarrow 1 < 1 \rightarrow \text{Contradict}$$

$$P_1 = P_2 \quad \therefore X$$

Theorem:- (Fixed point iteration theorem)

Assume that $g(x)$ and $g'(x)$ are continuous on a balanced interval $(a,b) = (p-\delta, p+\delta)$ that contains a unique Fixed point p and that the started value P_0 is chosen in this interval.

1. if $|g'(x)| \leq k < 1$ for all $x \in (a,b)$ then the FPI converge (attractive Fixed point)

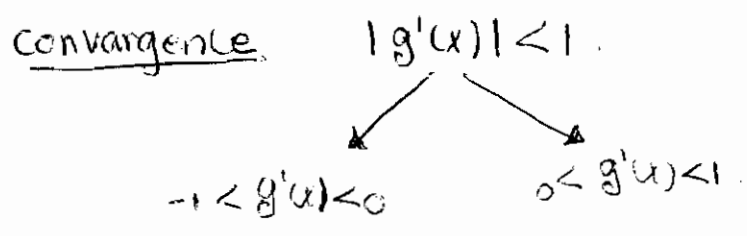
$$P_{n+1} = g(P_n)$$

2. if $|g'(x)| > 1$ for all $x \in (a,b)$ then the Fixed point iteration diverges (we call it repulsive Fixed point).

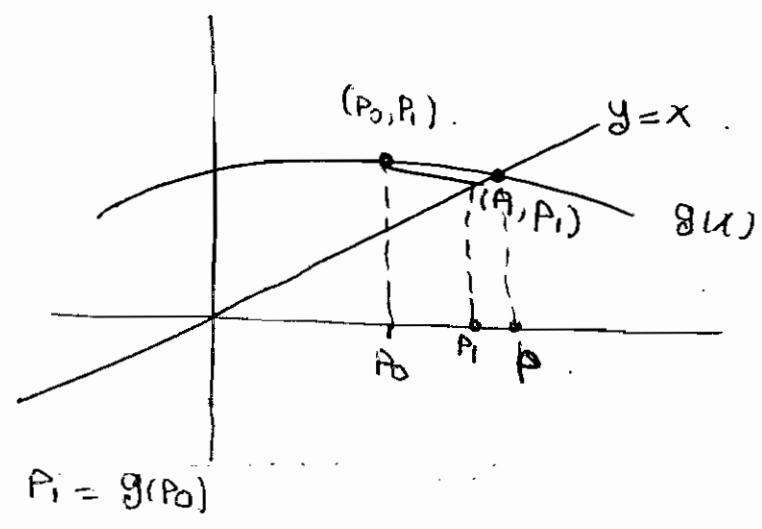
Note:-

if P is given we can replace the above two conditions by

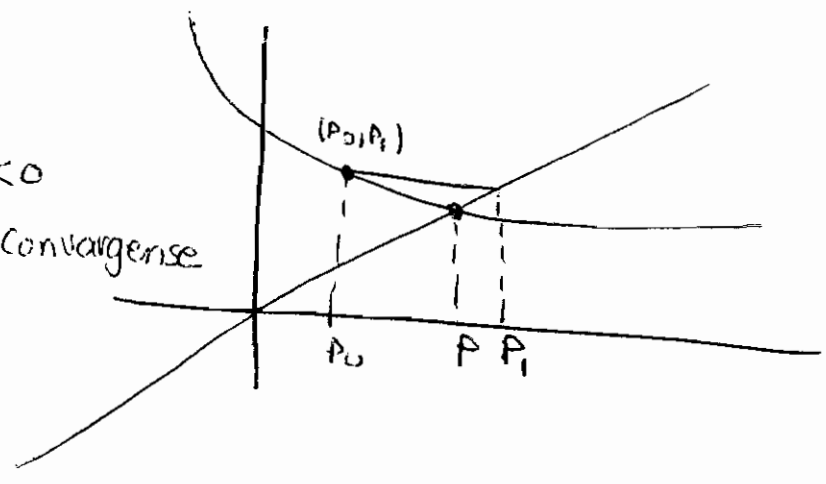
1. if $|g'(P)| < 1 \rightarrow$ the FPI converges.
2. if $|g'(P)| \geq 1 \rightarrow$ the FPI diverges.



$0 < g'(x) < 1$
monotone
convergence.



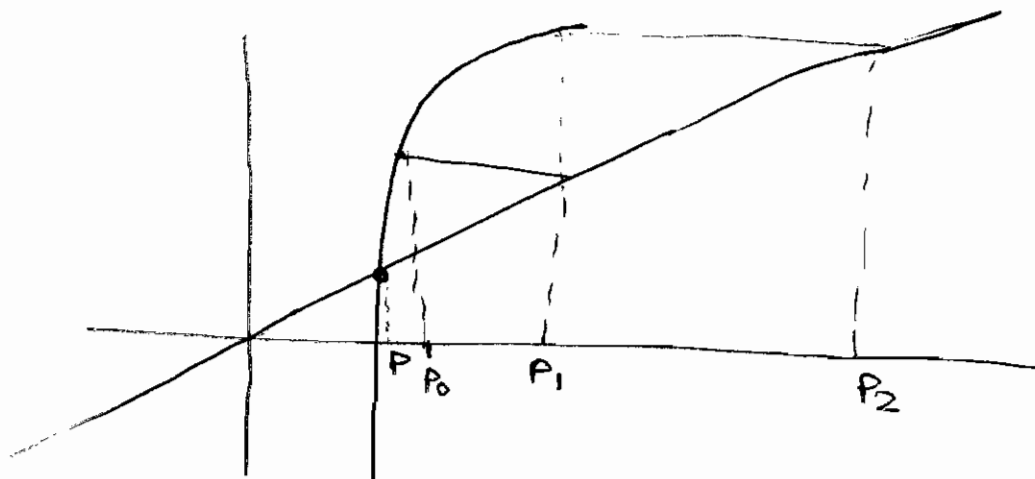
$-1 < g'(x) < 0$
alternating convergence



divergence $|g'(x)| > 1$

$g'(x) > 1$ $g'(x) < -1$

$g'(x) > 1$



example

investigate the nature of the FPI and show your answer by examples for

$$g(x) = 1 + x - \frac{x^2}{4}$$

Solution

$$x = g(x)$$

$$x = 1 + x - \frac{x^2}{4}$$

$$x^2 = 4$$

$$x = \pm 2 \text{ (Fixed points)}$$

when $x = 2$.

$$g'(x) = 1 - \frac{x}{2}$$

$|g'(2)| = 0 < 1 \rightarrow$ Convergence Fixed point. (attractive Fixed point)

to show that:-

$$\text{let } P_0 = 1.6$$

$$P_1 = g(1.6) = 1.96$$

$$P_2 = g(1.96) = 1.996$$

$$P_n \rightarrow 2$$

if $P_0 = 2.5$

$$P_1 = g(2.5)$$

$$\vdots$$

$$\rightarrow P_n \rightarrow 2$$

at $x = -2$

~~$|g'(-2)| = 2 > 1$~~ $|g'(-2)| = 2 > 1$ diverge \rightarrow FPI diverge. (Repulsive Fixed point).

$P_0 = -2.05$.

$P_1 = g(-2.05) = -2.1 \dots$

$P_2 = g(-2.1) = -2.2$.

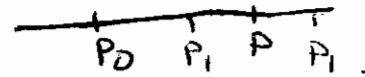
\vdots
 $P_n \rightarrow$ divergence.

Proof:-

by mean value.

$|P_1 - P| = |g(P_0) - g(P)| = |g'(c)| (P_0 - P) < (P_0 - P)$

$\rightarrow P_1$ is closer to P from P_0 .



P_0 عن P أقرب P_1

$|P_n - P| = |g(P_{n-1}) - g(P)| = |g'(c)| (P_{n-1} - P) < k (P_{n-1} - P) < k |P_{n-2} - P|$

$\rightarrow |P_n - P| < k^n |P_0 - P|$

$\rightarrow \lim_{n \rightarrow \infty} |P_n - P| = 0$

$\rightarrow \lim_{n \rightarrow \infty} P_n = P$

$\leq k \cdot k \cdot k |P_{n-3} - P|$
 \vdots
 $\leq k^n |P_0 - P|$

② $|P_1 - P| = |g'(c)| |P_0 - P| > |P_0 - P|$

Theorem:-

a. $|P_n - P| \leq k^n |P_0 - P|$
error.

فنقدر نطلع
upper bound for error
 \rightarrow we can found n

k is the upper bound
للمنتقة

$k = g'(P)$ إذا فنعرف P .

b. $|P_n - P| \leq \frac{k^n}{1-k} |P_1 - P_0|$ (exercise).

example:-

$x^3 - x + 5 = 0$.

Use Fixed point iteration to Find all the roots, Find k for each case.

- ~~$g(x) = x^3 - x + 5$~~
- ~~$x^3 - x + 5 = 0$~~
- ~~$x^3 - x + 5 = 0$~~

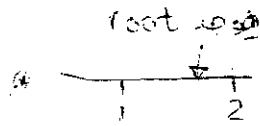
$$F(x) = x^3 - x + 5$$

$$F(0) = 5$$

$$F(-1) = 5$$

$$F(-2) = -11$$

$$F(2) = 1$$



$$x^3 = x + 5$$

$$x = \sqrt[3]{x+5} = (x+5)^{1/3}$$

$$g(x) = x$$

$$g'(x) = \frac{1}{3} (x+5)^{-2/3}$$

$$= \frac{1}{3 \sqrt[3]{(x+5)^2}} < 1$$

for all x
 $0 < x$

$$P_0 = 1.5$$

root position

for $x > 0$.

$$x+5 > 5$$

$$(x+5)^2 > 25$$

$$\sqrt[3]{(x+5)} > (25)^{1/3} > 2$$

$$\frac{1}{\sqrt[3]{(x+5)}} < \frac{1}{2}$$

$$\frac{1}{3 \sqrt[3]{(x+5)}} < \frac{1}{6}$$

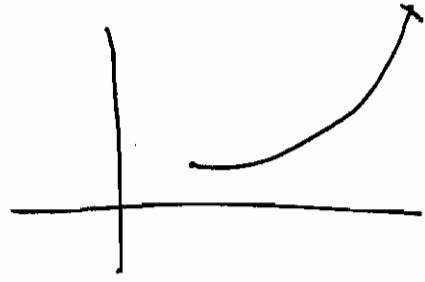
$$k_2 = \frac{1}{6}$$

Discussion

$$f(x) = 1 + e^{-\cos(x-1)}$$

[1, 2]

max point. نقطهٔ حداکثر



$$x^4 - 3x^2 - 3 = 0$$

$$P_0 = 1$$

10^{-2}
[1, 2]

$$x^4 = 3x^2 + 3$$

$$x = \sqrt[4]{3x^2 + 3}$$

$$P_1 = g(1) = \sqrt[4]{6} = 1.56508$$

$$P_2 = 1.79358$$

$$P_3 = 1.88595$$

$$P_4 = 1.92285$$

$$P_5 = 1.93751$$

$$P_6 = 1.94332$$

لرغبنا 5 iteration حتى
ثبتنا منزلتين

4/12.2)

$$P_n = P_{n-1} - \frac{P_{n-1}^5 - 7}{5P_{n-1}^4}$$

$$g(x) = x - \frac{x^5 - 7}{5x^4}$$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g(x) = x - \frac{x}{5} + \frac{7}{5x^4}$$

$$g(x) = \frac{4x}{5} + \frac{7}{5x^4}$$

$$g'(x) = \frac{4}{5} - \frac{28}{5x^5}$$

$$P = 7^{1/5}$$

$$P_n = g(P_{n-1})$$

$$x = 7^{1/5}$$

$$x^5 = 7$$

$$x^5 - 7 = 0$$

$$f(x) = x^5 - 7$$

أهم مآدة . 2.2, 2.4, 2.5
2.1.

$$g'(7^{1/5}) = \frac{4}{5} - \frac{28}{5(7^{1/5})^5}$$

$$= \frac{4}{5} - \frac{28}{5 \cdot 7} = \frac{4}{5} - \frac{4}{5} = 0. \quad \text{method أسرع}$$

newton method.

14
2.2

Solve

$$x = \tan x \quad \text{in } [4, 5].$$

$$g(x) = \sec^2 x > 1.$$

$$x = \tan^{-1} x$$

$$g(x) = \tan^{-1} x$$

$$g'(x) = \frac{1}{1+x^2} < 1$$

$$P_0 = 4.5.$$

$$P_1 = \tan^{-1}(4.5) = 1.352127$$

$$P_2 = \tan^{-1}(P_1) = 0.93$$

$$x = \tan x = \tan(x - \pi) = \tan(x + \pi)$$

$$x = \tan(x - \pi)$$

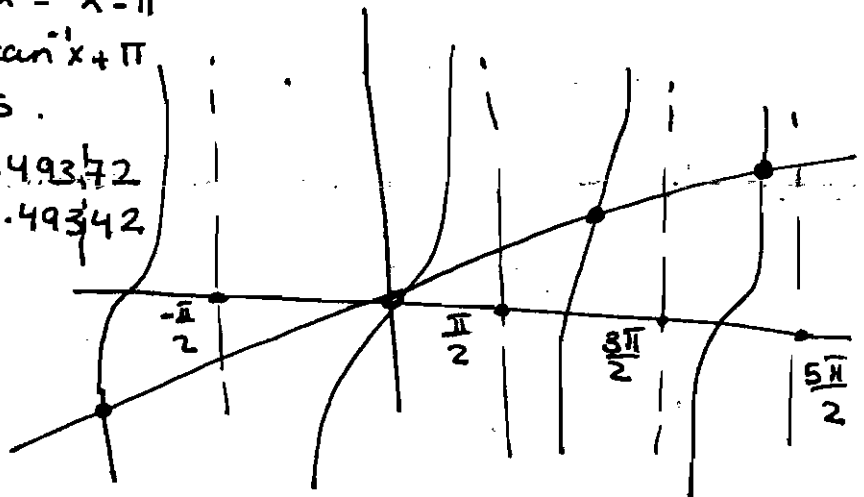
$$\tan^{-1} x = x - \pi$$

$$x = \tan^{-1} x + \pi$$

$$P_0 = 4.5.$$

$$P_1 = 4.49372$$

$$P_2 = 4.49342$$



14
2.1

Let $f(x) = (x-1)^{10}$

$$P = 1$$

$$P_n = 1 + \frac{1}{n}$$

Show that if $|c_n| < 10^{-3}$

but $|P - P_n| < 10^{-3}$ requires

for $n > 1$

$n > 1000$.

1. $|P - P_n| < \epsilon$

2. $|c_{n+1} - c_n| < \epsilon$

3. $|f(c_n)| < \epsilon$

$$F(P_n) = \left(\frac{1}{n}\right)^{10} < 10^{-3} \text{ for } n > 1.$$

$$|P - P_n| < 10^{-3} = \left|1 - 1 - \frac{1}{n}\right| < 10^{-3}$$

$$\left|-\frac{1}{n}\right| < 10^{-3}$$

$$\frac{1}{n} < 10^{-3} \Rightarrow n > 1000.$$

15
2.1

$$P_n = \sum_{k=1}^{\infty} \frac{1}{k}$$

show that P_n diverge even though $\lim_{n \rightarrow \infty} (P_n - P_{n-1}) = 0.$

$$P = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

$$\lim_{n \rightarrow \infty} P_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{n} = \sum_{n=0}^{\infty} \frac{1}{n} \text{ harmonic series (diverge)}$$

$$P_n - P_{n-1} = \left(1 + \frac{1}{2} + \dots + \frac{1}{n-1} + \frac{1}{n}\right) - \left(1 + \frac{1}{2} + \dots + \frac{1}{n-1}\right) = \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} (P_n - P_{n-1}) = \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

$$C_n = \frac{a_n + b_n}{2} \text{ stop.}$$

$$F(C_n) \leq \epsilon \text{ or } |C_n - C_{n-1}| \leq \epsilon$$

$$\text{stop if } F(C_n) \leq \epsilon \text{ and } \frac{C_n - C_{n-1}}{C_{n-1}} \leq 1 \times 10^{-6}.$$

Solve this equ

$$3x^2 - e^x = 0$$

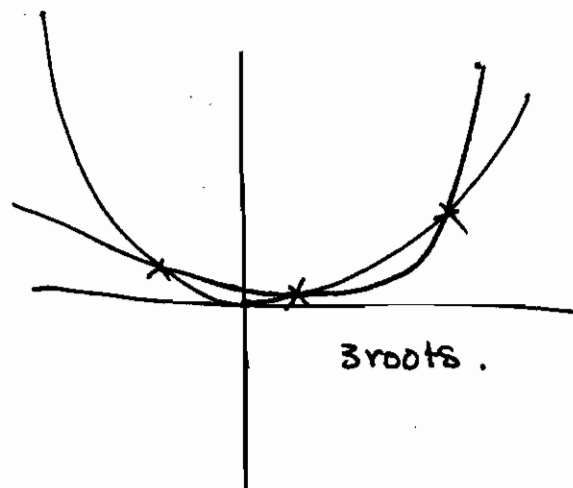
$$f(0) = -1$$

$$f(1) = 0.28 > 0$$

$$f(2) = 12 - e^2 > 0$$

$$f(3) = 27 - e^3 > 0$$

$$f(4) = 48 - e^4 < 0$$



Newton method

$$f'(P_0) = \frac{f(P_0) - 0}{P_0 - A}$$

$$P_0 - A = \frac{f(P_0)}{f'(P_0)}$$

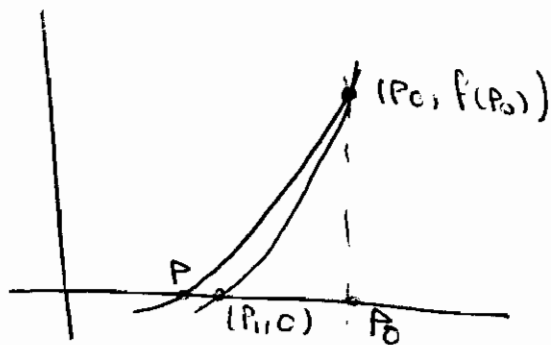
$$A = P_0 - \frac{f(P_0)}{f'(P_0)}$$

$$P_2 - P_1 = \frac{f(P_1)}{f'(P_1)}$$

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$$

$$x = x - \frac{f(x)}{f'(x)}$$

$g(x) \leftarrow$ Newton fixed point function



Th:- Newton Raphson theorem

assume $f \in C^2[a, b]$ and $\exists P \in [a, b]$ such that $f(P) = 0$, if $f'(P) \neq 0$ then there exist a $\delta > 0$ such that the sequence

$$\{P_{i\epsilon}\}_{i\epsilon=0}^{\infty} \text{ which is defined by } P_{i\epsilon} = g(P_{i\epsilon-1}) = P_{i\epsilon-1} - \frac{f(P_{i\epsilon-1})}{f'(P_{i\epsilon-1})}$$

will converge to P for any initial approximation $P_0 \in [P-\delta, P+\delta]$

example:-

estimate $5^{\frac{3}{7}}$

$$x = 5^{\frac{3}{7}}$$

$$x^7 = 5^3$$

$$f(x) = x^7 - 125$$

$$A = P_0 - \frac{f(P_0)}{f'(P_0)}$$

$$f'(x) = 7x^6$$

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$$

$$= P_n - \frac{P_n^7 - 125}{7P_n^6}$$

$$= \frac{6}{7}P_n + \frac{125}{7P_n^6}$$

$$P_0 = 2$$

$$P_1 = \frac{6}{7}(2) + \frac{125}{7(2)^6} = 1.71428$$

$$P_2 = \frac{6}{7}(1.71428) + \frac{125}{7(1.71428)^6} = 2.17$$

Proof the theorem

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2}$$

$$g'(x) = \frac{f(x)f''(x)}{(f'(x))^2}$$

$$g'(p) = \frac{f(p)f''(p)}{(f'(p))^2} = 0$$

→ by Fixed point iteration ^{theory} → the Fixed point iteration will converge.

• if $e_{n+1} \approx A e_n$ where $e_n = p - P_n$ (error)
 or $e_{n+1} \approx \frac{1}{100} e_n$ (error smaller than the first) (the best one)
 $e_{n+1} \approx \frac{1}{2} e_n$ results ↓

← $e_{n+1} \approx A e_n^2$ (أفضل)
 if $e_n = 0.01$
 $P_{n+1} = A(e_n)^2$

Definition

P is a root of multiplicity M of $f(x)$ if $f(x) = (x-P)^M h(x)$,
 $h(P) \neq 0$.

