Chapter 6

6.1

Th:- Central differance Formula of order $O(h^2)$ (f_1).

assum that $f \in C^2[a,b]$, and $x-h, x, x+h \in [a,b]$ then f

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

furthermore there exists a number $c \in [a,b]$ such that

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2 f^{(3)}(c)}{6}$$

where the error then $-\frac{h^2 f^{(3)}(c)}{6}$ is called the trancation error and is denoted by

$$E_{tranc}(f,h) \neq \text{ i.e } E_{tranc}(f,h) = \frac{h^2 f^{(3)}(c)}{6}$$

Let

t	d
0.1	13.21
0.2	20.55
0.3	24.12
0.4	29.79

$$V(0.2) = \frac{d(0.3) - d(0.1)}{2(0.1)} = \frac{24.12 - 13.21}{0.2} = 54.55$$

$$\text{error} = C(0.1)^2$$
$$= C(0.01) \rightarrow \text{error in the } 4^{th} \text{ digit.}$$

$$V(0.3) = \frac{d(0.4) - d(0.2)}{0.02}$$

V(0.4) = لا أعرف

V(0.1) = لا نعرف

$f(x) = \cos x$

$f'(0.8) = ??$ $\quad h = 0.01$

$$f'(0.8) \cong \frac{f(0.8+0.01) - f(0.8-0.01)}{2(0.01)} \cong \frac{\cos(0.81) - \cos(0.79)}{0.02}$$

$.001$

$$\cong \frac{0.68998933 - 0.7303895326}{0.02} = -\underbrace{0.717}3 44160.$$

Exact $= f'(0.8) = \sin(0.8) = -\underbrace{0.7173}56091$ ٤منازل تقريباً صحيحة

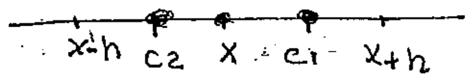
by Theorem $c(h^2)^2 = \cancel{\text{Error}} = c(0.01)^2 = c(0.0001)$

(٠.٠١)

اربع منازل صحيحة وفقاً "للنظرية".

- Derivation

Using Taylor expansion at x,



$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(c_1), \quad c_1 \in (x, x+h).$$

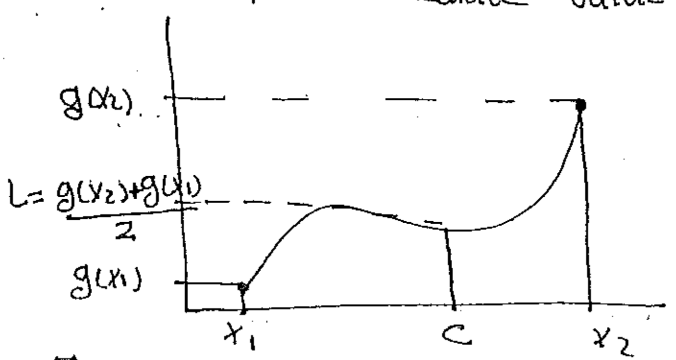
$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(c_2), \quad c_2 \in (x-h, x).$$

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{h^3}{6}\left(f'''(c_1) - f'''(c_2)\right)$$

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{h^3}{6}\left(2f'''(c)\right) \quad c \in (c_1, c_2).$$

$$\underbrace{\frac{f(x+h) - f(x-h)}{2h}}_{F_1} \approx \underbrace{\frac{h^2 f'''(c)}{6}}_{\text{Error}} = f'(x).$$

IVP (Intermediate Value property)



$\Rightarrow \exists\, c \in (x_1, x_2)$ such that $g(c) = \frac{g(x_1) + g(x_2)}{2}$

$\rightarrow g(x_1) + g(x_2) = 2g(c)$

Section 6.1

Central differance Formula of $O(h^4)$

assume $f \in C^5[a,b]$ and $x-2h, x-h, x+h, x+2h \in [a,b]$ then

$$f'(x) = \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h}$$

with error

$$E_{tranc}(f,h) = \frac{h^4 f^{(5)}(c)}{30} \approx ch^4$$

Example 1

Let

t	d
0.1	13.25
0.2	18.53
0.3	21.25
0.4	24.30
0.5	27.12

فقط يمكن استخراجها عند 0.3

$$V(0.3) = \frac{-d(0.5) + 8d(0.4) - 8d(0.2) + d(0.1)}{12(0.1)}$$

$$= \frac{-27.12 + 8(24.30) - 8(18.53) + 13}{12}$$

Example 2

$f(x) = \cos x$

$f'(0.8)$ using $h = 0.01$

$$f'(0.8) = \frac{-\cos(0.82) + 8\cos(0.81) - 8\cos(0.79) + \cos(0.78)}{0.12}$$

$f'(0.8) = -0.717356108$

Compare to exact $-\sin(0.8) = -0.717356091$

error $c(0.01)^4 = c(10^{-8