- $f(x) = x^3 - 3x + 2$

1 is a root of $f(x)$

what is the multiplicity of 1?

$$\begin{array}{r} x-1 \overline{) x^2 + x - 2} \\ \underline{-x^3 + x^2} \\ x^2 - 3x + 2 \\ \underline{-x^2 + x} \\ -2x + 2 \\ \underline{+2x - 2} \\ 0 \end{array}$$

$$\begin{array}{r} x-1 \overline{) x^2 + x - 2} \\ \underline{-x^2 + x} \\ 2x - 2 \\ \underline{2x - 2} \\ 0 \end{array}$$

$$f(x) = (x-1)(x^2 + x - 2)$$

1 has multiplicity 2 (quadratic root) $M=2$
-2 is a simple root ($M=1$).

Theory:-

P is a root of multiplicity M of $f(x)$ iff.

$$f(P) = 0, f'(P) = 0, \dots, f^{(M-1)}(P) = 0 \text{ but}$$

$$f^{(M)}(P) \neq 0$$

Example:-

$$f(x) = x^3 - 3x + 2$$

$$f(1) = 0$$

$$f'(x) = 3x^2 - 3$$

$$f'(1) = 0$$

$$f''(x) = 6x$$

$$f''(1) = 6$$

$$M = 2$$

$$e_{n+1} \approx A e_n$$

$$\frac{e_{n+1}}{e_n} \approx A$$

$$\lim_{n \rightarrow \infty} \frac{e_{n+1}}{e_n} = A \rightarrow \text{linear convergence.}$$

$$\text{if } e_{n+1} \approx A e_n^2$$

$$\lim_{n \rightarrow \infty} \frac{e_{n+1}}{e_n^2} \approx A \rightarrow \text{quadratic convergence.}$$

Definition:- Order of Convergence

assume $P_n \rightarrow P$ and $e_n = P - P_n$, if there exists two positive numbers A, R such that

$$\lim_{n \rightarrow \infty} \frac{|e_{n+1}|}{|e_n|^R} = A$$

Then the sequence is said to converge to P with order of convergence R , A is called the Asymptotic error constant.

if $R=1$, we call it linear convergence.

if $R=2$, we call it quadratic convergence.

example:-

show that $P_n = \frac{1}{n^3}$ converges to $\downarrow P$
 0 linearly??

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|e_{n+1}|}{|e_n|} &= \lim_{n \rightarrow \infty} \frac{|0 - \frac{1}{(n+1)^3}|}{|0 - \frac{1}{n^3}|} = \lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)^3}}{\frac{1}{n^3}} = \lim_{n \rightarrow \infty} \frac{n^3}{(n+1)^3} \\ &= \left(\lim_{n \rightarrow \infty} \frac{n}{n+1} \right)^3 = 1. \end{aligned}$$

$\frac{1}{n^3} \downarrow \rightarrow 0$ linearly
converge to

Example:-

$$f(x) = x^{101} - x^{100} - x + 1$$

$$f(1) = 0$$

$$f'(x) = 101x^{100} - 100x^{99} - 1$$

$$f'(1) = 101 - 100 - 1 = 0$$

$$f''(x) = (101)(100)x^{99} - (100)(99)x^{98}$$

$$f''(1) \neq 0$$

$$M = 2$$

Theorem:- Convergence of newton method

if we use newton iteration,

1. if P is a simple root, then

$$\lim_{n \rightarrow \infty} \frac{|e_{n+1}|}{|e_n|^2} = \left| \frac{f''(P)}{2f'(P)} \right|$$

$$\left[\begin{array}{l} P \text{ is a simple root} \\ \text{Convergence is quadratic} \\ A = \left| \frac{f''(P)}{2f'(P)} \right|, R = 2 \end{array} \right]$$

2. if P has multiplicity $M > 1$, then

$$\lim_{n \rightarrow \infty} \frac{|e_{n+1}|}{|e_n|} = \frac{M-1}{M}$$

$$\left[\begin{array}{l} \text{convergence is linear} \\ A = \frac{M-1}{M}, R = 1 \end{array} \right]$$

example

$$f(x) = x^3 - 3x + 2$$

$$f'(x) = 3x^2 - 3$$

$$f(x) = (x-1)^2(x+2)$$

$$f''(x) = 6x$$

-2 is a simple roots

convergence is fast $R = 2$ $\left(\frac{|e_{n+1}|}{|e_n|^2} = \left| \frac{f''(P)}{2f'(P)} \right| \right)$

$$A = \left| \frac{f''(-2)}{2f'(-2)} \right| = \left| \frac{-12}{2(9)} \right| = \frac{2}{3}$$

$$P = 1, M = 2$$

linear convergence ($P = 1$)

$$\lim_{n \rightarrow \infty} \frac{|e_{n+1}|}{|e_n|} = \frac{1}{3}$$

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$$

$$P_0 = -2.4$$

n	P_n	$e_n \xrightarrow{P-A}$	$\frac{ e_{n+1} }{ e_n }$
0	-2.4	0.4	
1	-2.0761904	0.0761904	0.4761 ----
2	-2.00359601	0.003596	0.6194 ---
3	-2.000008589	0.000008589	0.6642 ↓ $\frac{2}{3} \approx A$

fast convergence.

$$P_0 = 1.2$$

n	P_n	e_n	$\frac{ e_{n+1} }{ e_n }$
0	1.2	-0.2	
1	1.103030	-0.10303	0.515 ----
2	1.0523564	-0.052356	0.5081
3	1.0264008	-0.026400811	0.4962 ↓ $A \approx \frac{1}{2}$

slow convergence.

$$A \rightarrow \frac{1}{2}$$

Theory:- accelerated newton method

if α is a root of multiplicity M then the iteration

$$P_{n+1} = P_n - \frac{M f(P_n)}{f'(P_n)}$$
 will converge quadratically to α .

Ex:-

For the previous example. $f(x) = (x-1)^2(x+2)$

1 has multiplicity 2, if we use the accelerated newton iteration

$$P_{n+1} = P_n - \frac{2 f(P_n)}{f'(P_n)}$$
 will get quadratic convergence!

$$P_0 = 1.2$$

n	P_n	e_n	$\frac{ e_{n+1} }{ e_n ^2}$
0	1.2	-0.2	
1	1.0060606	-0.00606	0.15
2	1.00006087	-0.00006087	0.15

Secant method:-

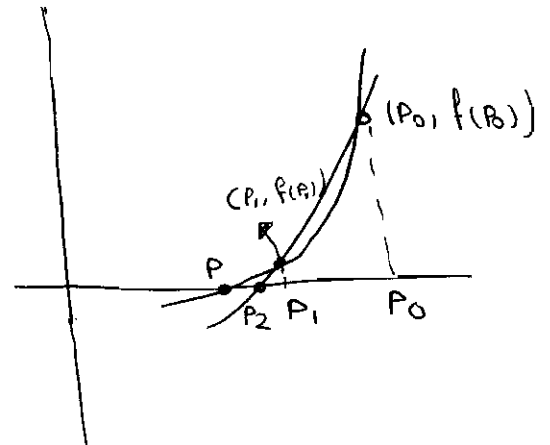
$$\frac{f(A) - 0}{A - B} = \frac{f(A_1) - f(A_0)}{A_1 - A_0}$$

$$A - B = \frac{f(A)(A_1 - A_0)}{f(A_1) - f(A_0)}$$

$$A_2 = A_1 - \frac{f(A_1)(A_1 - A_0)}{f(A_1) - f(A_0)}$$

$$A_3 = A_2 - \frac{f(A_2)(A_2 - A_1)}{f(A_2) - f(A_1)}$$

$$A_n = A_{n-1} - \frac{f(A_{n-1})(A_{n-1} - A_{n-2})}{f(A_{n-1}) - f(A_{n-2})}$$



Theorem:-

if we use secant method to get $A_n \rightarrow p$ then.

$$\lim_{n \rightarrow \infty} \frac{|e_{n+1}|}{|e_n|^{1.618}} = \left| \frac{f''(p)}{2f'(p)} \right|^{0.618}$$

$$\rightarrow R = 1.618 = \frac{1 + \sqrt{5}}{2}$$

Ex:-

$$f(x) = (x+2)(x-1)^2$$

$$P_0 = -2.6, P_1 = -2.4$$

and we use secant method.

n	P_n	e_n	$\frac{ e_{n+1} }{ e_n ^{1.618}}$
0	-2.6	0.6	
1	-2.4	0.4	
2	-2.106598	0.106598	
3	-2.02264	0.02264	
4	-2.00151	0.00151	

False position method

<u>Speed</u>	1
<u>Coast</u>	1
<u>Convergence</u>	very accurate

Secant method

1.6
1
depends on P_0, P_1

Newton method

2
2
depends on P_0

• 2.6 Fixed point iteration For system of equation

$$\begin{aligned}x^2 \cos y + y \sin x &= 10 \\ y \ln x + x^2 \cos y &= 5\end{aligned}$$

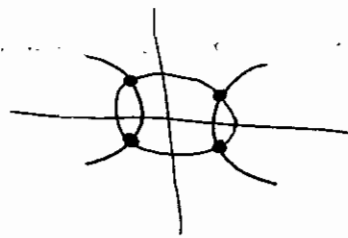
$$x^2 - y^2 = 1$$

$$x^2 + y^2 = 2$$

$$2x^2 = 3$$

$$x^2 = \frac{3}{2}$$

$$x = \pm \sqrt{\frac{3}{2}}$$



$$x^2 - y^2 = x + 3$$

$$x^2 + y^2 = e^x - 1$$

$$2x^2 = x + 3 + e^x - 1$$

$$2x^2 - x - e^x - 2 = 0$$

$$x = g_1(x, y)$$

$$y = g_2(x, y)$$

$$(P_0, Q_0)$$

$$P_1 = g_1(P_0, Q_0)$$

$$P_2 = g_1(P_1, Q_1)$$

$$Q_1 = g_2(P_0, Q_0)$$

$$Q_2 = g_2(P_1, Q_1)$$

$$P_{n+1} = g_1(P_n, Q_n)$$

$$Q_{n+1} = g_2(P_n, Q_n)$$

Definition:-

(P, Q) is a Fixed Point of the system

$$x = g_1(x, y), y = g_2(x, y) \text{ if } P_x = g_1(P, Q) \text{ and } Q_y = g_2(P, Q)$$

Def:-

Fixed point iteration for the system

$$x = g_1(x, y), y = g_2(x, y) \text{ is given } (P_0, Q_0) \text{ then}$$

$$P_{n+1} = g_1(P_n, Q_n)$$

$$Q_{n+1} = g_2(P_n, Q_n) \quad n=1, 2, 3, \dots$$

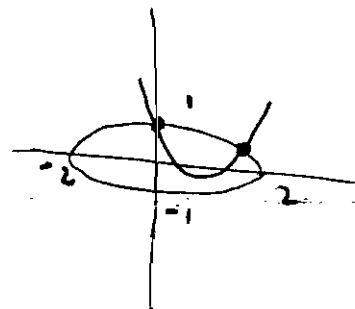
Ex:-

$$f_1(x, y) = x^2 - 2x - y + 0.5 = 0$$

$$f_2(x, y) = x^2 + 4y^2 - 4 = 0$$

estimate the solutions?

$$\rightarrow x^2 + 4y^2 = 4$$
$$\frac{x^2}{4} + y^2 = 1$$



$$x = \frac{x^2 - y + 0.5}{2} = g_1(x, y)$$

$$y = \frac{-x^2 - 4y^2 + 8y + 4}{8} = g_2(x, y)$$

$$(P_0, Q_0) = (0, 1)$$

$$P_1 = g_1(0, 1) = \frac{0 - 1 + 0.5}{2} = -0.25$$

$$Q_1 = g_2(0, 1) = \frac{0 - 4 + 8 + 4}{8} = 1$$

$$P_4 = -0.2221680$$

$$Q_4 = 0.9938121$$

$$P_5 = -0.222194$$

$$Q_5 = 0.9938095$$

$$(P_0, Q_0) = (2, 0) \text{ (diverges)}$$

$$P_1 = g_1(2, 0) = 2.25$$

$$Q_1 = g_2(2, 0) = 0$$

$$\text{Let } g_1(x, y) = \frac{-x^2 + 4x + y - 0.5}{2}$$

$$g_2(x, y) = \frac{-x^2 - 4y^2 - 11x + 4}{8}$$

$$(P_0, Q_0) = (2, 1) \rightarrow \text{Converge}$$

$$(2, 1) \rightarrow (1.900, 0.311)$$

The: Fixed point iteration For system of equation:-

assume $g_1(x,y)$, $g_2(x,y)$ and their partial derivative are continuous on a region that contains the Fixed point (P, Q) , if the starting point (P_0, Q_0) is choosing sufficiently closed to (P, Q) and.

$$\left| \frac{dg_1}{dx} \right| + \left| \frac{dg_1}{dy} \right| < 1 \text{ and } \left| \frac{dg_2}{dx} \right| + \left| \frac{dg_2}{dy} \right| < 1 \text{ in that region}$$

then the FPI will Converge.

• Note:-

if (P, Q) is given we apply the condition at (P, Q) only.

to proof
لتحقيق الشرط

Fixedpoint \rightarrow we talk about g 's
newton \rightarrow we talk about F .

if $|x| < 0.5$ and $0.5 < y < 1.5$ أولنا نختار الفترة

$$\left| \frac{dg_1}{dx} \right| + \left| \frac{dg_1}{dy} \right| = |x| + 0.5 < 1$$

أكبر قيمة
0.5

$$\left| \frac{dg_2}{dx} \right| + \left| \frac{dg_2}{dy} \right| = \frac{|x|}{4} + |1-y| < \frac{1}{8} + 0.5 < 1$$

أكبر قيمة
0.5 أكبر قيمة
1.5

حتى نشبه ان النقطة المتناثرة divergance \leftarrow نختار فترة لا تحقق الشرطين السابقين او لا تحقق شرط واحد من الاقله.

example (linear system)

$$\begin{aligned} 3x + 2y + 7z &= 10 & \rightarrow & x = \frac{10 - 2y - 7z}{3} = g_1(x, y, z) \\ 2x + 4y - z &= 4 & \rightarrow & y = \frac{4 + z - 2x}{4} = g_2(x, y, z) \\ x + 5y + 10z &= 15 & \rightarrow & z = \frac{15 - x - 5y}{10} = g_3(x, y, z). \end{aligned}$$

$$p_1 = g_1(p_0, q_0, r_0)$$

$$q_1 = g_2(p_0, q_0, r_0)$$

$$r_1 = g_3(p_0, q_0, r_0)$$

$$\rightarrow p_1 = g_1(p_1, q_0, r_0)$$

$$q_1 = g_2(p_1, q_0, r_0)$$

$$r_1 = g_3(p_1, q_0, r_0)$$

Gauss-Seidel
method

$$\bullet p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$$

$$f_1(x, y) = 0$$

$$f_2(x, y) = 0$$

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} p_n \\ q_n \end{pmatrix} - \underset{\substack{\downarrow \\ \text{Jacobian:} \\ (p_n, q_n)}}{\mathcal{J}^{-1}} \begin{pmatrix} f_1(p_n, q_n) \\ f_2(p_n, q_n) \end{pmatrix}$$

$$h: (x, y) \rightarrow (f_1(x, y), f_2(x, y))$$

$$h' = \mathcal{J} = \begin{pmatrix} \frac{df_1}{dx} & \frac{df_1}{dy} \\ \frac{df_2}{dx} & \frac{df_2}{dy} \end{pmatrix}$$

$$\vec{p}_{n+1} = \vec{p}_n - \mathcal{J}^{-1} \#$$

2.7 Newton method

given $F_1(x, y) = 0, F_2(x, y) = 0$

and $F_1(p, q) = 0, F_2(p, q) = 0$.

starting with (p_0, q_0) close to (p, q) then using Taylor expansion in two dimension at (p_0, q_0)

$$F_1(x, y) \approx F_1(p_0, q_0) + \left. \frac{dF_1}{dx} \right|_{(p_0, q_0)} (x - p_0) + \left. \frac{dF_1}{dy} \right|_{(p_0, q_0)} (y - q_0)$$

$$F_2(x, y) \approx F_2(p_0, q_0) + \left. \frac{dF_2}{dx} \right|_{(p_0, q_0)} (x - p_0) + \left. \frac{dF_2}{dy} \right|_{(p_0, q_0)} (y - q_0)$$

substitute (p, q) above

$$0 = F_1(p_0, q_0) + \left. \frac{dF_1}{dx} \right|_{(p_0, q_0)} (p - p_0) + \left. \frac{dF_1}{dy} \right|_{(p_0, q_0)} (q - q_0)$$

$$0 = F_2(p_0, q_0) + \left. \frac{dF_2}{dx} \right|_{(p_0, q_0)} (p - p_0) + \left. \frac{dF_2}{dy} \right|_{(p_0, q_0)} (q - q_0)$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} F_1(p_0, q_0) \\ F_2(p_0, q_0) \end{bmatrix} + \begin{bmatrix} \left. \frac{dF_1}{dx} \right|_{(p_0, q_0)} & \left. \frac{dF_1}{dy} \right|_{(p_0, q_0)} \\ \left. \frac{dF_2}{dx} \right|_{(p_0, q_0)} & \left. \frac{dF_2}{dy} \right|_{(p_0, q_0)} \end{bmatrix} \begin{bmatrix} p - p_0 \\ q - q_0 \end{bmatrix}$$

$$-\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}_{(p_0, q_0)} = \mathcal{J}_{(p_0, q_0)}^{-1} \begin{bmatrix} p - p_0 \\ q - q_0 \end{bmatrix} \rightarrow \text{Direct method.}$$

$$-\mathcal{J}_{(p_0, q_0)}^{-1} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} p - p_0 \\ q - q_0 \end{bmatrix}$$

$$\begin{bmatrix} p_0 \\ q_0 \end{bmatrix} - \mathcal{J}_{(p_0, q_0)}^{-1} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}_{(p_0, q_0)} = \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} \quad \text{inverse way.}$$

- Inverse method.

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \underset{(p_n, q_n)}{J}^{-1} \begin{bmatrix} f_1(p_n, q_n) \\ f_2(p_n, q_n) \end{bmatrix}$$

- Direct method

$$-\begin{bmatrix} f_1(p_n, q_n) \\ f_2(p_n, q_n) \end{bmatrix} = \underset{(p_n, q_n)}{J} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

$$\Delta x = p_{n+1} - p_n \rightarrow p_{n+1} = \Delta x + p_n$$

$$\Delta y = q_{n+1} - q_n \rightarrow q_{n+1} = \Delta y + q_n$$

- example

Solve using Newton method.

- inverse method.

$$x^2 - 2x - y = 0.5 \rightarrow f_1 = x^2 - 2x - y - 0.5 = 0 = f_1(x, y)$$

$$x^2 + 4y^2 = 4 \rightarrow x^2 + 4y^2 - 4 = 0 = f_2(x, y)$$

$$(p_0, q_0) = (2, 0.25)$$

$$J = \begin{pmatrix} 2x-2 & -1 \\ 2x & 8y \end{pmatrix}_{(2, 0.25)} = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}$$

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0.25 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix}$$

$$f_1(2, 0.25) = 0.25$$

$$f_2(2, 0.25) = 0.25$$

$$= \begin{pmatrix} 2 \\ 0.25 \end{pmatrix} - \frac{1}{8} \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix} \begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix} = \begin{pmatrix} 1.90625 \\ 0.3125 \end{pmatrix}$$

$$\begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1.90625 \\ 0.3125 \end{pmatrix} - \begin{pmatrix} 1.8125 & -1 \\ 3.8125 & 2.5 \end{pmatrix}^{-1} \begin{pmatrix} 0.008789 \\ 0.024414 \end{pmatrix}$$

$$= \begin{pmatrix} 1.900691 \\ 0.31213 \end{pmatrix}$$

→ Direct method

$$-\begin{pmatrix} f_1(2, 0.25) \\ f_2(2, 0.25) \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

$$-\begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

$$\Delta x = \frac{\begin{vmatrix} -0.25 & -1 \\ -0.25 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix}} = \frac{-0.75}{8} = -0.09375$$

$$\begin{aligned} p_1 &= \Delta x + p_0 \\ &= -0.09375 + 2 \\ &= 1.90625 \end{aligned}$$

$$\Delta y = \frac{\begin{vmatrix} -0.25 & 2 & -0.25 \\ -0.25 & 4 & -0.25 \end{vmatrix}}{8} = \frac{-0.5 + 1}{8} = \frac{0.5}{8} = 0.0625$$

$$\Delta y = q_1 + q_0$$

$$\begin{aligned} q_1 &= \Delta y + q_0 \\ &= 0.0625 + 0.25 \\ &= 0.3125 \end{aligned}$$

discussion

2.4

$$\boxed{10} \quad f(x) = (x-p)^m h(x).$$

$$\leftrightarrow f(p) = 0, f'(p) = 0 \dots f^{(m-1)}(p) = 0 \quad \text{but } f^{(m)}(p) \neq 0.$$

$$f(p) = 0.$$

$$f'(x) = m(x-p)^{m-1} h(x) + (x-p)^m h'(x).$$

$$f'(p) = 0.$$

$$f(p) = 0.$$

$(x-p)$ is a factor of $f(x)$.

$(x-p)^2$ is a factor of $f'(x)$.

$$\boxed{8} \quad g(x) = x - \frac{mf(x)}{f'(x)} \quad \text{it will converge quadratically to } p.$$

p is a root of multiplicity m for $f(x)$.

$$g'(p) = 0 \quad \text{بناشبهه ال}$$

$$f(x) = (x-p)^m h(x), \quad h(p) \neq 0.$$

$$f'(x) = m(x-p)^{m-1} h(x) + (x-p)^m h'(x).$$

$$g(x) = x - \frac{m(x-p)^m h(x)}{m(x-p)^{m-1} h(x) + (x-p)^m h'(x)}$$

$$= x - \frac{m(x-p) h(x)}{m h(x) + (x-p) h'(x)}$$

$$g'(x) = 1 - \frac{(m h(x) + (x-p) h'(x)) (m h(x) + (x-p) h'(x)) - m(x-p) h(x) h'(x)}{[m h(x) + (x-p) h'(x)]^2}$$

$$g'(p) = 1 - \frac{(m h(p))^2}{m h(p)^2}$$

$$= 0.$$

6

a. $P_n = 10^{-2^n} \rightarrow 0$ quadratically

$$\lim_{n \rightarrow \infty} \frac{|e_{n+1}|}{|e_n|^2} = \lim_{n \rightarrow \infty} \frac{10^{-2^{n+1}}}{(10^{-2^n})^2} = \lim_{n \rightarrow \infty} \frac{10^{-2^{n+1}}}{10^{-2^{n+1}}} = 1$$

b. $P_n = 10^{-n^k} \rightarrow 0$ quadratically.

$$\lim_{n \rightarrow \infty} \frac{|e_{n+1}|}{|e_n|^2} = \lim_{n \rightarrow \infty} \frac{10^{-(n+1)^k}}{(10^{-n^k})^2} = \lim_{n \rightarrow \infty} \frac{10^{-(n+1)^k}}{10^{-2(n^k)}}$$

$$= \lim_{n \rightarrow \infty} \frac{10^{2(n^k)}}{10^{(n+1)^k}} = \lim_{n \rightarrow \infty} \frac{10^{n^k} \cdot 10^{n^k}}{10^{(n+1)^k}} \rightarrow \infty$$

$\leftarrow 10$

20

2.3

$$1564,000 = 1,000,000 e^2 + \frac{435,000}{2} (e^2 - 1)$$

$$1564 = 1000 e^2 + \frac{435}{2} (e^2 - 1)$$

$$f(x) = 1000 e^x + \frac{435}{2} (e^x - 1) - 1564 = 0$$

Chapter 3

linear systems:-

Iterative methods:-

1. Fixed point iteration
2. Gauss - Seidel method
3. Newton Method.

Direct methods:- (A is nonsingular).

1. Gaussian Elimination $[A:b] \rightarrow [U|c]$ + Back substitution.
2. Gauss - Jordan $[A|b] \rightarrow [I|X]$.
3. inverse method $x = A^{-1}b$
4. Cramer's $x_i = \frac{|A_i|}{|A|}$
5. L - U Factorization.

section 3.3

back substitution

$$3x_1 + 2x_2 + 4x_3 = 9$$

$$4x_2 + 6x_3 = 10$$

$$10x_3 = 10$$

$$\begin{bmatrix} 3 & 2 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} 9 \\ 10 \\ 10 \end{bmatrix}$$

$$10x_3 = 10 \rightarrow \boxed{x_3 = 1}$$

$$4x_2 + 6x_3 = 10$$

$$4x_2 = 10 - 6$$

$$4x_2 = 4 \rightarrow \boxed{x_2 = 1}$$

$$3x_1 + 2x_2 + 4x_3 = 9$$

$$3x_1 = 9 - 4 - 2$$

$$3x_1 = 3 \rightarrow \boxed{x_1 = 1}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,n-2} & a_{1,n-1} & a_{1,n} \\ \vdots & & & & & \\ & & & a_{n-2,n-2} & a_{n-2,n-1} & a_{n-2,n} \\ & & & & a_{n-1,n-1} & a_{n-1,n} \\ & & & & & a_{n,n} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_{n-2} \\ b_{n-1} \\ b_n \end{bmatrix}$$

$a_{n,n}x_n = b_n$
 $x_n = \frac{b_n}{a_{n,n}}$
 $a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = b_{n-1}$
 $x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}$

$$x_{n-2} = \frac{b_{n-2} - a_{n-2,n-1}x_{n-1} - a_{n-2,n}x_n}{a_{n-2,n-2}}$$

$x_k = b_k$

$$x_{1k} = \frac{b_{1k} - a_{1k,k+1}x_{k+1} - a_{1k,k+2}x_{k+2} - \dots - a_{1k,n}x_n}{a_{1k,k}}$$

$$x_{1k} = \frac{b_{1k} - \sum_{j=k+1}^n a_{1k,j}x_j}{a_{1k,k}} \quad k = n, n-1, \dots$$

33 Cost

Steps	+/-	x/÷
1	0	1
2	1	2
3	2	3
...
k	k-1	k
...
n	n-1	n
Total	$\frac{(n-1)(n)}{2}$	$\frac{n(n+1)}{2}$

$$\text{Total cost} = \frac{n^2 - n}{2} + \frac{n^2 + n}{2} = n^2$$

3.4 Gaussian Elimination

$$Ax = b$$

$$[A|b] \rightarrow [O|C] + \text{back sub.}$$

Row operations:-

1. multiply any row by a nonzero constant
2. switch any two rows
3. Replace any row by adding to it a nonzero multiple of another row

$$\text{row } r := \text{row } r + C \text{ row } p$$

$$= \text{row } r - m_{r,p} \text{ row } p$$

$$; m_{r,p} = \frac{a_{r,p}}{a_{p,p}} \quad r > p$$

Example

Solve:-

$$x_1 + 2x_2 + x_3 + 4x_4 = 13$$

$$2x_1 + 4x_3 + 3x_4 = 28$$

$$4x_1 + 2x_2 + 2x_3 + x_4 = 20$$

$$-3x_1 + x_2 + 3x_3 + 2x_4 = 6$$

Pivot element	1	2	1	4	13
	2	0	4	3	28
	4	2	2	1	20
	-3	1	3	2	6

Pivot Row

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{2}{1} = 2$$

$$m_{31} = \frac{a_{31}}{a_{11}} = \frac{4}{1} = 4$$

$$m_{41} = \frac{a_{41}}{a_{11}} = \frac{-3}{1} = -3$$

$R_2 - 2R_1$	0	-4	2	-5	2
$R_3 - 4R_1$	0	-6	-2	-15	-32
$R_4 + 3R_1$	0	7	6	14	45

Pivot Row

$$m_{32} = \frac{a_{32}^{(2)}}{a_{22}^{(2)}} = \frac{6}{-4} = -1.5$$

$$m_{42} = \frac{a_{42}^{(2)}}{a_{22}^{(2)}} = \frac{7}{-4} = -1.75$$

$R_3 - 1.5R_2$	0	0	-5	-7.5	-35
$R_4 + 1.75R_2$	0	0	9.5	15.25	48.5

Pivot Row

$$m_{43} = \frac{a_{43}^{(3)}}{a_{33}^{(3)}} = \frac{9.5}{-5} = -1.9$$

$R_4 - 1.9R_3$	0	0	0	-9	-18
----------------	---	---	---	----	-----

$$x_4 = 2$$

$$x_3 = 4$$

$$x_2 = -1$$

$$x_1 = 3$$

Coast

Step	+/-	x/÷	
1	4x3	3 + 4x3	
2	3x2	2 + 3x2	
3	2x1	1 + 2x1	
total	20	26	46

in general for nxn matrix

Step	+/-	x/÷
1	$(n-1)n$	$(n-1)n + n - 1$
2	$(n-2)(n-1)$	$(n-2)(n-1) + n - 2$
3	$(n-3)(n-2)$	$(n-3)(n-2) + n - 3$
⋮		
P	$(n-P)(n-P+1)$	$(n-P)(n-P+1) + n - P$
last step →	$(n-1)$	

$$\text{total +/-} : \sum_{P=1}^{n-1} (n-P)(n-P+1)$$

$$\text{x/÷} : \sum_{P=1}^{n-1} (n-P)(n-P+1) + (n-P)$$

$$\sum_{P=1}^{n-1} (n-P)(n-P+1) = \sum_{P=1}^{n-1} (n-P)^2 + (n-P)$$

Let $k = n - p$

if $p = 1 \rightarrow k = n - 1$

$p = n - 1 \rightarrow k = 1$

$$\therefore \sum_1^{n-1} k^2 + k$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^n k = \frac{(n-1)n}{2}$$

$$\therefore \sum_1^{n-1} k^2 + k = \frac{n(n+1)(2n+1)}{6} + \frac{(n-1)n}{2}$$

total $x/\%$ = $\frac{(n-1)(n)(2n-1)}{6} + \frac{n(n-1)}{2} + \frac{n(n-1)}{2}$

qalsi ~~qalsi~~
 \therefore Grand Total = $2 \left[\frac{(n^2-n)(2n-1)}{6} + \frac{n(n-1)}{2} \right] + \frac{n(n-1)}{2}$

$$= 2 \left[\frac{2n^3 - 3n^2 + n}{6} + \frac{3n^2 - 3n}{6} \right] + \frac{n^2 - n}{2}$$

$$= \frac{2n^3 - 2n}{3} + \frac{n^2 - n}{2}$$

$$= \frac{4n^3 - 4n + 3n^2 - 3n}{6} = \frac{4n^3 + 3n^2 - 7n}{6} \approx \frac{2}{3} n^3$$

• Coast for Gaussian

Coast = $\frac{4n^3 + 3n^2 - 7n}{6} + (n^2)$ ← Coast For back substitution.

$$= \frac{4n^3 + 9n^2 - 7n}{6}$$

$$\approx \frac{2}{3} n^3$$

• Algorithm

will store the Augmented matrix in $n+1$ column.

$$\left[\begin{array}{cccc|c} a_{1,1}^{(1)} & a_{1,2}^{(1)} & \dots & a_{1,n}^{(1)} & a_{1,n+1}^{(1)} \\ a_{2,1}^{(1)} & a_{2,2}^{(1)} & \dots & a_{2,n}^{(1)} & a_{2,n+1}^{(1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n,1}^{(1)} & a_{n,2}^{(1)} & \dots & a_{n,n}^{(1)} & a_{n,n+1}^{(1)} \end{array} \right]$$

and will construct an equivalent upper triangular U

$$\left[\begin{array}{cccc|c} a_{1,1}^{(1)} & a_{1,2}^{(1)} & \dots & a_{1,n}^{(1)} & a_{1,n+1}^{(1)} \\ 0 & a_{2,2}^{(2)} & \dots & a_{2,n}^{(2)} & a_{2,n+1}^{(2)} \\ 0 & 0 & a_{3,3}^{(3)} & \dots & a_{3,n+1}^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n,n+1}^{(n)} \end{array} \right]$$

Step 1 store the coefficient in array

Step 2 switch rows if necessary so that $a_{1,1}^{(1)} \neq 0$

find $m_{r,1} = \frac{a_{r,1}^{(1)}}{a_{1,1}^{(1)}}$ for $r=2$ to n .

for c from 2 to $n+1$.

set $a_{r,c}^{(2)} = a_{r,c}^{(1)} - m_{r,1} a_{1,c}^{(1)}$

we get

$$\left[\begin{array}{cccc|c} a_{1,1}^{(1)} & a_{1,2}^{(1)} & \dots & a_{1,n}^{(1)} & a_{1,n+1}^{(1)} \\ 0 & a_{2,2}^{(2)} & \dots & a_{2,n}^{(2)} & a_{2,n+1}^{(2)} \\ 0 & 0 & a_{3,3}^{(3)} & \dots & a_{3,n+1}^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n,n+1}^{(n)} \end{array} \right]$$

in general

P+1 step find $a_{p,p}^{(p)} \neq 0$ From $r = p+1$ to N

$m_{r,p} = \frac{a_{r,p}^{(p)}}{a_{p,p}^{(p)}}$ and $a_{r,p}^{(p+1)} = 0$

For $c = p+1$ to $n+1$

$a_{r,c}^{(p+1)} = a_{r,c}^{(p)} - m_{r,p} a_{p,c}^{(p)}$

we have 3 loop

• Error

$0.37205 * (7) = 2.60435 \approx 2.6044$ تقریباً بعد صغیر امن
 $0.12345 * (7) = 0.86415 \approx 0.86415$ حد التقریب بعد کبیر

- Gaussian elimination with pivoting:-

to avoid propagation of error we use the pivot element to be the largest in the remaining of the column in $|a_{k-p}| = \max[|a_{pp}|, |a_{p+1,p}|$

$\dots |a_{n-1,p}|, |a_{n,p}|]$ and switch row p with row k if $k > p$

Example:-

$(1.000, 1.000)$ is a solution to $1.133x_1 + 5.281x_2 = 6.414$
 $24.14x_1 - 1.210x_2 = 22.93$

Solve the above by Gaussian with pivoting and without pivoting

• without pivoting

$$\left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 24.14 & 1.210 & 22.93 \end{array} \right] \quad m_{21} = \frac{24.14}{1.133} = 21.31$$

$$\rightarrow \left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 0 & -113.7 & -113.8 \end{array} \right] \quad \begin{array}{l} x_2 = 1.001 \\ x_1 = 0.9956 \end{array}$$

with pivoting

$$\left[\begin{array}{cc|c} 24.14 & -1.210 & 22.93 \\ 1.133 & 5.281 & 6.414 \end{array} \right] \quad m_{21} = \frac{1.133}{24.14} = 0.0464$$

$$\rightarrow \left[\begin{array}{cc|c} 24.14 & -1.210 & 22.93 \\ 0 & 5.338 & 5.338 \end{array} \right] \quad \begin{array}{l} x_1 = 1.000 \\ x_2 = 1.000 \end{array}$$

$$Ax=b$$

1. Gaussian $[A|b] \rightarrow [U|c]$ + backsubstitution

2. Gauss - Jordan Elimination

$$\left[\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ * & * & * & * \\ 0 & * & * & * \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \alpha_1 \\ 0 & 1 & 0 & \alpha_2 \\ 0 & 0 & 1 & \alpha_3 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & * & * \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array} \right]$$

Step	+/-	* / ÷
1	3×2	$3 + 3 \times 2$
2	2×2	$2 + 2 \times 2$
3	1×2	$1 + 1 \times 2$
i		
n	max	

Solve

$$3x_1 + 2x_2 + 4x_3 = 9$$

$$x_1 - 2x_2 + 3x_3 = 2$$

$$3x_1 + 4x_2 - x_3 = 6$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 4 & 9 \\ 1 & -2 & 3 & 2 \\ 3 & 4 & -1 & 6 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2/3 & 4/3 & 3 \\ 1 & -2 & 3 & 2 \\ 3 & 4 & -1 & 6 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2/3 & 4/3 & 3 \\ 0 & -8/3 & 5/3 & -1 \\ 0 & 2 & -5 & -3 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2/3 & 4/3 & 3 \\ 0 & 1 & -5/8 & 3/8 \\ 0 & 2 & -5 & -3 \end{array} \right]$$

Exercise

Find the total cost for Gauss Jordan elimination.

Ex 3. Inverse method.

$$[A \setminus I] \rightarrow [I \setminus A^{-1}]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & A^{-1} \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} * & * & * & 1 & 0 & 0 \\ * & * & * & 0 & 1 & 0 \\ * & * & * & 0 & 0 & 1 \end{array} \right]$$

$$Ax = b \\ x = A^{-1}b$$

multiplication cost = $2n^2 - n$.

Coast

Step	+/-	* / ÷
1	5×2	$5 + 5 \times 2$
2	4×2	$4 + 4 \times 2$
3	3×2	$3 + 3 \times 2$
	\vdots	$(2n-p)(n-1) + (2n-p)$
	$(2n-p)(n-1)$	

$(2n-p)(n-1) \rightarrow$

Step	+/-	* / ÷
1	$(2n-1) \times (n-1)$	$(2n-1) + (2n-1)(n-1)$
2	$(2n-2)(n-1)$	$(2n-2) + (2n-2)(n-1)$
\vdots	$(2n-p)(n-1)$	$(2n-p) + (2n-p)(n-1)$
n	$n(n-1)$	$n + n(n-1)$

$$\text{Coast} = \frac{16n^3 - 9n^2 - n}{6} \approx 2 \frac{2}{3} n^3$$

4. Cramer's method

$$x_i = \frac{|A_i|}{|A|}$$

Find the coast of Cramer's method for 3×3 matrix.

$$x_1 = \frac{|A_1|}{|A|}$$

$$x_2 = \frac{|A_2|}{|A|}$$

$$x_3 = \frac{|A_3|}{|A|}$$

$$4 \downarrow \quad | \quad 3 \times 3 \quad + \quad 3 \downarrow \\ \text{determinant} \quad \text{cost}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$\text{Coast} = 4 \times (14) + 3 = 59$$

$$4 \times 3 + 2 = 14$$

3.6 L-U Factorization

$$Ax = b$$

$$LUX = b$$

1. $LY = b \rightarrow$ Forward substitution

2. $UX = Y \rightarrow$ backward substitution (cost = n^2)

$$[A] \rightarrow \begin{bmatrix} a_{11}^{(1)} & \dots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \dots & a_{2n}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & a_{nn}^{(n)} \end{bmatrix} \quad L = \begin{bmatrix} 1 & & & \\ m_{21} & 1 & & \\ m_{31} & m_{32} & 1 & \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix}$$

Ex:-

Solve using L-U Factorization.

$$4x_1 + 3x_2 - x_3 = -2$$

$$-2x_1 + 4x_2 + 5x_3 = 20$$

$$x_1 + 2x_2 + 6x_3 = 7$$

No switch in Row

$$\begin{bmatrix} 4 & 3 & -1 \\ -2 & -4 & 5 \\ 1 & 2 & 6 \end{bmatrix} \quad \begin{matrix} m_{21} = -\frac{2}{4} = -\frac{1}{2} \\ m_{31} = \frac{1}{4} = 0.25 \end{matrix}$$

$$\begin{matrix} R_2 + 0.5R_1 \\ R_3 - 0.25R_1 \end{matrix} \rightarrow \begin{bmatrix} 4 & 3 & -1 \\ 0 & -2.5 & 4.5 \\ 0 & 1.25 & 6.25 \end{bmatrix} \quad m_{32} = \frac{1.25}{-2.5} = -0.5$$

$$\begin{matrix} R_3 + 0.5R_2 \end{matrix} \rightarrow \begin{bmatrix} 4 & 3 & -1 \\ 0 & -2.5 & 4.5 \\ 0 & 0 & 8.5 \end{bmatrix} = U \quad \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0.25 & -0.5 & 1 \end{bmatrix} = L$$

$$LY = b$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0.25 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 20 \\ 7 \end{bmatrix}$$

$$y_1 = -2$$

$$y_2 = 19$$

$$y_3 = 17$$

Forward substitution

$$UX = y$$

$$\begin{bmatrix} 4 & 3 & -1 \\ 0 & -2.5 & 4.5 \\ 0 & 0 & 8.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 19 \\ 17 \end{bmatrix}$$

$$x_3 = \frac{17}{8.5} = 2$$

$$x_2 = -4$$

$$x_1 = 3$$

• total cost = backsubstitution

Row operations multiplication) $\frac{n^3 - n}{3}$ + n^2 + $\frac{n^2 - n}{6}$ + $\frac{2n^3 - 3n^2 + n}{6}$ Row operation (addition)

Forward substitution

$$= \frac{2n^3 - 2n}{6} + \frac{6n^2}{6} + \frac{6n^2 - 6n}{6} + \frac{2n^3 - 3n^2 + n}{6}$$

$$= \frac{4n^3 + 9n^2 - 7n}{6}$$

Step	+/-	* / ÷
1	$(n-1)(n-1)$	$(n-1) + (n-1)(n-1)$
2	$(n-2)(n-2)$	$(n-2) + (n-2)(n-2)$
P	$(n-p)^2$	$(n-p) + (n-p)^2$

$$\text{total} = \sum (n-p)^2 + \sum (n-p) + \sum (n-p)^2$$

$$= 2 \sum (n-p)^2 + \sum (n-p)$$

$$= 2 \frac{(n-1)(n)(2n-1)}{6} + \frac{(n-1)(n)}{2}$$

$$= 2 \frac{(2n^3 - 3n^2 + n)}{6} + \frac{n^2 - n}{2}$$

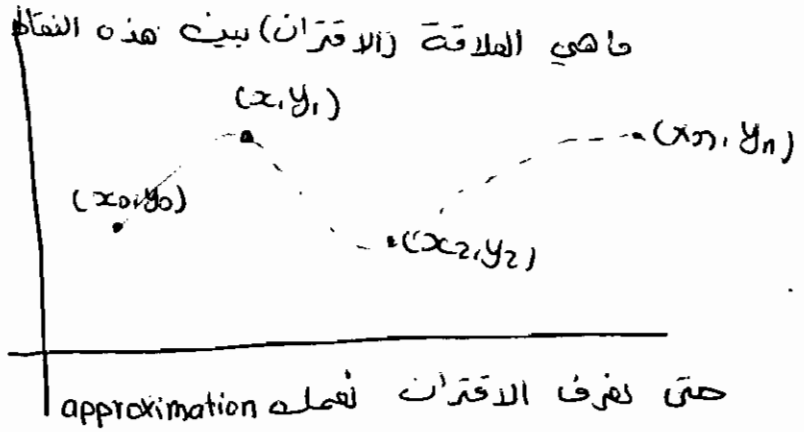
$$= \frac{4n^3 - 6n^2 + 2n + 3n^2 - 3n}{6}$$

$$= \frac{4n^3 - 3n^2 - n}{6}$$

Interpolation by polynomials:-

Given $(x_0, y_0) (x_1, y_1) (x_2, y_2) \dots (x_n, y_n)$

x_i	y_i
x_0	y_0
x_1	y_1
x_2	y_2
\vdots	\vdots
x_n	y_n



Curve Fitting أخذ خط يمر بين النقط حتى تكون العلاقة بينه النقط اصغر ما يمكن

interpolation :- is estimation of the unknown function by poly which passes through all given points

$$P_n(x_i) = f(x_i)$$

P_n is the approximation polynomial
 f is the unknown function

n Polonomial من الدرجة n

← إذا عندنا $(n+1)$ points

Example

$(1, 2), (3, 5), (7, 10)$

$$P_2(x) = Ax^2 + Bx + C$$

$$P_2(1) = A + B + C = 2$$

$$P_2(3) = 9A + 3B + C = 5$$

$$P_2(7) = 49A + 7B + C = 10$$

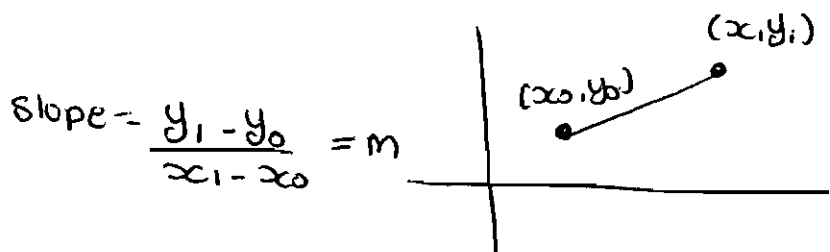
← النقاط الاقتران المرغوب

• given $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$.

we need to Find the polynomial $P_n(x)$ which satisfies

$$P_n(x_i) = y_i, \quad i = 0, \dots, n$$

• given $(x_0, y_0), (x_1, y_1)$.



$$y - y_0 = m(x - x_0)$$

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0} (x - x_0) \rightarrow y = y_0 + \frac{y_1 - y_0}{x_1 - x_0} (x - x_0)$$

$$= \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$

$$P_1(x) = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$

~~$$P_1(x) = \frac{x - (x_1 + x_2)}{x_0 - (x_1 + x_2)} y_0 + \dots$$~~

$$P_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

$$P_3(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} y_1 + \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} y_3$$

$$P_n(x) = \underbrace{\frac{(x - x_1)(x - x_2) \dots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)}}_{L_{n,0}} y_0 + \underbrace{\frac{(x - x_0)(x - x_2) \dots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n)}}_{L_{n,1}} y_1 + \dots + \underbrace{\frac{(x - x_0)(x - x_{k-1})(x - x_{k+1}) \dots (x - x_n)}{(x_{k-1} - x_0)(x_{k-1} - x_{k-2}) \dots (x_{k-1} - x_{k+1}) \dots (x_{k-1} - x_n)}}_{L_{n,k}} y_k + \dots$$

Lagrange coefficient polynomial

$$P_1(x) = \sum_{k=0}^1 L_{1,k}(x) y_k$$

$$P_n(x) = \sum_{k=0}^n L_{n,k}(x) y_k \quad \leftarrow \text{Lagrange Polynomial}$$

Proof \rightarrow $P_n(x_i) = y_i$

$$n=1 \rightarrow P_1(x_0) \stackrel{??}{=} y_0$$

$$P_1(x_0) = y_0$$

$$P_1(x_1) = y_1$$

$$n=2 \rightarrow$$

$$P_2(x_0) = y_0$$

$$P_2(x_1) = y_1$$

$$P_2(x_2) = y_2$$

828
85

$$n=k \rightarrow P_n(x_k) = y_k$$

Example 1

given $f(x) = \cos x$ on $[0, 1.2]$

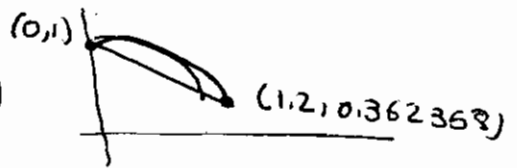
Find $P_1(x)$, $P_2(x)$, $P_3(x)$ and compare the answers

$P_1(0.35)$, $P_2(0.35)$, $P_3(0.35)$ to the exact.

to find $A(x)$

$$(x_0, y_0) = (0, \cos 0) = (0, 1)$$

$$(x_1, y_1) = (1.2, \cos 1.2) = (1.2, 0.362358)$$



$$P_1(x) = \frac{x-x_1}{x_0-x_1} y_0 + \frac{x-x_0}{x_1-x_0} y_1$$

$$= \frac{x-1.2}{0-1.2} (1) + \frac{x-0}{1.2-0} (0.362358)$$

$$P_1(x) = -0.833333(x-1.2) + 0.301965x$$

$$P_1(0.35) = -0.833333(0.35-1.2) + 0.301965(0.35)$$

$$= 0.8140208$$

$$\text{exact} = \cos(0.35) = 0.9393727$$

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$(x_0, y_0), (x_1, y_1), (x_2, y_2)$$

$$(0, \cos 0), (\cancel{0.6}, \cos 0.6), (1.2, \cos(1.2))$$

$$(0.6, \cos 0.6)$$

$$(0, 1), (0.6, 0.825336), (1.2, 0.362258)$$

$$P_2(x) = \frac{(x-0.6)(x-1.2)}{(0-0.6)(0-1.2)} (1) + \frac{(x-0)(x-1.2)}{(0.6-0)(0.6-1.2)} (0.825336) + \frac{(x-0)(x-0.6)}{(1.2-0)(1.2-0.6)} (0.362258)$$

$$= 0.38889(x-0.6)(x-1.2) - 2.292599x(x-1.2) + 0.503275x(x-0.6)$$

$$P_2(0.35) = 0.9233150528$$

$$h_2 = \frac{1.2-0}{3} = 0.4$$

$$(0, 1), (0.4, 0.921061), (0.8, 0.696707), (1.2, 0.362258)$$

$$P_3(x) = \frac{(x-0.4)(x-0.8)(x-1.2)}{(0-0.4)(0-0.8)(0-1.2)} (1) + \frac{(x-0)(x-0.8)(x-1.2)}{(0.4-0)(0.4-0.8)(0.4-1.2)} (0.921061)$$

$$+ \frac{(x-0)(x-0.4)(x-1.2)}{(0.8-0)(0.8-0.4)(0.8-1.2)} (0.696707) + \frac{(x-0)(x-0.4)(x-0.8)}{(1.2-0)(1.2-0.4)(1.2-0.8)} (0.362258)$$

$$= -2.60417(x-0.4)(x-0.8)(x-1.2) \dots$$

$$= 0.939607167$$

4.3 Lagrange interpolating polynomial.

given $(x_0, y_0) (x_1, y_1) \dots, (x_n, y_n)$

$$P_n(x) = \sum_{k=0}^n L_{n,k}(x) y_k = L_{n,0}(x) y_0 + L_{n,1}(x) y_1 + \dots + L_{n,n}(x) y_n$$

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)(x-x_2) \dots (x-x_{k-1})(x-x_{k+1}) \dots (x-x_n)}{(x_k-x_0)(x_k-x_1)(x_k-x_2) \dots (x_k-x_{k-1})(x_k-x_{k+1}) \dots (x_k-x_n)}$$

Theory:-

if $f(x) = P_n(x) + E_n(x)$ (n+1)

$$E_n(x) = \frac{(x-x_0)(x-x_1) \dots (x-x_n)}{(n+1)!} f^{(n+1)}(c)$$

for the
previous
example

$$E_2(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{3!} f^{(3)}(c)$$

$$(0, 1), (0.6, 0.825336), (1.2, 0.362326)$$

$$E_2(x) = \frac{x(x-0.6)(x-1.2)}{6} f^{(3)}(c)$$

$$E_2(0.35) = \frac{(0.35)(0.35-0.6)(0.35-1.2)}{6} f^{(3)}(c)$$

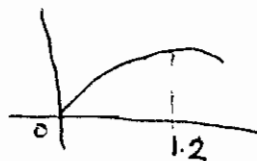
$$|E_2(x)| \leq \frac{|x(x-0.6)(x-1.2)|}{6} \max_{x_0 \leq x \leq x_n} |f^{(3)}(c)|$$

$$f(x) = \cos x \quad [0, 1.2]$$

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$



$$\max |f'''(x)| = \sin(1.2)$$

$$0 < x \leq 1.2 = 0.9320$$

$$|E_2(x)| \leq \frac{|x(x-0.6)(x-1.2)|}{6} \quad (0.9320).$$

$$|E_2(0.35)| \leq \frac{0.35(0.25)(0.85)}{6} \quad (0.9320)$$

$$= 0.01155$$

$$P_2(0.35) = 0.93315.$$

$$\text{Exact} = 0.9393727.$$

$$\text{Error} = 0.0062$$

Find an upperbound for $E_2(x)$ for all x .

x_{\max} ← $\frac{1}{2}(a+b)$ ← $\frac{1}{2}(0.6+1.2)$ ← 0.9
 max upper bound. ← $\frac{1}{2}(a+b)$

$g'(x) = 0 \rightarrow$ uniform bound.

Theorem:- Uniform bound

For uniform partition $h = \frac{b-a}{n} = \frac{x_n - x_0}{n}$

$n = \text{integer}$

$$\rightarrow x_k = x_0 + k h$$

$$x_n = x_0 + n h$$

$$\text{Let } M_n = \max_{a \leq x \leq b} |f^{(n)}(x)|$$

$$\text{then } 1. |E_1(x)| \leq \frac{h^2 M_2}{8} \quad \text{For all } x \in [x_0, x_1]$$

$$2. |E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}} \quad \text{for all } x \in [x_0, x_2]$$

$$3. |E_3(x)| \leq \frac{h^4 M_4}{24} \quad \text{for all } x \in [x_0, x_3]$$

Ex:-

Using the theorem for the previous example

$$|E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}} = \frac{(0.6)^3 (0.9320)}{9\sqrt{3}} = 0.03587$$

↓
Upper bounded
for all x

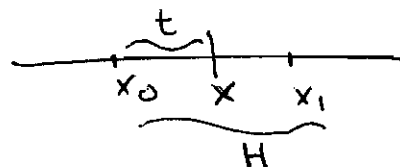
Proof

$$|E_1(x)| \leq \frac{h^2 M_2}{8} \quad \text{for all } x \in [x_0, x_1]$$

to show $|E_1(x)| \leq \frac{h^2 M_2}{8}$

$$E_1(x) = \frac{(x-x_0)(x-x_1)}{2!} M_2$$

$$h(x) = (x-x_0)(x-x_1)$$



Let $t = x - x_0$

$$h(x) = t(x-x_1) = t(t-h)$$

$$x - x_1 \neq \\ = (x_0 + t) - (x_0 + h)$$

$$h(t) = t(t-h)$$

$$h(t) = t^2 - th$$

$$h'(t) = 2t - h = 0$$

$$t = \frac{h}{2} \rightarrow \text{critical point}$$

$$h\left(\frac{h}{2}\right) = \frac{h}{2} \left(\frac{h}{2} - h\right)$$

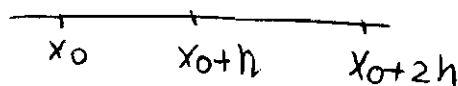
$$= \frac{h}{2} \left(-\frac{h}{2}\right) = -\frac{h^2}{4}$$

$$\max |h(x)| = \frac{h^2}{4}$$

$$|E_1(x)| \leq \frac{h^2}{4} \cdot \frac{M_2}{2} = \frac{h^2 M_2}{8}$$

for
proof

$$|E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}}$$



let $x = x_0 + th$
 $0 < t < 2$
sq-

Theorem

$$f(x) = P_n(x) + E_n(x).$$

then

$$E_n(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_n)}{(n+1)!} f^{(n+1)}(c).$$

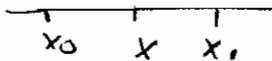


proof: For $n=1$.

To ~~show~~ show that the error.

$$E_1(x) = f(x) - P_1(x),$$

is equal to $\frac{(x-x_0)(x-x_1)}{2!} f^{(2)}(c).$



Let $h(t) = f(t) - P_1(t) - E_1(x) \frac{(t-x_0)(t-x_1)}{(x-x_0)(x-x_1)}$

$h(x)$ are continuous and differentiable.

$$h(x_0) = f(x_0) - P_1(x_0) - E_1(x) \frac{(x_0-x_0)(x_0-x_1)}{(x-x_0)(x-x_1)}$$

$$h(x_1) = f(x_1) - P_1(x_1) - 0 = 0$$

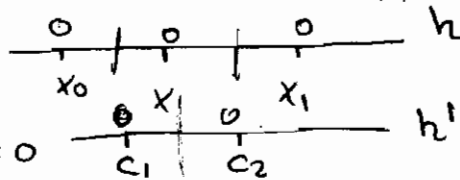
$$h(x) = f(x) - P_1(x) - E_1(x) \frac{(x-x_0)(x-x_1)}{(x-x_0)(x-x_1)}$$

$$= \underbrace{f(x) - P_1(x) - E_1(x)}_{=0} = 0$$

$h(x) = 0$

Using MVT on (x_0, x) , $\exists c \in (x_0, x)$.

such that



$$h'(c_1) = \frac{h(x) - h(x_0)}{x - x_0} = 0$$

Similarly $\exists c_2 \in (x, x_1)$ such that

$$h'(c_2) = \frac{h(x_1) - h(x)}{x_1 - x} = 0$$

similarly $\exists c \in (c_1, c_2)$ such that

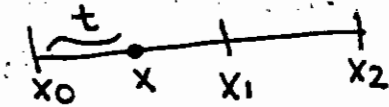
$$h''(c) = \frac{h'(c_2) - h'(c_1)}{c_2 - c_1} = \frac{0 - 0}{c_2 - c_1} = 0$$

• proof $|E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}}$

$$x = x_0 + t$$

$$x_1 = x_0 + h$$

$$x_2 = x_0 + 2h$$



$$|E_2(x)| = \frac{(x-x_0)(x-x_1)(x-x_2) F'''(\xi)}{3!}$$

$$|E_2(x)| \leq \frac{(x-x_0)(x-x_1)(x-x_2) M_3}{6}$$

$$x - x_0 = t$$

$$x - x_1 = t - h$$

$$x - x_2 = t - 2h$$

$$|E_2(x)| \leq \frac{(t)(t-h)(t-2h) M_3}{6}$$

$$\phi(t) = t(t-h)(t-2h)$$

$$= (t^2 - th)(t-2h)$$

$$= t^3 - 2ht^2 - ht^2 + 2h^2t = t^3 - 3ht^2 + 2h^2t$$

$$\phi'(t) = 3t^2 - 6ht + 2h^2$$

$$\phi'(t) = 0$$

$$t = \frac{6 \pm \sqrt{36 - 4 \times 3 \times 2}}{6} h$$

$$t = 0.42264973 h$$

$$t = 1.577350269 h$$

$$\phi(t) = 0.384900179 h^3$$

$$|E_2(x)| \leq \frac{0.384900179 h^3 M_3}{6}$$

$$|E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}}$$

• proof $|E_3(x)| \leq \frac{h^4 M_4}{24}$

$$E_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)^4}{4!} f^{(4)}(\xi)$$

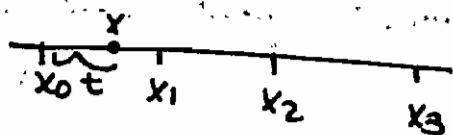
$$|E_3(x)| \leq \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3) M_4}{24}$$

$$x = x_0 + t$$

$$x_1 = x_0 + h$$

$$x_2 = x_0 + 2h$$

$$x_3 = x_0 + 3h$$



$$x - x_0 = t$$

$$x - x_1 = t - h$$

$$x - x_2 = t - 2h$$

$$x - x_3 = t - 3h$$

$$|E_3(x)| \leq \frac{(t)(t-h)(t-2h)(t-3h) M_4}{24}$$

$$\begin{aligned} \text{Let } \phi(t) &= t(t-h)(t-2h)(t-3h) \\ &= (t^2 - th)(t-2h)(t-3h) \\ &= (t^3 - 2t^2h - t^2h + 2th^2)(t-3h) \\ &= (t^3 - 3t^2h + 2th^2)(t-3h) \\ &= t^4 - 6ht^3 + 11h^2t^2 - 6h^3t \end{aligned}$$

$$\phi'(t) = 4t^3 - 18ht^2 + 22h^2t - 6h^3$$

$$\phi'(t) = 0$$

$$t = 2.618033989h$$

$$= 0.381966011h$$

$$= 0.05h$$

$$\phi(t) = 1 - 1 = 0$$

For $\phi(t)$ the max = h^4 @ $t = 2.618033989h$

$$|E_3(x)| \leq \frac{h^4 M_4}{24}$$

$$h'(t) = f'(t) - P_1'(t) - E_1(x) \left(\frac{(t-x_0) + (t-x_1)}{(x_0-x_1)(x-x_1)} \right)$$

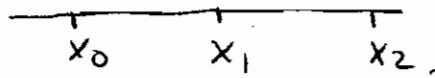
$$h''(t) = f''(t) - 0 - \frac{E_1(x)(2)}{(x_0-x_1)(x-x_1)}$$

\downarrow
 $P_1''(t)$ because
 the function
 is linear

$$\rightarrow h''(c) = f''(c) - \frac{2E_1(x)}{(x-x_0)(x-x_1)} = 0$$

$$E_1(x) = \frac{(x-x_0)(x-x_1)}{2} f''(c)$$

for $n=2$



Exercise

$$h(t) = f(t) - P_2(t) - E_2(x) \frac{(t-x_0)(t-x_1)(t-x_2)}{(x-x_0)(x-x_1)(x-x_2)}$$

$$h(x_0) = f(x_0) - P_2(x_0) - 0$$

$$h(x_1) = f(x_1) - P_2(x_1) - 0$$

$$h(x_2) = f(x_2) - P_2(x_2) - 0$$

$$h(x) = f(x) - P_2(x) - E_2(x)(1)$$

$$= 0$$

4.4 Newton interpolation polynomial

given $x_0, x_1, x_2, \dots, x_n$.

$(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$.

$$P_n(x) = f(x)$$

$$P_1(x) = a_0 + a_1(x - x_0)$$

$$P_2(x) = P_1(x) + a_2(x - x_0)(x - x_1)$$

$$P_3(x) = P_2(x) + a_3(x - x_0)(x - x_1)(x - x_2)$$

$$P_n(x) = P_{n-1}(x) + a_n(x - x_0)(x - x_1) \dots (x - x_{n-1})$$

$$P_1(x_0) = a_0 + a_1(x_0 - x_0)$$

$$P_1(x_0) = a_0 = f(x_0) = y_0 \rightarrow \boxed{a_0 = y_0}$$

$$P_1(x_1) = \underset{a_0}{f(x_0)} + a_1(x_1 - x_0) = f(x_1)$$

$$\boxed{a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}} = F[x_0, x_1] \quad \text{First divided difference.}$$

$$f(x_2) = P_2(x_2) = f(x_0) + F[x_0, x_1](x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$a_2 = \frac{f(x_2) - f(x_0) - F[x_0, x_1](x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{(x_2 - x_0)} = F[x_0, x_1, x_2]$$

Then

$$a_{1k} = F[x_0, x_1, x_2, \dots, x_{1k}] \quad \text{1st } k^{\text{th}} \text{ divided difference.}$$

Definition

$$f[x_k] = f(x_k) \quad \text{zero}^{\text{th}} \text{ divided difference.}$$

$$f[x_{k-1}, x_k] = \frac{f[x_k] - f[x_{k-1}]}{x_k - x_{k-1}} = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \quad \text{1st divided difference.}$$

2nd divided difference.

$$f[x_{k-2}, x_{k-1}, x_k] = \frac{f[x_{k-1}, x_k] - f[x_{k-2}, x_k]}{x_k - x_{k-2}}$$

3rd divided difference

$$f[x_{k-3}, x_{k-2}, x_{k-1}, x_k] = \frac{f[x_{k-2}, x_{k-1}, x_k] - f[x_{k-3}, x_{k-2}, x_k]}{x_k - x_{k-3}}$$

example:

$$(1, 3), (2, 5), (4, 7), (8, 11), (9, 15)$$

$$\begin{aligned} f[2, 4, 8] &= \frac{f[4, 8] - f[2, 4]}{8 - 2} = \frac{\frac{f(8) - f(4)}{8 - 4} - \frac{f(4) - f(2)}{4 - 2}}{6} \\ &= \frac{\frac{11 - 7}{4} - \frac{7 - 5}{2}}{6} = 0 \end{aligned}$$

$$\begin{aligned} f[1, 2, 4, 8] &= \frac{f[2, 4, 8] - f[1, 2, 4]}{8 - 1} \\ &= \frac{\frac{f(4, 8) - f(2, 4)}{8 - 2} - \left[\frac{f(2, 4) - f(1, 2)}{4 - 1} \right]}{8 - 1} \end{aligned}$$

another way

x_{k2}	$f(x_{k2})$	1 st divided	2 nd divided	3 rd divided
	$f[x_{k2}]$	$f[x_{k2-1}, x_{k2}]$	$f[x_{k2-2}, x_{k2-1}, x_{k2}]$	$f[x_{k2-3}, x_{k2-2}, x_{k2-1}, x_{k2}]$
x_0	$f(x_0)$ a_0	/	/	/
x_1	$f(x_1)$	$f[x_0, x_1]$ a_1	/	/
x_2	$f(x_2)$	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$ a_2	/
x_3	$f(x_3)$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$ a_3
x_4	$f(x_4)$	$f[x_3, x_4]$	$f[x_2, x_3, x_4]$	$f[x_1, x_2, x_3, x_4]$
x_5	$f(x_5)$	$f[x_4, x_5]$	$f[x_3, x_4, x_5]$	$f[x_2, x_3, x_4, x_5]$
x_6	$f(x_6)$	$f[x_5, x_6]$	$f[x_4, x_5, x_6]$	$f[x_3, x_4, x_5, x_6]$
\vdots				

example

Find Newton interp $P_1, P_2, P_3, P_4, \dots$ for the following table

x_{k2}	$f(x_{k2})$	1 st	2 nd	3 rd	4 th
1	-3 a_0	/	/	/	/
2	0	3 a_1	/	/	/
3	15	15	6 a_2	/	/
4	48	33	9	1 a_3	/
5	105	57	12	1	0
6	192	87	15	1	0

$$P_1(x) = a_0 + a_1(x-x_0) = -3 + 3(x-1)$$

$$P_2(x) = P_1(x) + a_2(x-x_0)(x-x_1) = -3 + 3(x-1) + 6(x-1)(x-2)$$

$$P_3(x) = P_2(x) + 1(x-1)(x-2)(x-3)$$

$$P_4 = P_3$$

$$P_5 = P_4 = P_3$$

note:-

Error for newton interpolation polynomial equal to the error for lagrange because they uses the same Polynomial.

Example:-

Estimate $f(5.5)$ using Newton int polynomial P_1, P_2, P_3, P_4 for the following table.

x_i	$f(x_i)$	1 st	2 nd	3 rd	4 th
1	3	/	/	/	/
3	4.5	0.75	/	/	/
4.25	6		0.138462	/	/
5.75	7.25			0.05722	/
6	8				0.10287

$$P_1(x) = a_0 + a_1(x-x_0)$$

$$= 3 + 0.75(x-1)$$

$$f(5.5) \approx P_1(5.5)$$

$$= 3 + 0.75(5.5-1)$$

$$= 6.375$$

$$P_2(x) = P_1(x) + a_2(x-1)(x-3)$$

$$= 0.13846(x-1)(x-3) + P_1(x)$$

$$f_2(5.5) \approx P_2(5.5)$$

$$P_3(x) = P_2(x) + 0.05722(x-1)(x-3)(x-4.25)$$

$$\approx 6.375 + 0.13846(5.5-1)(5.5-3)$$

$$f_3(5.5) \approx P_3(5.5)$$

$$\approx 7.93267 + 0.05722(5.5-1)(5.5-3)(5.5-4.25)$$

$$\approx 8.7373$$

$$P_4(x) = P_3 + 0.10287(x-1)(x-3)(x-4.5)(x-5.75)$$

$$f(5.5) \approx P_4(5.5)$$

$$\approx 8.7373 + 0.10287(4.5)(2.5)(1.25)(-0.25)$$

$$\approx 8.375.$$

Chapter Five

5.1+5.2

Best Fit

given $(x_0, y_0), (x_1, y_1) \dots (x_n, y_n)$ to Find the best Fitting Curve

i.e

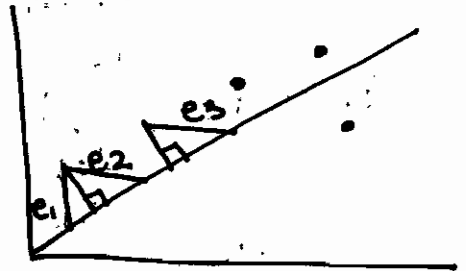
the curve with smallest distance to the given point

$$\text{if } e_k = f(x_k) - y_k$$

$$\text{max error } E_\infty(f) = \|f\|_\infty = \max_{0 \leq k \leq n} |e_k|$$

$$\text{Average error} = E_1(f) = \|f\|_1 = (\sum |e_k|) / n$$

$$\text{Root Mean Square error} = E_2(f) = \|f\|_2 = (\sum |e_k|^2 / n)^{1/2}$$



Example 5.1

Compare the max error, Average error and RMS error for the linear approximation $f(x) = -1.6x + 8.6$ to the data

$(-1, 10), (0, 9), (1, 7), (2, 5), (3, 4), (4, 3), (5, 0), (6, -1)$

x_k	y_k	$f(x_k)$	$ e_k $	e_k^2
-1	10	10.2	0.2	0.04
0	9	8.6	0.4	0.16
1	7	7	0	0
2	5	5.4	0.4	0.16
3	4	3.8	0.2	0.04
4	3	2.2	0.8	0.64
5	0	0.6	0.6	0.36
6	-1	-1	0	0

$$\begin{aligned} \sum e_k &= 2.6 \\ E_\infty(f) &= 0.8 \\ E_1(f) &= \frac{\sum |e_k|}{n} \\ &= \frac{2.6}{8} \\ &= 0.325 \\ \sum e_k^2 &= 1.4 \\ E_2(f) &= \left(\frac{\sum e_k^2}{n} \right)^{1/2} \\ &= 0.42 \end{aligned}$$

- to Find the best Fitting curve we need to minimize the least square error (RMS)

$$E_2(f) = \left(\frac{\sum_{k=1}^n |f(x_k) - y_k|^2}{n} \right)^{1/2}$$

$$n E_2^2(f) = \sum_{k=1}^n (f(x_k) - y_k)^2$$

$$E(f) = \sum_{k=1}^n (f(x_k) - y_k)^2$$

1. To Find The best Fitting line $F(x) = Ax + B$

$$E(A, B) = \sum_{k=1}^n |(Ax_k + B) - y_k|^2$$

$$\frac{dE}{dA} = \sum_{k=1}^n 2 |(Ax_k + B) - y_k| \cdot x_k = 0 \quad \dots (1)$$

$$\frac{dE}{dB} = \sum_{k=1}^n 2 |(Ax_k + B) - y_k| \cdot 1 = 0 \quad \dots (2)$$

$$(1) \text{ } \omega \quad A \sum_{k=1}^n x_k^2 + B \sum_{k=1}^n x_k = \sum_{k=1}^n y_k x_k \quad \dots$$

$$(2) \text{ } \omega \quad A \sum_{k=1}^n x_k + nB = \sum_{k=1}^n y_k \quad \dots$$

Normal equations

Example:-

Find the best Fitting line $F(x) = Ax + B$ For the data

$(-1, 10) (0, 9) (1, 7) (2, 5) (3, 4) (4, 3) (5, 0) (6, -1)$.

x_k	y_k	x_k^2	$x_k y_k$	
-1	10	1	-10	
0	9	0	0	
1	7	1	7	
2	5	4	10	
3	4	9	12	
4	3	16	12	
5	0	25	0	
6	-1	36	-6	
Σ	20	37	92	25

$$92A + 20B = 25$$

$$20A + 8B = 37$$

$$A = \frac{\begin{vmatrix} 25 & 20 \\ 37 & 8 \end{vmatrix}}{\begin{vmatrix} 92 & 20 \\ 20 & 8 \end{vmatrix}} \approx -1.61$$

$$B = \frac{\begin{vmatrix} 92 & 25 \\ 20 & 37 \end{vmatrix}}{\begin{vmatrix} 92 & 20 \\ 20 & 8 \end{vmatrix}} \approx 8.64$$

Example

For the following data find the best curve of the form

$$y = Ax^2$$

$$E(A) = \sum_{k=1}^n (Ax_k^2 - y_k)^2$$

$$\frac{dE}{dA} = 2 \sum_{k=1}^n (Ax_k^2 - y_k) \cdot x_k^2 = 0$$

$$A = \frac{\sum_{k=1}^n y_k x_k^2}{\sum_{k=1}^n x_k^4}$$

التكملة :
مثل :
الباقي :

$$A = \frac{85}{2276} = 0.037346$$

Example

Find the best fitting parabola $f(x) = Ax^2 + Bx + C$

$$E(A, B, C) = \sum_{k=1}^n [(Ax_k^2 + Bx_k + C) - y_k]^2$$

$$\frac{dE}{dA} = 0 = 2 \sum_{k=1}^n [(Ax_k^2 + Bx_k + C) - y_k] \cdot 2x_k$$

$$\frac{dE}{dB} = 0 = 2 \sum_{k=1}^n [(Ax_k^2 + Bx_k + C) - y_k] \cdot x_k$$

$$\frac{dE}{dC} = 0 = 2 \sum_{k=1}^n [(Ax_k^2 + Bx_k + C) - y_k] \cdot 1$$

5.2

linearization

$$f(x) \rightarrow Ax+B$$

Example:

Find the best Fitting curve of the form $f(x) = Ce^{Dx}$ for the following table. (0, 1.5), (1, 2.5), (2, 3.5), (3, 5), (4, 7.5)

$$y = ce^{Dx}$$

$$\ln y = \ln c + Dx$$

$$\ln y = Dx + \ln c$$

$$Y = AX + B$$

$$Y = \ln y$$

$$X = x$$

$$D = A$$

$$c = e^B$$

x_k	y_k	X_k	$Y_k = \ln y_k$	x_k^2	$X_k Y_k$
0	1.5	0	0.405465	0	0
1	2.5	1	0.916291	1	0.916291
2	3.5	2	1.252763	4	2.505526
3	5	3	1.609438	9	4.828313
4	7.5	4	2.014903	16	8.059612
Σ		10	6.198860	30	16.309742

table 5.4 From the text book.

$$30A + 10B = 16.309742$$

$$10A + 5B = 6.198860$$

$$A = \frac{\begin{vmatrix} 16.309742 & 10 \\ 6.198860 & 5 \end{vmatrix}}{\begin{vmatrix} 30 & 10 \\ 10 & 5 \end{vmatrix}} = 0.3912023$$

$$B = 0.457367$$

$$D = A \approx 0.39$$

$$c = e^B = e^{0.457367} \approx 1.58$$

$$f(x) = 1.58 e^{0.39x} = Ce^{Dx}$$

Example 3:-

1. $y = \frac{D}{x+c}$

$$\frac{1}{y} = \frac{x}{D} + \frac{c}{D}$$

$$Y = AX + B$$

$$Y = \frac{1}{y}, X = x, A = \frac{1}{D}, B = \frac{c}{D}$$

\downarrow \downarrow

$$D = \frac{1}{A} \qquad c = BD$$

$$y = \frac{D}{x+c}$$

$$y = \frac{1}{x} D +$$

$$A = \frac{D}{x} \Rightarrow \frac{1}{x} = \frac{A}{D}$$
$$Ax = \frac{D}{x}$$
$$A = \frac{D}{x^2}$$

2. $y = \frac{x}{A+Bx}$

$$\frac{1}{y} = \frac{A}{x} + B$$

$$Y = AX + B$$

$$Y = \frac{1}{y}$$

$$X = \frac{1}{x}$$

$$A = A$$

$$B = B$$

3. $y = ce^{-Dx}$

$$\frac{y}{x} = ce^{-Dx}$$

$$\ln\left(\frac{y}{x}\right) = \ln c - Dx$$

$$\ln\left(\frac{y}{x}\right) = -Dx + \ln c$$

$$Y = AX + B$$

$$Y = \ln\left(\frac{y}{x}\right)$$

$$X = x$$

$$A = -D \rightarrow D = -A$$

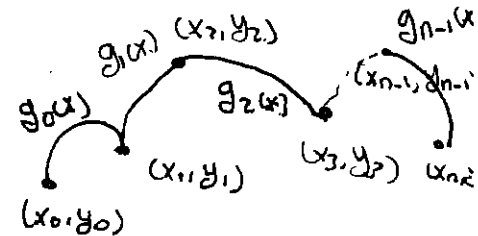
$$B = \ln c \rightarrow c = e^B$$

Section 5.3

Cubic spline

given $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$

The cubic spline is a function $g(x)$ such that it is a cubic polynomial between every two nodes and its of this form $g_i(x) = a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i$ on $[x_i, x_{i+1}]$ for $i=0, 1, \dots, n-1$ and that satisfies



$$1. g_i(x_i) = y_i \quad i=0, 1, \dots, n-1, \quad g_{n-1}(x_n) = y_n$$

$(n+1)$ conditions.

$$2. g_i(x_{i+1}) = g_{i+1}(x_{i+1}) \quad i=0, \dots, n-2$$

$(n-1)$ conditions

$$g_0(x_1) = g_1(x_1)$$

$$g_1(x_2) = g_2(x_2)$$

$$g_{n-2}(x_{n-1}) = g_{n-1}(x_{n-1})$$

$$3. g_i'(x_{i+1}) = g_{i+1}'(x_{i+1}) \quad i=0, \dots, n-2 \rightarrow (n-1) \text{ condition.}$$

$$4. g_i''(x_{i+1}) = g_{i+1}''(x_{i+1}) \quad i=0, \dots, n-2 \rightarrow (n-1) \text{ condition.}$$

so we have $(n+1) + (3(n-1)) = 4n-2$ conditions.

→ eq Since $g_i(x_i) = y_i \Rightarrow$
 $d_i = y_i$

→ equation (2) gives

$$y_{i+1} = g_{i+1}(x_{i+1}) = a_i(x_{i+1}-x_i)^3 + b_i(x_{i+1}-x_i)^2 + c_i(x_{i+1}-x_i) + d_i$$

$$* = a_i h_i^3 + b_i h_i^2 + c_i h_i + d_i$$

where $h_i = (x_{i+1} - x_i)$

$$g_i'(x) = 3a_i(x-x_i)^2 + 2b_i(x-x_i) + c_i$$

if I have n function
 $\rightarrow 4n$ unknowns.

- continuous
- رتبه المتصلة
- نفس النقط

$$g''_c(x) = 6a_c(x-x_c) + 2b_c$$

if $S_c = g''_c(x_c)$

substitute $\rightarrow b_c = \frac{S_c}{2}$

Using the same equation.

$$g''_{c+1}(x_{i+1}) = 6a_c(x_{i+1}-x_c) + 2b_c$$

$$S_{i+1} = 6a_c h_i + S_c$$

$$a_c = \frac{S_{i+1} - S_c}{6h_i}$$

substitute in *

$$C_i = \frac{y_{i+1} - y_i}{h_i} - \frac{2h_i S_i + h_i S_{i+1}}{6}$$

• Considering the equation

$$g''_c(x_i) = g''_{c+1}(x) \text{ we get}$$

$$h_{i-1} S_{c-1} + 2(h_{i-1} + h_i) S_c + h_i S_{c+1} = 6 [f(x_c, x_{c+1}) - f(x_{c-1}, x_c)]$$

for $i=1, \dots, n-1$

$$\begin{bmatrix}
 S_0 & & & & & & & & & & \\
 h_0 & 2(h_0+h_1) & h_1 & 0 & \dots & 0 & 0 & 0 & & & \\
 0 & h_1 & 2(h_1+h_2) & h_2 & \dots & 0 & 0 & 0 & & & \\
 0 & 0 & h_2 & 2(h_2+h_3) & \dots & 0 & 0 & 0 & & & \\
 & & & & \ddots & & & & & & \\
 & & & & & h_{n-3} & 2(h_{n-3}, h_{n-2}) & h_{n-2} & 0 & & \\
 & & & & & h_{n-2} & 2(h_{n-2}, h_{n-1}) & h_{n-1} & & &
 \end{bmatrix}
 \begin{bmatrix}
 S_0 \\
 S_1 \\
 S_2 \\
 \vdots \\
 S_{n-1} \\
 S_n
 \end{bmatrix}
 = 6
 \begin{bmatrix}
 f[x_1, x_2] - f[x_0, x_1] \\
 f[x_2, x_3] - f[x_1, x_2] \\
 f[x_3, x_4] - f[x_2, x_3] \\
 \vdots \\
 f[x_n, x_{n+1}] - f[x_{n-1}, x_n]
 \end{bmatrix}$$

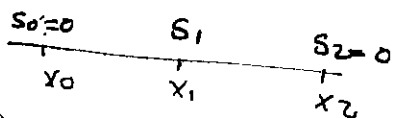
$(n-1)$ equations & $(n+1)$ unknown, we need two more condition.

1. Natural Spline $S_0 = S_n = 0$

we get $(n-1)$ equations with $(n-1)$ unknowns

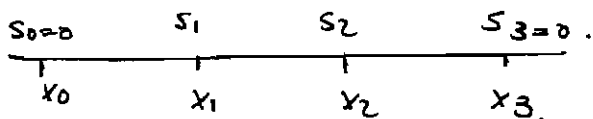
When $n=1$ x ما بينغ لذن كطرفى
 لا يوجد matrix لذن يوجد $\frac{0}{x_0} \quad \frac{0}{x_1}$
 حلاله .

When $n=2$



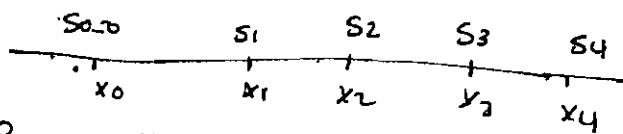
$$2(h_0 + h_1) S_1 = 6 [f[x_1, x_2] - f[x_0, x_1]]$$

When $n=3$



$$\begin{bmatrix} 2(h_0 + h_1) & & & \\ & h_1 & & \\ & & 2(h_1 + h_2) & \\ h_1 & & & \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = 6 \begin{bmatrix} f[x_1, x_2] - f[x_0, x_1] \\ f[x_2, x_3] - f[x_1, x_2] \end{bmatrix}$$

When $n=4$



$$\begin{bmatrix} 2(h_0 + h_1) & & & & \\ & h_1 & & & \\ & & 2(h_1 + h_2) & & \\ h_1 & & & h_2 & \\ & & h_2 & & 2(h_2 + h_3) \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = 6 \begin{bmatrix} f[x_1, x_2] - f[x_0, x_1] \\ f[x_2, x_3] - f[x_1, x_2] \\ f[x_3, x_4] - f[x_2, x_3] \end{bmatrix}$$

Example

Find the natural Spline For the given table.

	x_i	y_i
h_0	0	2
	1	4.4366
h_1	1.5	6.7134
h_2	2.25	13.9130

$$h_0=1, h_1=0.5, h_2=0.75$$

$$f[0,1] = 2.4366$$

$$f[1,1.5] = 4.5536$$

$$f[1.5,2.25] = 9.5995$$

$$\begin{bmatrix} 2(1.5) & 0.5 \\ 0.5 & 2(1.25) \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4.5536 - 2.4366 \\ 9.5995 - 4.5536 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 0.5 \\ 0.5 & 2.5 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 12.7020 \\ 30.2754 \end{bmatrix}$$

$$s_0 = 0$$

$$s_3 = 0$$

$$s_1 = 2.292, s_2 = 11.6618$$

$$a_i = \frac{s_{i+1} - s_i}{6h_i}$$

$$a_0 = \frac{s_1 - s_0}{6h_0} = \frac{2.292 - 0}{6(1)} = 0.3820$$

$$a_1 = ??$$

$$a_2 = ??$$

$$b_i = \frac{s_i}{2}$$

$$b_0 = \frac{s_0}{2} = 0$$

$$b_1 = \frac{s_1}{2} = 1.146$$

$$b_2 = \frac{s_2}{2} = 5.8259$$

$$c_i = \dots$$

$$c_0 = 2.0546$$

$$c_1 = 3.2005$$

$$c_2 = 6.6866$$

$$d_i = y_i$$

$$d_0 = 2$$

$$d_1 = 4.4215$$

$$d_2 = 6.7130$$

$$g_0(x) = 0.3820(x-0)^3 + 0(x-0)^2 + 2.054(x-0) + 2.000 \quad \text{on } [0, 1]$$

$$g_1(x) = 3.1199(x-1)^3 + 1.146(x-1)^2 + 5.205(x-1) + 4.4366 \quad \text{on } [1, 1.5]$$

$$g_2(x) = -2.5895(x-1.5)^3 + 5.8259(x-1.5)^2 + 6.6866(x-1.5) + 6.7134 \quad \text{on } [1.5, 2.2]$$

$$f(0.66) = 3.4659 \quad \text{Exact} = 3.34343$$

$$f(1.75) = 8.7087 \quad \text{Exact} = 8.4467$$

Natural Spline

$$s_0 = 0 \quad s_n = 0$$

$n=3$

A horizontal line with four nodes labeled x_0, x_1, x_2, x_3 from left to right. Above the nodes are labels $s_0=0, s_1, s_2, s_3=0$.

$$\begin{bmatrix} 2(h_0+h_1) & h_1 \\ h_1 & 2(h_1+h_2) \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = 6 \begin{bmatrix} F(x_1, x_2) - F(x_0, x_1) \\ F(x_2, x_3) - F(x_1, x_2) \end{bmatrix}$$

$n=4$

A horizontal line with five nodes labeled x_0, x_1, x_2, x_3, x_4 from left to right. Above the nodes are labels $s_0=0, s_1, s_2, s_3, s_4=0$.

$$\begin{bmatrix} 2(h_0+h_1) & h_1 & 0 \\ h_1 & 2(h_1+h_2) & h_2 \\ 0 & h_2 & 2(h_2+h_3) \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = 6 \begin{bmatrix} F(x_1, x_2) - F(x_0, x_1) \\ F(x_2, x_3) - F(x_1, x_2) \\ F(x_3, x_4) - F(x_2, x_3) \end{bmatrix}$$

Clamped Spline

$$F'(x_0) = A$$

$$F'(x_n) = B$$

$$(1) \rightarrow 2h_0s_0 + h_0s_1 = 6 [F(x_0, x_1) - A]$$

$$(2) \rightarrow h_{n-1}s_{n-1} + 2h_{n-1}s_n = 6 [B - F(x_{n-1}, x_n)]$$

$$\underline{n=1}$$

$$\begin{bmatrix} 2h_0 & h_0 \\ h_0 & 2h_0 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \end{bmatrix} = 6 \begin{bmatrix} f(x_0, x_1) - A \\ B - f(x_0, x_1) \end{bmatrix}$$

$$g_0(x) = a_0(x-x_0)^3 + b_0(x-x_0)^2 + c_0(x-x_0) + d_0$$

نعوض في النقاط وكذلك المشتقة عند الأطراف و بالتالي نعرف الجاهيل

$$\underline{n=2}$$

$$\begin{bmatrix} 2h_0 & h_0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 \\ 0 & h_1 & 2h_1 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ s_2 \end{bmatrix} = 6 \begin{bmatrix} F(x_0, x_1) - A \\ F(x_1, x_2) - F(x_0, x_1) \\ B - F(x_1, x_2) \end{bmatrix}$$

$$g(x) = \begin{cases} a_0(x-x_0)^3 + b_0(x-x_0)^2 + c_0(x-x_0) + d_0 & x_0 \leq x \leq x_1 \\ a_1(x-x_1)^3 + b_1(x-x_1)^2 + c_1(x-x_1) + d_1 & x_1 \leq x \leq x_2 \end{cases}$$

$$f'(x_0) = A$$

$$f'(x_1) = B$$

$$f(x_0) = d_0 = y_0$$

$$f(x_1) = d_1 = y_1$$

$$g_0(x_1) = g_1(x_1)$$

$$g_0'(x_1) = g_1'(x_1)$$

$$g_0''(x_1) = g_1''(x_1)$$

$$g_1(x_2) = y_2$$

$$g_0(x_0) = y_0$$

$$g_1(x_1) = y_1$$

$$g_1(x_2) = y_2$$

For $n=3$

$$\begin{bmatrix} 2h_0 & h_0 & 0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 \\ 0 & 0 & h_2 & 2h_2 \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = 6 \begin{bmatrix} F(x_0, x_1) - A \\ F(x_1, x_2) - F(x_0, x_1) \\ F(x_2, x_3) - F(x_1, x_2) \\ B - F(x_2, x_3) \end{bmatrix}$$

Q2) Clamped spline

$(0,0) (1,1) (2,2)$

$S'(0)=1$, $S'(2)=1$

$$g(x) = \begin{cases} g_0(x) = a_0(x-0)^3 + b_0(x-0)^2 + c_0(x-0) + d_0 & \text{on } [0,1] \\ g_1(x) = a_1(x-1)^3 + b_1(x-1)^2 + c_1(x-1) + d_1 & \text{on } [1,2] \end{cases}$$

$$g(x) = \begin{cases} g_0(x) = a_0x^3 + b_0x^2 + c_0x + d_0 & \text{on } [0,2] \\ g_1(x) = a_1(x-1)^3 + b_1(x-2)^2 + c_1(x-1) + d_1 & \text{on } [1,2] \end{cases}$$

$$g_0(0) = d_0 = 0$$

$$g_1(2) = d_1 = 1$$

$$g_0'(x) = 3a_0x^2 + 2b_0x + c_0$$

$$g_0'(0) = c_0 = 1$$

$$g_1'(x) = 3a_1(x-1)^2 + 2b_1(x-1) + c_1$$

$$= 3a_1 + 2b_1 + c_1 = 2$$

$$g_0'(1) = g_1'(1)$$

$$3a_0x^2 + 2b_0x + c_0 = 3a_1(x-1)^2 + 2b_1(x-1) + c_1$$

$$3a_0 + 2b_0 + 1 = 3a_1(0) + 2b_1(0) + c_1$$

$$g_0''(1) = g_1''(1)$$

$$6a_0x + 2b_0 = 6a_1(x-1)^2 + 2b_1$$

$$6a_0 + 2b_0 = 2b_1$$

$$g_1(2) = 2$$

$$a_1 + b_1 + c_1 = 2 - 1$$

$$a_1 + b_1 + c_1 = 1$$

$$g_0'(2) = g_1'(2)$$

$$a_0 + b_0 + c_0 + d_0 = d_1$$

$$a_0 + b_0 + c_0 = 1$$

0	0	//	///
1	1	1	///
1	2	1	0

$$h_0 = 1$$

$$h_1 = 1$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$s_0 = 0$$

$$s_1 = 0$$

$$s_2 = 0$$

Q15) Cubic-poly [a, b]

$f(x)$ its own clamped spline but it cannot be its own Free spline ?

$$f(a) =$$

Cubic $s_0, s_1 \neq \text{Zero}$

its not natural

المشتقة الثانية \neq صفر
المشتقة الأولى \neq صفر

$$a_3 \neq 0$$

$$g(x) = a_3 x^3 + b x^2 + c_2 x + d_2$$

$$g'(x) = 3a_3 x^2 + 2bx + c_2$$

$$g''(x) = 6a_3 x + 2b_2$$

منفرد عند الطرفين
ولكن $a_3 = 0$

4 - Unknowns

4 equ (condition)

$$y_0 = a_0(x-x_0)^3 + b_0(x-x_0)^2 + c_0(x-x_0) + d_0$$

أربع نقاط مشتركة وبالتالي poly نفرد

Step 6

Th:- Central difference Formula of order $O(h^2)$ (f_1).

assum that $f \in C^2[a, b]$, and $x-h, x, x+h \in [a, b]$ then

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

furthermore there exists a number $c \in [a, b]$ such that

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2 f^{(3)}(c)}{6}$$

where the error then $-\frac{h^2 f^{(3)}(c)}{6}$ is called the truncation error and is denoted by

$$E_{\text{trunc}}(f, h) \approx \text{i.e. } E_{\text{trunc}}(f, h) = \frac{h^2 f^{(3)}(c)}{6}$$

Let

t	d
0.1	13.21
0.2	20.55
0.3	24.12
0.4	29.79

$$V(0.2) = \frac{d(0.3) - d(0.1)}{2(0.1)} = \frac{24.12 - 13.21}{0.2} = 54.55$$

$$\begin{aligned} \text{error} &= C(0.1)^2 \\ &= C(0.01) \rightarrow \text{error in the 4}^{\text{th}} \text{ digit.} \end{aligned}$$

$$V(0.3) = \frac{d(0.4) - d(0.2)}{0.02}$$

$$V(0.4) = \text{لا أعرف}$$

$$V(0.1) = \text{لا أعرف}$$

• $f(x) = \cos x$

$f'(0.8) = ??$

$h = 0.01$

$$f'(0.8) \approx \frac{f(0.8+0.01) - f(0.8-0.01)}{2(0.01)} \approx \frac{\cos(0.81) - \cos(0.79)}{0.02}$$

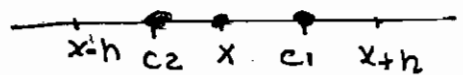
$$\approx \frac{0.689498933 - 0.7303895326}{0.02} = -0.717344160$$

Exact = $f'(0.8) = \sin(0.8) = -0.717356091$ ϵ منازك تقريباً
 مسيحة

by Theorem $C(h^2)^2 = \text{Error} = C(0.01)^2 = C(0.0001)$
 ابع منازك مسيحة وفقاً للتقريب.

Derivation

Using Taylor expansion at x ,



$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(c_1), \quad c_1 \in (x, x+h)$$

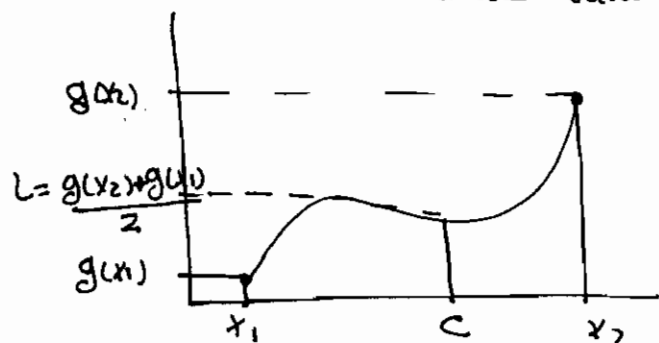
$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(c_2), \quad c_2 \in (x-h, x)$$

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{h^3}{6} (f'''(c_1) - f'''(c_2))$$

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{h^3}{6} (2f'''(c)) \quad c \in (c_1, c_2)$$

$$\underbrace{\frac{f(x+h) - f(x-h)}{2h}}_{F_1} \approx \underbrace{\frac{h^2 f'''(c)}{6}}_{\text{Error}} = f'(x)$$

I.V.P (Intermediate value property)



$\Rightarrow \exists c \in (x_1, x_2)$ such that $g(c) = \frac{g(x_1) + g(x_2)}{2}$
 $\rightarrow g(x_1) + g(x_2) = 2g(c)$

Section 6.1

Central difference Formula of $O(h^4)$

assume $f \in C^5[a, b]$ and $x-2h, x-h, x+h, x+2h \in [a, b]$ then

$$f'(x) = \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h}$$

with error

$$E_{\text{trunc}}(f, h) = \frac{h^4 f^{(5)}(\xi)}{30} \approx Ch^4$$

Example 1

Let

t	d
0.1	13.25
0.2	18.53
0.3	21.25
0.4	24.30
0.5	27.12

فقط بيّن
استخراج
عند 0.3

$$V(0.3) = \frac{-d(0.5) + 8d(0.4) - 8d(0.2) + d(0.1)}{12(0.1)}$$

$$= \frac{-27.12 + 8(24.30) - 8(18.53) + 13.25}{12}$$

Example 2

$$f(x) = \cos x$$

$$f'(0.8) \text{ using } h=0.01$$

$$f'(0.8) = \frac{-\cos(0.82) + 8\cos(0.81) - 8\cos(0.79) + \cos(0.78)}{0.12}$$

$$f'(0.8) = -0.717356108$$

Compare to exact $-\sin(0.8) = -0.717356091$

$$\text{error } C(0.01)^4 = C(10^{-8})$$

error = 1×10^{-7} بيتا $\sqrt{\text{منزلة}}$
near to 10^{-8}

Derivation

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \frac{h^4}{4!} f^{(4)}(x) + \frac{h^5}{5!} f^{(5)}(x)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) + \frac{h^4}{4!} f^{(4)}(x) - \frac{h^5}{5!} f^{(5)}(x)$$

نفر f''

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{2h^3}{3!} f'''(x) + \frac{2h^5}{5!} f^{(5)}(x)$$

$$(1) - 8(f(x+h) - f(x-h)) = 16hf'(x) + \frac{16h^3}{3!} f'''(x) + \frac{16h^5}{5!} f^{(5)}(x)$$

$$(2) - - f(x+2h) - f(x-2h) = 4hf'(x) + \frac{16h^3}{3!} f'''(x) + \frac{64h^5}{5!} f^{(5)}(x)$$

نفرح (1) و (2)

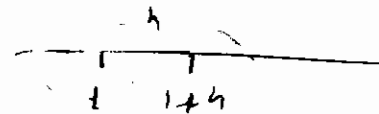
$$-f(x+2h) + 8(f(x+h) - 8f(x-h) + f(x-2h)) = 12f'(x) - \frac{48h^5}{120} f^{(5)}(x)$$

$$\frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h} + \frac{1}{30} h^4 f^{(5)}(x) = f'(x)$$

F_2 Etrac (f, h)

$$-f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) \cong \frac{f(x+h) - f(x)}{h}$$



when h is smaller we get best estimation for $f'(x)$?

Example

$$f(x) = e^x$$

$$f'(x) = e^x$$

$$f'(1) = e$$

$$f'(1) \cong \frac{f(1+h) - f(1)}{h} = \frac{e^{1+h} - e^1}{h} \xrightarrow{??} e$$

هل هذا صحيح يقرب
عند $h \rightarrow 0$

h	$D_n = e^{4h} - e/h$
0.1	2.858841960
0.01	2.731918700
0.001	2.719642000
0.0001	2.718420000
10^{-5}	2.718300000 → the best h
10^{-6}	2.719000000
10^{-7}	;
10^{-10}	00000000

• Notation

$$f(x+h) = y_1 + e_1$$

$$f(x-h) = y_{-1} + e_{-1}$$

⋮

$$f(x+kh) = y_k + e_k$$

$$f(x+h) = \cos(0.81) = \underbrace{0.689498433}_{y_1} \text{ (is not exact (have error))}$$

$$= y_1 + e_1$$

$$|e_1| < 0.5 * 10^{-10} < 0.5 * 10^{-9}$$

$$F_1 = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{6} f^{(3)}(c)$$

$$= \frac{(y_1 + e_1) - (y_{-1} + e_{-1})}{2h} - \frac{h^2}{6} f^{(3)}(c)$$

$$= \underbrace{\frac{y_1 - y_{-1}}{2h}}_{\text{Round off error}} + \underbrace{\frac{e_1 - e_{-1}}{2h}}_{\text{truncation error}} - \frac{h^2}{6} f^{(3)}(c)$$

Round off error

truncation error

$E_{\text{round}}(f, h)$

$E_{\text{trunc}}(f, h)$

$$\begin{aligned} \text{Total error} = E_{\text{tot}}(f, h) &= E_{\text{round}}(f, h) + E_{\text{trunc}}(f, h) \\ &= \frac{e_1 - e_{-1}}{2h} - \frac{h^2 f^{(3)}(c)}{6} \end{aligned}$$

$$|E_{\text{tot}}(f, h)| = \left| \frac{e_1 - e_{-1}}{2h} \right| + \left| \frac{h^2 f^{(3)}(c)}{6} \right| \quad \text{if } |e_k| < \epsilon$$

$$\leq \underbrace{\frac{2\epsilon}{2h} + \frac{h^2 M_3}{6}}_{g(h)} \quad M_3 = \max |f^{(3)}(x)|$$

$$g(h) = \frac{\epsilon}{h} + \frac{h^2}{6} M$$

$$g'(h) = -\frac{\epsilon}{h^2} + \frac{h}{3} M = 0$$

$$\frac{h}{3} M = \frac{\epsilon}{h^2}$$

$$h^3 = \frac{3\epsilon}{M}$$

$$h = \left(\frac{3\epsilon}{M} \right)^{1/3} \text{ best } h$$

• $f(x) = \cos x$, $\epsilon = 0.5 \times 10^{-9}$

$$h = \left(\frac{3 \times 0.5 \times 10^{-9}}{M} \right)^{1/3} = 0.001144714$$

① max for $|f^{(3)}(x)|$

$$h = 0.001 \text{ best } h$$

• Find best h for F_2 .

F_2

$$F_2'(x) = \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h} + \frac{h^4 f^{(5)}(c)}{30}$$

$$E_{\text{tot}} = \frac{-e_2 + 8e_1 - 8e_{-1} + e_{-2}}{12h} + \frac{h^4 f^{(5)}(c)}{30}$$

$$|E(f, h)| \leq \frac{|e_2| + 8|e_1| + 8|e_{-1}| + |e_{-2}|}{12h} + \frac{h^4 M}{30}$$

$$M = \max |f^{(5)}(x)| \quad a \leq x \leq b$$

$$\leq \frac{18\epsilon}{12h} + \frac{h^4 M}{30} = \frac{3\epsilon}{2h} + \frac{h^4 M}{30} = g(h)$$

$$|e_k| < \epsilon$$

$$g'(h) = -\frac{3E}{2h^2} + \frac{4h^3 M}{30} = 0$$

$$\frac{2}{15} h^3 M = \frac{3E}{2h^2}$$

$$h^5 = \frac{45E}{4M}$$

$$\text{Optimal } h = \left(\frac{45E}{4M} \right)^{1/5}$$

$$- f(x) = \cos x$$

$$M = 1$$

$$E = 0.5 * 10^{-9}$$

$$h = \left(\frac{45 * 0.5 * 10^{-9}}{4 * 1} \right)^{1/5} = 0.022 \dots$$

$$\text{Optimal } h = 0.01$$

Section 6.2

High order derivations

• $O(h^2)$

$$1. f''(x) \approx \frac{f_1 - 2f_0 + f_{-1}}{h^2}$$

$$f_k = f(x + kh)$$

$$2. f'''(x) \approx \frac{f_2 - 2f_1 + 2f_{-1} - f_{-2}}{2h^3}$$

⋮

• $O(h^4)$

$$1. f''(x) \approx \frac{-f_2 + 16f_1 - 30f_0 + 16f_{-1} - f_{-2}}{12h^2}$$

$$2. f'''(x) \approx \dots$$

$$3. f^{(4)}(x) \approx \dots$$

حين المعلومات نحدد فاذا نستخدم.

$$f_1 = f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \frac{h^4}{4!} f^{(4)}(c)$$

$$f_{-1} = f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) + \frac{h^4}{4!} f^{(4)}(c)$$

$$f_1 + f_{-1} = 2f_0 + h^2 f''(x) + \frac{h^4}{12} f^{(4)}(c)$$

where $f_0 = f(x)$

$$\underbrace{\frac{f_1 - 2f_0 + f_{-1}}{h^2}}_{\text{Formula}} - \underbrace{\frac{h^2}{12} f^{(4)}(c)}_{\text{trancation error}} = f''(x)$$

Formula trancation error

Best h:

$$E_{\text{tot}}(f, h) = E_{\text{round}}(f, h) + E_{\text{trunc}}(f, h)$$

$$E_{\text{tot}}(f, h) = \frac{e_1 - 2e_0 + e_{-1}}{h^2} - \frac{h^2 f^{(4)}(c)}{12}$$

if $|e_n| < \epsilon$, and $M = \max_{a \leq x \leq b} |f^{(4)}(x)|$

then

$$|E_{\text{tot}}| \leq \frac{4\epsilon}{h^2} + \frac{h^2 M}{12} = g(h)$$

$$g'(h) = -\frac{8\epsilon}{h^3} + \frac{hM}{6} = 0$$

$$\frac{hM}{6} = \frac{8\epsilon}{h^3}$$

$$h^4 = \frac{48\epsilon}{M}$$

$$h = \left(\frac{48\epsilon}{M}\right)^{1/4}$$

- Example

$$f(x) = \cos x$$

$f''(0.8)$ using $h=0.01$.

$$f''(0.8) \approx \frac{\cos(0.81) - 2\cos(0.8) + \cos(0.79)}{(0.01)^2} \approx -0.696690006$$

$$\text{Exact} = -\cos(0.8) = -0.697067$$

EXAMPLE

t	d
0.0	0.989992
0.1	0.999135
0.2	0.998295
0.3	0.987480

$$V(0) = ??$$

$$V(0.1) \approx \checkmark$$

$$V(0.2) = \checkmark$$

$$V(0.3) = ??$$

$$a(0) = ??$$

$$a(0.1) = \frac{d(0.2) - 2d(0.1) + d(0.0)}{(0.1)^2}$$

$$= \frac{0.998295 - 2(0.999135) + 0.989992}{0.01}$$

$$a(0.2) = \checkmark = \frac{d(0.3) - 2d(0.2) + d(0.1)}{(0.1)^2}$$

- Forward difference Formula's of $O(h^2)$

$$f'(x) \approx \frac{-3f_0 + 4f_1 - f_2}{2h}$$

$$f''(x) \approx \frac{2f_0 - 5f_1 + 4f_2 - f_3}{h^2}$$

- Backward difference Formula's of $O(h^2)$

$$f'(x_0) \approx \frac{3f_0 + 4f_{-1} + f_{-2}}{2h}$$

$$f''(x_0) \approx \frac{2f_0 - 5f_{-1} + 4f_{-2} - f_{-3}}{h^2}$$

$$f'(x_2) = \frac{3f_2 + 4f_1 + f_0}{2h}$$

$$f''(x_2) \approx \frac{2f_2 - 5f_1 + 4f_0 - f_{-1}}{h^2}$$

Example

$$f(x) = \cos x$$

$$h = 0.01$$

• Forward

$$f'(0.8) = \frac{-3 \cos(0.8) + 4 \cos(0.81) - \cos(0.82)}{2(0.01)}$$

• Backward

$$f'(0.8) = \frac{3 \cos(0.8) - 4 \cos(0.79) + \cos(0.78)}{2(0.01)}$$

• Forward

$$f''(0.8) = \frac{2 \cos(0.8) - 5 \cos(0.81) + 4 \cos(0.82) - \cos(0.83)}{(0.01)^2}$$

• Backward

$$f''(0.8) = \frac{2 \cos(0.8) - 5 \cos(0.79) + 4 \cos(0.78) - \cos(0.77)}{(0.01)^2}$$

• Using the table

$$V(0) = \frac{-3d(0) + 4d(0.1) - d(0.2)}{2(0.1)}$$

يعني استخدام Forward

$$+ V(0.1) = \frac{-3d(0.1) + 4d(0.2) - d(0.3)}{2(0.1)}$$

central

$$V(0.2) \quad \text{central} \quad \text{يعني استخدام Forward}$$

$$V(0.3) = \frac{3d(0.3) - 4d(0.2) + d(0.1)}{2(0.1)}$$

backward

t	d
0.0	0.989992
0.1	0.999135
0.2	0.998295
0.3	0.998...

$$a(0) \cong \frac{2d(0) - 5d(0.1) + 4d(0.2) - d(0.3)}{(0.1)^2}$$

$$a(0.1) = \text{Central}$$

$$a(0.2) = \text{Central}$$

$$a(0.3) = \frac{2d(0.3) - 5d(0.2) + 4d(0.1) - d(0)}{(0.1)^2}$$

• derive $f'(x_2) = \frac{3f_2 - 4f_1 + f_0}{2h} \quad O(h^2)$

$$f_1 = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(c)$$

$$f_2 = f(x) + 2hf'(x) + 2h^2 f''(x) + \frac{8}{6} h^3 f'''(c)$$

$$3f_2 = 3f(x) + 6hf'(x) + 6h^2 f''(x) + 4h^3 f'''(c)$$

$$4f_1 = 4f(x) + 4hf'(x) + 2h^2 f''(x) + \frac{4}{3} h^3 f'''(c)$$

$$3f_2 - 4f_1 = -f(x) + 2hf'(x)$$

$$-f_{-1} = f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(c)$$

$$f_{-2} = f(x-2h) = f(x) - 2hf'(x) + 2h^2 f''(x) - \frac{8}{6} h^3 f'''(c)$$

$$-4f_{-1} = -4f_0 + 4hf'(x) - 2h^2 f''(x) + \frac{4}{6} h^3 f'''(c)$$

$$-4f_{-1} + f_{-2} = -3f_0 + 2hf'(x) + 0 - \frac{4}{6} h^3 f'''(c)$$

$$\frac{3f_0 - 4f_{-1} + f_{-2}}{2h} + \frac{2}{3} h^2 f'''(c) = f'(x)$$

Formula

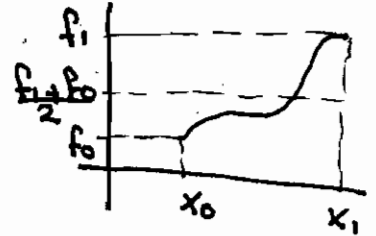
Error

$$f(x) = f_0$$

7.1 Newton Cotes Formula's:-

1. Trapezoidal Rule:-

$$\int_{x_0}^{x_1} f(x) dx = \frac{h}{2} (f_0 + f_1) \quad \text{with error} = -\frac{h^3}{12} f^{(2)}(c)$$



2. Simpson's $\frac{1}{3}$ Rule:-

$$\int_{x_0}^{x_2} f(x) dx = \frac{h}{3} (f_0 + 4f_1 + f_2) \quad \text{with error} = -\frac{h^5}{90} f^{(4)}(c)$$

3. Simpson's $\frac{3}{8}$ Rule:-

$$\int_{x_0}^{x_3} f(x) dx = \frac{3h}{8} (f_0 + 3f_1 + 3f_2 + f_3) \quad \text{with error} = -\frac{3h^5}{80} f^{(4)}(c)$$

Example 1:-

Estimate $\int_0^1 (1 + e^{-x} \sin x) dx$ using the three rules.

1. Trapezoidal = $\frac{h}{2} (f_0 + f_1) = \frac{1}{2} (f(0) + f(1)) = \frac{1}{2} (1 + 0.72159) = 0.86079$

2. Simpson $\int_0^1 f(x) dx = \frac{h}{3} (f_0 + 4f_1 + f_2) = \frac{1/2}{3} (f(0) + 4f(1/2) + f(1))$
 $h = \frac{x_n - x_0}{n} = \frac{x_2 - x_0}{2} = \frac{1-0}{2} = \frac{1}{2}$

$$= \frac{1}{6} (1 + 4(1.55152) + 0.72159) = 1.32128$$

3. Simpson $\frac{3}{8}$ $\int_0^1 f(x) dx = \frac{3h}{8} (f_0 + 3f_1 + 3f_2 + f_3)$
 $h = \frac{x_n - x_0}{n} = \frac{x_3 - x_0}{3} = \frac{1-0}{3} = \frac{1}{3}$

$$= \frac{3(1/3)}{8} (f(0) + 3f(1/3) + 3f(2/3) + f(1))$$

$$= 1.31440 \quad \text{أقرب واحدة للجواب}$$

Exact answer = 1.3082606

Example 2

t	v(t)
x	f(x)
1	20.1
2	22.5
3	25.6
4	28.9

$$\int_1^4 f(x) dx = ??$$

by Simpson's 3/8 Rule (because we have 4 points)

$$\int_1^4 f(x) dx = \frac{3(4)}{8} (f(1) + 3f(2) + 3f(3) + f(4))$$

or by trapezoidal

$$\int_1^4 f(x) dx = \frac{4}{2} (f(1) + f(4))$$

Example

Derive trapezoidal error or Rule.

we use $f(x)$ and $\int_{x_0}^{x_1} f(x) dx \approx \int_{x_0}^{x_1} P(x) dx$

$$= \int_{x_0}^{x_1} \left(\frac{x-x_1}{x_0-x_1} y_0 + \frac{x-x_0}{x_1-x_0} y_1 \right) dx \quad \begin{matrix} x = x_0 + ht \\ x_1 = x_0 + h \end{matrix} \quad dx = h dt$$

$$= \int_0^1 \left(\frac{h(t-1)}{-h} y_0 + \frac{ht}{h} y_1 \right) h dt = -y_0 h \int_0^1 (t-1) dt + h y_1 \int_0^1 t dt$$

$$= \frac{y_0 h}{2} + \frac{h y_1}{2} = \frac{h}{2} (y_0 + y_1)$$

$$\text{Error} = \int_{x_0}^{x_1} E(x) dx = \int_{x_0}^{x_1} \frac{(x-x_0)(x-x_1)}{2!} f^{(2)}(c) dx$$

$$= \int_0^1 h(t) h(t-1) \frac{f^{(2)}(c)}{2} h dt = \frac{f^{(2)}(c)}{2} \int_0^1 h(t) h(t-1) h dt$$

$$= \frac{h^3 f^{(2)}(c)}{2} \int_0^1 (t^2 - t) dt = \frac{-h^3 f^{(2)}(c)}{12}$$

Def:-

The degree of precision or accuracy of a quadrature Formula is the largest positive integer n is such that the Formula is exact for x^k , $k=0,1,2,\dots$

Example:-

Find the degree of accuracy of Simpson's method:-

$$\frac{h}{3} (f_0 + 4f_1 + f_2)$$

$$\begin{array}{ccc} 0 & 1 & 2 \\ x_0 & x_1 & x_2 \end{array}$$

<u>F(x)</u>	<u>Formula</u>	<u>Exact</u>	<u>Error</u>
$x^0 = 1$	$\frac{1}{3} (f_0 + 4f_1 + f_2)$ $\frac{1}{3} (1 + 4(1) + 1) = 2$	$\int_0^2 1 dx = 2$	0
$x^1 = x$	$\frac{1}{3} (0 + 4(1) + 2) = 2$	$\int_0^2 x dx = \frac{x^2}{2} \Big _0^2 = 2$	0
x^2	$\frac{1}{3} (0 + 4(1) + 4) = \frac{8}{3}$	$\int_0^2 x^2 dx = \frac{x^3}{3} \Big _0^2 = \frac{8}{3}$	0
x^3	$\frac{1}{3} (0 + 4(1) + 8) = 4$	$\int_0^2 x^3 dx = \frac{x^4}{4} \Big _0^2 = 4$	0
x^4	$\frac{1}{3} (0 + 4(1) + 16) = \frac{20}{3}$	$\int_0^2 x^4 dx = \frac{x^5}{5} \Big _0^2 = \frac{32}{5}$	$\frac{32}{5} - \frac{20}{3} \neq 0$

degree of accuracy of Simpson's is 3

Note:-

degree of accuracy of trapezoidal is 1

degree of accuracy of Simpson $\frac{1}{3}$ is 3

degree of accuracy of Simpson $\frac{3}{8}$ is 3

Theory:-

Error = $k f^{(n+1)}(\xi)$, k is the degree of accuracy

Example

For Simpson's $\frac{1}{3}$ method

$$\text{Error} = k f^{(4)}(\xi)$$

$$f(x) = x^4$$

$$f'(x) = 4x^3$$

$$f''(x) = 12x^2$$

$$f'''(x) = 24x$$

$$f^{(4)}(x) = 24$$

$$\text{Error} = k f^{(4)}(\xi)$$

$$\frac{32}{5} - \frac{20}{3} = k(24)$$

$$\frac{96-100}{15} = 24k$$

$$k = \frac{-4}{15 \times 24} = -\frac{1}{90}$$

- if $f(x) = (x-x_0)^4$

$$\int_{x_0}^{x_2} f(x) dx$$

$$\text{Error} = \text{Exact} - \text{Formula}$$

$$\text{Exact} = \int_{x_0}^{x_2} f(x) dx = \int_{x_0}^{x_2} (x-x_0)^4 dx = \left. \frac{(x-x_0)^5}{5} \right|_{x_0}^{x_2} = \frac{32h^5}{5}$$

$$\text{Formula} = \frac{h}{3} [f(x_0) + 4f(x+h) + f(x_2)] = \frac{20}{3} h^5$$

$$\text{Error} = -\frac{1}{90} h^5$$

$\frac{7}{6.1}$

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{2h}$$

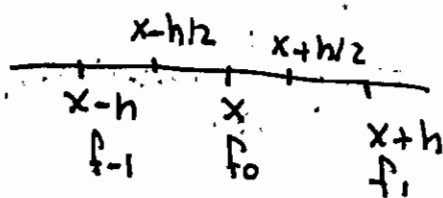
$$f_x(x, y) = \frac{f(x+h, y) - f(x-h, y)}{2h}$$

$$f_y(x, y) = \frac{f(x, y+h) - f(x, y-h)}{2h}$$

$\frac{10}{6.2}$

$$f'(x + \frac{h}{2}) = \frac{f_1 - f_0}{h}$$

$$f'(x - \frac{h}{2}) = \frac{f_0 - f_{-1}}{h}$$



$$f''(x) = (f'(x))' = \frac{f'(x+h/2) - f'(x-h/2)}{2(h/2)}$$

$$= \frac{\frac{f_1 - f_0}{h} - \frac{f_0 - f_{-1}}{h}}{h}$$

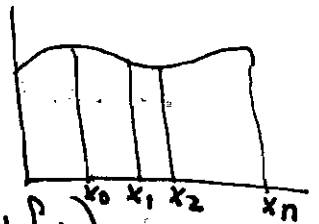
$$f''(x) = \frac{f_1 - 2f_0 + f_{-1}}{h^2}$$

$$f'''(x) = (f''(x))' = (f'''(x))'$$

7.2 Composite Rules

1. Composite trapezoidal Rule

$$\int_{x_0}^{x_n} f(x) dx = \int_{x_0}^{x_1} f(x) dx + \int_{x_1}^{x_2} f(x) dx + \dots + \int_{x_{n-1}}^{x_n} f(x) dx$$



$$= \frac{h_1}{2} (f_0 + f_1) + \frac{h_2}{2} (f_1 + f_2) + \dots + \frac{h_n}{2} (f_{n-1} + f_n)$$

$$h_k = h$$

$$= \frac{h}{2} (f_0 + 2f_1 + 2f_2 + \dots + 2f_{n-1} + f_n)$$

$$= \frac{h}{2} \sum_{k=1}^n (f_{k-1} + f_k) = T(f, h)$$

EXAMPLE

t	v(t)
0	10
1	12
2	13
3	15

$$D(3) = \frac{1}{2} (f(0) + 2f(1) + 2f(2) + f(3))$$

EXAMPLE

t	v(t)
1	10
3	15
4	20
5	21

$$D(5) = \frac{2}{2} (f(1) + f(3)) + \frac{1}{2} (f(3) + f(4)) + \frac{1}{2} (f(4) + f(5))$$

• Error For Composite trapezoidal.

$$\text{Error} = -\frac{h^3}{12} f''(c_1) - \frac{h^3}{12} f''(c_2) + \dots - \frac{h^3}{12} f''(c_n)$$

$$= -\frac{h^3}{12} (f''(c_1) + f''(c_2) + \dots + f''(c_n))$$

$$= -\frac{h^3}{12} (n f''(c)) \quad h = \frac{b-a}{n}$$

$$= -\frac{h^3}{12} \left(\frac{b-a}{n} f''(c) \right)$$

$$E_T(f, h) = -\frac{(b-a) f''(c) h^2}{12} \approx O(h^2)$$

• EXAMPLE

Find the number m at step size h so that $|E_T(f, h)| \leq 5 \times 10^{-9}$

of the approximation $\int_2^7 \frac{dx}{x} = T(f, h)$

where m is the number of trapezoidal composite

$$m = n$$

$$|E_T(f, h)| \leq 5 \times 10^{-9}$$

$$\frac{(b-a) f''(c) \left(\frac{b-a}{n}\right)^2}{12} \leq 5 \times 10^{-9}$$

$$f(x) = \frac{1}{x} = x^{-1}$$

$$f'(x) = -\frac{1}{x^2}$$

$$f''(x) = \frac{2}{x^3}$$

$$\max |f''(x)| = \frac{2}{8} = \frac{1}{4}$$

$$2 \leq x \leq 7$$

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$$\frac{(b-a) f^{(2)}(c) \left(\frac{b-a}{n}\right)^2}{12} \leq 5 \times 10^{-9}$$

$$\frac{5(0.25) \left(\frac{5}{n}\right)^2}{12} \leq 5 \times 10^{-9}$$

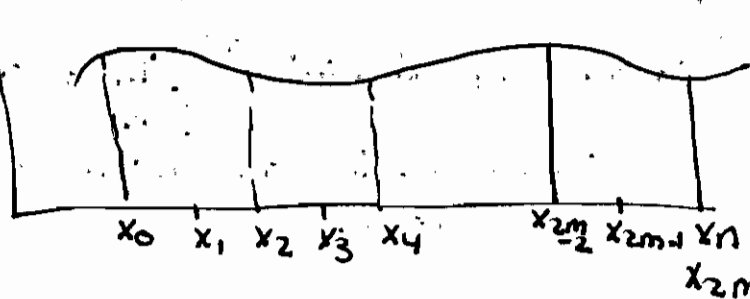
$$n \geq \sqrt{\frac{5 \times 0.25 \times 25}{12 \times 5 \times 10^{-9}}} = 22821.77$$

$$n = 22822$$

$$h = \frac{b-a}{n} = \frac{5}{22822} = 0.000219$$

2. Composite Simpson's $\frac{1}{3}$ Rule.

$$\int_{x_0}^{x_{2m}} f(x) dx = \int_{x_0}^{x_2} f(x) dx + \int_{x_2}^{x_4} f(x) dx + \int_{x_4}^{x_6} f(x) dx + \dots + \int_{x_{2m-2}}^{x_{2m}} f(x) dx$$



$$= \frac{h_1}{3} (f_0 + 4f_1 + f_2) + \frac{h_2}{3} (f_2 + 4f_3 + f_4) + \dots + \frac{h_m}{3} (f_{2m-2} + 4f_{2m-1} + f_{2m})$$

$n = 2m$
 $m = \frac{n}{2}$

$h_k = h$

$$= \frac{h}{3} (f_0 + 4f_1 + f_2 + f_2 + 4f_3 + 2f_4 + \dots + 2f_{2m-2} + 4f_{2m-1} + f_{2m})$$

$$= \frac{h}{3} \sum_{k=1}^m (f_{2k-2} + 4f_{2k-1} + f_{2k}) = S(f, h)$$

• Error For Composite Simpson

$$\begin{aligned} E_S(f, h) &= -\frac{h^5}{90} f^{(4)}(c_1) - \frac{h^5}{90} f^{(4)}(c_2) - \dots - \frac{h^5}{90} f^{(4)}(c_m) \\ &= -\frac{h^5}{90} (f^{(4)}(c_1) + f^{(4)}(c_2) + \dots + f^{(4)}(c_m)) \\ &= -\frac{h^5}{90} (m f^{(4)}(c)) \end{aligned}$$

$$= -\frac{h^5}{90} \left(\frac{b-a}{2h} f^{(4)}(c) \right)$$

$$= -\frac{(b-a)h^4}{180} f^{(4)}(c)$$

$$\approx Ch^4$$

$$M = \frac{b-a}{2h}$$

EXAMPLE

Find the number \underline{m} and step size h that $|E_S(f;h)| \leq 5 \times 10^{-9}$ of the approximation $\int_2^7 \frac{dx}{x} = S(f;h)$.

$$|E_S(f;h)| \leq 5 \times 10^{-9}$$

$$\left| \frac{b-a}{180} \left(\frac{b-a}{2m} \right)^4 f^{(4)}(c) \right| < 5 \times 10^{-9}$$

$$\frac{5 \cdot \left(\frac{5}{2m} \right)^4 \cdot 0.75}{180} < 5 \times 10^{-9}$$

$$m > \sqrt[4]{\frac{5 \cdot 5^4 \cdot 0.75}{2^4 \cdot 180 \cdot 5 \cdot 10^{-9}}} = 112.9$$

$$f(x) = \frac{1}{x} = x^{-1}$$

$$f'(x) = -\frac{1}{x^2}$$

$$f''(x) = \frac{2}{x^3}$$

$$f'''(x) = -\frac{6}{x^4}$$

$$f^{(4)}(x) = \frac{24}{x^5}$$

$$\max_{2 \leq x \leq 7} |f^{(4)}(x)| = \frac{24}{x^5} \Big|_{x=2}$$

$$= \frac{24}{32} = 0.75$$

7.5 Gauss - Legendre Formulas

2 points Formula

$$\int_a^b f(x) dx \approx w_1 f(x_1) + w_2 f(x_2)$$

عندنا 2 نقاط x_1 و x_2 و w_1 و w_2

we assume degree of precision 3

1. $E(1) = 0 \rightarrow \int_{-1}^1 1 dx = 2 \rightarrow \text{Formula} = w_1(1) + w_2(1) \rightarrow w_1 + w_2 = 2$
2. $E(x) = 0 \rightarrow \int_{-1}^1 x dx = 0 \rightarrow \text{Formula} = w_1 x_1 + w_2 x_2 \rightarrow w_1 x_1 + w_2 x_2 = 0$
3. $E(x^2) = 0 \rightarrow \int_{-1}^1 x^2 dx = \frac{2}{3} \rightarrow \text{Formula} = w_1 x_1^2 + w_2 x_2^2 \rightarrow w_1 x_1^2 + w_2 x_2^2 = \frac{2}{3}$
4. $E(x^3) = 0 \rightarrow \int_{-1}^1 x^3 dx = 0 \rightarrow \text{Formula} = w_1 x_1^3 + w_2 x_2^3 \rightarrow w_1 x_1^3 + w_2 x_2^3 = 0$

$$\text{Exact} = \int_{-1}^1 f(x) dx$$

$$\text{Formula} = w_1 f(x_1) + w_2 f(x_2)$$

$$\text{Exact} = \text{Formula}$$

حل المعادلات

$$w_1 x_1^3 = -w_2 x_2^3$$

$$w_1 x_1 = -w_2 x_2$$

$$x_1^2 = x_2^2$$

$$\text{منه } x_1 = x_2 \text{ or } \boxed{x_1 = -x_2}$$

$$w_1 x_1 + w_2 (-x_1) = 0$$

$$x_1 (w_1 - w_2) = 0$$

$$w_1 - w_2 = 0$$

$$\boxed{w_1 = w_2}$$

$$w_1 * w_2 = 2$$

$$2w_1 = 2$$

$$\boxed{w_1 = 1}, \boxed{w_2 = 1}$$

$$1(x_1)^2 + 1(x_1)^2 = \frac{2}{3}$$

$$2x_1^2 = \frac{2}{3}$$

$$x_1^2 = \frac{1}{3}$$

$$\rightarrow \boxed{x_1 = \pm \frac{1}{\sqrt{3}}}$$

$$\int_{-1}^1 f(x) dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= G_2(f)$$

Gauss - Legendre

2 points Formula

EXAMPLE

Estimate $\int_{-1}^1 \frac{1}{x+2} dx$ using

$$\frac{1}{(x+2)} = \frac{-1}{x+2}$$

1. $T(f, 2) = \frac{2}{2} (f(-1) + f(1)) = 1.3333$

2. $S(f, 2) = \frac{1}{3} (f(-1) + 4f(0) + f(1)) = 1.1111$

3. $G_2(f) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$
 $= 1.09091$

Exact = 1.09861

$E = \text{Exact} - \text{Formula} = \frac{24}{(x+2)^5}$
 $= 1.09091 - 1.09861$

The Error For $G_2(f) = \frac{f^{(4)}(c)}{135}$

Error = $\frac{1}{135} f^{(4)}(c)$

$f(x) = x^4$

- Gauss Legendre 3 points Formula should have 5 degree of accuracy

$$\int_{-1}^1 f(x) dx = w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) + \dots$$

$$G_3(f) = \frac{5}{9} f\left(-\frac{\sqrt{3}}{3}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\frac{\sqrt{3}}{3}\right)$$

Error = $\frac{f^{(6)}(c)}{15.750}$

• Gauss - Legendre Formulas

- Gauss Legendre two pts Formula

$$G_2(f) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \text{ with error} = \frac{1}{135} f^{(4)}(c)$$

has 3 - degree of accuracy

- Gauss Legendre three pts Formula

$$G_3(f) = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) \text{ with error} = \frac{1}{15750}$$

has 5 - degree of accuracy

$$G_n(f) = w_1 f(x_1) + w_2 f(x_2) + \dots + w_n f(x_n)$$

has $2n-1$ degree of accuracy

- $G_8(f) \rightarrow \text{error} = \frac{f^{(16)}(c) 2^{17} (8!)^4}{(16!)^3 17!}$ Very accurate formula

- Theorem

if x_n 's are the pts of Gauss Legendre Formula, and w_n 's are the weights in $[-1, 1]$ to apply the formula on $[a, b]$ we use the transformation.

$$t = \frac{a+b}{2} + \frac{b-a}{2} x \quad ; \quad [-1, 1] \rightarrow [a, b]$$

$$dt = \frac{b-a}{2} dx$$

$$\int_a^b f(t) dt = \frac{b-a}{2} \sum_{k=1}^n w_k f\left(\frac{a+b}{2} + \frac{b-a}{2} x_k\right)$$

$$= \frac{b-a}{2} (w_1 f\left(\frac{a+b}{2} + \frac{b-a}{2} x_1\right) + w_2 f\left(\frac{a+b}{2} + \frac{b-a}{2} x_2\right) + \dots)$$

EXAMPLE

Use $G_3(f)$ to estimate $\int_1^5 \frac{1}{t} dt$

$$\begin{aligned} G_3(f) &= 2 \left[\frac{5}{9} f\left(3 + 2\left(-\sqrt{\frac{3}{5}}\right)\right) + \frac{8}{9} f(3 + 2(0)) + \frac{5}{9} f\left(3 + 2\sqrt{\frac{3}{5}}\right) \right] \\ &= 1.602694 \end{aligned}$$

- Gauss Legendre Formula are very accurate.

Chapter 9

Numerical solution of 1st order ODE's

1st order ODE

$$- y'(t) = f(t, y(t))$$

$$y(t_0) = y_0$$

$$- y' = \frac{t-y}{2}$$

$$y(0) = 1$$

$$- t^2 y' + \sin t y^2 = \cos t$$

$$y(t_1) = y(t_0 + h)$$

$$= y(t_0) + h y'(t_0) + \frac{h^2}{2!} y''(c)$$

$$\rightarrow y(t_1) \approx y(t_0) + h y'(t_0) \text{ with error} = \frac{h^2}{2!} y''(c)$$

$$= y_0 + h f(t_0, y_0) \quad (\text{section 9.2})$$

$$y(t_1) \approx y(t_0) + h y'(t_0) + \frac{h^2}{2} y''(t_0) + \frac{h^3}{3!} y'''(c) \quad (\text{Section 9.4})$$

9.2 Euler method

$$\text{consider } y' = f(t, y)$$

$$y(t_0) = y_0$$

we will approximate the solution using set of points (t_k, y_k) where

$$\underbrace{y_k}_{\text{Estimate}} = \underbrace{y(t_k)}_{\text{estimation at } t_k}$$

- we will use n subintervals of $[a, b]$

$$h = \frac{b-a}{n}, \quad t_k = a + h k$$

$k=1, \dots, n$

- Using Taylor expansion of $y(t_1)$ at t_0 ,

$$y(t_1) = y(t_0+h) = y(t_0) + h y'(t_0) + \frac{h^2}{2} y''(c)$$

$$\rightarrow y_1 = y_0 + h f(t_0, y_0) \text{ with step error} = \frac{h^2}{2} y''(c)$$

notice that $y_1 \approx y(t_1)$

$$y_2 = y_1 + h f(t_1, y_1)$$

$$y_3 = y_2 + h f(t_2, y_2)$$

\vdots

$$y_{n+1} = y_n + h f(t_n, y_n) \text{ Euler method, step error} = \frac{h^2}{2} y''(c)$$



$t_0 =$
 $t_1 = t_0 + h$
 $t_2 = t_0 + 2h$

EXAMPLE

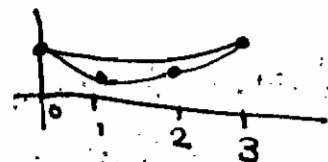
Estimate the solution of $y' = \frac{t-y}{2}$, $y(0) = 1$, on $[0, 3]$

$$y_1 = y_0 + h f(t_0, y_0)$$

$$= 1 + 1 f(0, 1) = 1 + (-0.5) = 0.5$$

$$y_2 = y_1 + h f(t_1, y_1)$$

$$= 0.5 + 1 f(1, 0.5) = 0.5 + \frac{1-0.5}{2} = 0.75$$



$$y_3 = y_2 + h f(t_2, y_2)$$

$$= 0.75 + 1 f(2, 0.75) = 0.75 + \frac{2-0.75}{2} = 1.375$$

Total error = $E(y(h), h)$

$$= \frac{y''(c_1) h^2}{2} + \frac{y''(c_2) h^2}{2} + \dots + \frac{y''(c_n) h^2}{2}$$

$$= \frac{h^2}{2} (y''(c_1) + y''(c_2) + \dots + y''(c_n))$$

$$= \frac{h^2}{2} (n y''(c))$$

$$= \frac{h^2}{2} \left(\frac{b-a}{h} y''(c) \right) = \frac{(b-a) h}{2} y''(c) \approx ch$$

$$E(y(h), \frac{h}{2}) = c \left(\frac{h}{2} \right) = \frac{1}{2} ch = \frac{1}{2} E(y(h), h) \text{ نصف الخطأ السابق}$$

9.4 Taylor method

Derive a formula of total error $O(h^2)$ to solve

$$y' = \frac{t-y}{2} \text{ on } [0, 3]$$

$$y_0 = 1, \quad h = 1$$

$$y_1 = y_0 + hf(t_0, y_0) + \frac{h^2}{2} f''(t_0, y_0)$$

$$y(t_1) = \underbrace{y_0 + hf(t_0, y_0)}_{y_1} + \frac{h^2}{2} y''(t_0) + \underbrace{\frac{h^3}{3!} y'''(c)}_{\text{Error}}$$

$$y_1 = y_0 + hf(t_0, y_0) + \frac{h^2}{2} \frac{df(t_0, y_0)}{dt}$$

$$\text{Step error} = \frac{h^3}{6} y'''(c)$$

$$y_{k+1} = y_k + hf(t_k, y_k) + \frac{h^2}{2} y''(t_k)$$

$$\text{Total error} = E(y(h), h) = \frac{ch^2}{6} = \frac{y'''(c)(b-a)h^2}{6}$$

• Solving the example.

$$\begin{aligned} y_1 &= y_0 + hf(t_0, y_0) + \frac{h^2}{2} y''(t_0) \\ &= 1 + 1f(0, 1) + \frac{1}{2} y''(0) \\ &= 1 + \frac{0-1}{2} + \frac{1}{2} \left(\frac{1}{2} - \frac{(0-1)}{4} \right) \\ &= 1 - 0.5 + 0.25 + 1/8 = 0.875 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + hf(t_1, y_1) + \frac{h^2}{2} y''(t_1) \\ &= y_1 + hf(t_1, y_1) + \frac{h^2}{2} \frac{d}{dt} (f(1, 0.875)) \\ &= 0.875 + hf(1, 0.875) + \frac{1}{2} \left(\frac{1}{2} - \frac{1-0.875}{4} \right) \\ &= 0.875 + 0.0625 + 0.25 - 0.03125 \\ &= 1.15625 \end{aligned}$$

$$y'(t) = \frac{t-y}{2}$$

$$\begin{aligned} y''(t) &= \frac{1}{2} - \frac{y'}{2} \\ &= \frac{1}{2} - \frac{(t-y)}{4} \end{aligned}$$

$$y'(t) = f(t, y(t))$$

$$y(t_0) = y_0$$

- Use Taylor method of order 4 to estimate the solution of $y' = \frac{t-y}{2}$, $y(0) = 1$ on $[0, 3]$, $h = 1$

$$y(t_1) = \underbrace{y(t_0) + hy'(t_0) + \frac{h^2}{2!} y''(t_0) + \frac{h^3}{3!} y'''(t_0) + \frac{h^4}{4!} y^{(4)}(t_0)}_{y_1} + \underbrace{\frac{h^5}{5!} y^{(5)}(c)}_{\text{step error}}$$

$$\text{- total error} = E(y(t_1), h) = Ch^4$$

$$y_{k+1} = y_k + hy'(t_k) + \frac{h^2}{2!} y''(t_k) + \frac{h^3}{3!} y'''(t_k) + \frac{h^4}{4!} y^{(4)}(t_k)$$

$$y_1 = y_0 + hy'(t_0) + \frac{h^2}{2} y''(t_0) + \frac{h^3}{3!} y'''(t_0) + \frac{h^4}{4!} y^{(4)}(t_0)$$

$$y'(t) = \frac{t-y}{2}, \quad y'(0) = \frac{0-1}{2} = -\frac{1}{2} \quad (y_0 = 1)$$

$$y''(t) = \frac{1}{2}(1-y') = \frac{1}{2} \left(1 - \frac{t-y}{2}\right) = \frac{1}{2} - \frac{t-y}{4}$$

$$y''(0) = \frac{1}{2} - \left(-\frac{1}{4}\right) = 0.75$$

$$y'''(t) = \frac{1}{2}(-y'') = -\frac{1}{2} \left(\frac{1}{2} - \frac{t-y}{4}\right)$$

$$y'''(0) = -\frac{1}{2}(0.75) = -0.375$$

$$y^{(4)}(t) = -\frac{1}{2} y''' = -\frac{1}{2} \left(-\frac{1}{2} \left(\frac{1}{2} - \frac{t-y}{4}\right)\right) =$$

$$y^{(4)}(0) = -\frac{1}{2}(-0.375) = 0.1875$$

$$y_1 = 1 + 1(0.5) + \frac{1}{2}(0.75) + \frac{1}{6}(-0.375) + \frac{1}{24}(0.1875) = 0.8203125$$

$$y_2 = y_1 + hy'(t_1) + \frac{h^2}{2} y''(t_1) + \frac{h^3}{3!} y'''(t_1) + \frac{h^4}{4!} y^{(4)}(t_1)$$

$$t_1 = 1, \quad y_1 = 0.8203125$$

$$y_2 = 1.1045 \dots$$

$$y_3 = 1.670 \dots$$

$$E(y(b), h) = Ch^4$$

$$E(y(b), h/2) = C(h/2)^4 = Ch^4/16$$

$$E(y(b), 10^{-2}h) = C(10^{-2}h)^4 = C(10^{-8})h^4$$

taylor → one evaluation

- Modified Method :- (Huen's Method)
Euler

$$y'(t) = f(t, y(t))$$

$$y(t_0) = y_0$$

$$1. \int_{t_0}^{t_1} y'(t) dt = \int_{t_0}^{t_1} f(t, y(t)) dt \rightarrow \text{Using trapezoidal}$$

$$y(t_1) - y(t_0) = \frac{h}{2} (f(t_0, y_0) + f(t_1, y(t_1)))$$

$$\text{Error} = \frac{-h^3 y''(c)}{12}$$

$$y(t_1) \approx y_0 + \frac{h}{2} (f(t_0, y_0) + f(t_1, y(t_1)))$$

↓
زغل قيمة معروفة

$$y(t_1) = y_0 + \frac{h}{2} (f(t_0, y_0) + f(t_1, y_0 + hf(t_0, y_0)))$$

$$- y_{k+1} = y_k + \frac{h}{2} (f(t_k, y_k) + f(t_{k+1}, y_k + hf(t_k, y_k)))$$

Total error for Huen's Method = Ch^2

• EXAMPLE

Solve Using Huen's Method with $h=1$

$$y' = \frac{t-y}{2} = f(t, y)$$

$$y(0) = 1$$

$f(t, y)$

$$y_1 = y_0 + \frac{h}{2} (f(t_0, y_0) + f(t_1, y_0 + hf(t_0, y_0)))$$

$$= 1 + \frac{1}{2} (f(0, 1) + f(1, 1 + (1)f(0, 1)))$$

$$= 1 + \frac{1}{2} (-0.5 + f(1, 1 - 0.5)) = 1 + \frac{1}{2} (-0.5 + \frac{1-0.5}{2}) = 0.875$$

$$\begin{aligned}
 y_2 &= y_1 + \frac{h}{2} (f(t_1, y_1) + f(t_2, y_1 + hf(t_1, y_1))) \\
 &= 0.875 + \frac{1}{2} (f(1, 0.875) + f(2, 0.875 + (1)f(1, 0.875))) \\
 &= 0.875 + \frac{1}{2} \left(\left(\frac{1-0.875}{2} \right) + f(2, 0.875(1-0.875/2)) \right) \\
 &= 1.171875
 \end{aligned}$$

$$y_3 = 1.732422$$

- if we have a period of $[0, 0.5]$, $h = \frac{1}{4}$

$$y_1 = 1 + 1/2(-0.25) = 0.875$$

0	0.25	0.5
t_0	t_1	t_2

on $[0, 1]$, $h = \frac{1}{2}$

0	0.5	1
t_0	t_1	t_2

9.5 RK4: Range-Kutta Method of order 4

$$y'(t) = f(t, y(t))$$

$$y(t_0) = y_0$$

$$\int_{t_0}^{t_1} y'(t) dt = \int_{t_0}^{t_1} f(t, y(t)) dt$$

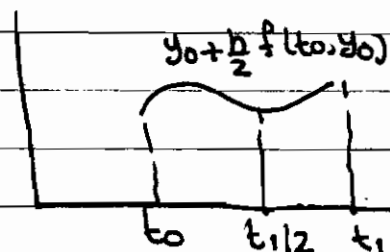
$$y_{k+1} = y_k + \frac{h}{6} (f_0 + 2f_1 + 2f_2 + f_3) \quad \text{where}$$

$$f_0 = f(t_0, y_0)$$

$$f_1 = f(t_0 + h/2, y_0 + h/2 f_0)$$

$$f_2 = f(t_0 + h/2, y_0 + h/2 f_1)$$

$$f_3 = f(t_1, y_0 + h f_2)$$



$$f_1 = \frac{f_1 + f_2}{2}$$

$$f_2(t_{1/2}, y_{1/2}) = y_0 + \frac{h}{2} f$$

$$f_1(t_{1/2}, y_{1/2}) = y_0 + \frac{h}{2} f(t_0)$$

EXAMPLE

Solve $y' = \frac{t-y}{2}$, $y(0) = 1$, $[0, 3]$, $h = \frac{1}{4}$

$$y_1 = y_0 + \frac{h}{6} (f_0 + 2f_1 + 2f_2 + f_3)$$

$$f_0 = f(t_0, y_0) = f(0, 1) = -0.5$$

$$f_1 = f(t_0 + h/2, y_0 + h/2 f_0) = f\left(\frac{1}{8}, 1 + \frac{1}{8}(-0.5)\right) = -0.40625$$

$$f_2 = f(t_0 + h/2, y_0 + h/2 f_1) = f\left(\frac{1}{8}, 1 + \frac{1}{8}(-0.40625)\right) = -0.4121094$$

$$f_3 = f(t_1, y_0 + h f_2) = f(1/4, 1 + \frac{1}{4}(-0.4121094)) = -0.3234863$$

$$y_1 = 1 + \frac{1/4}{6} ((-0.5) + 2(-0.40625) + 2(-0.4121094) + (-0.3234863))$$

$$= 0.8974915 \quad \text{Comparing to the exact } 0.8974917$$

$$y_2 = y_1 + h/6 (f_0 + 2f_1 + 2f_2 + f_3).$$

$$f_0 = f(t_1, y_1) = f(1/4, 0.8974915) = \dots$$

$$f_1 = f(t_1 + h/2, y_1 + h/2 f_0) = f(3/8, 0.8974915 + 1/8 f_0) = \dots$$

$$f_2 = f(t_2, y_1 + h f_1) = \dots$$

$$f_3 = f(t_2, y_1 + h f_2) = \dots$$

$$y_2 = \dots$$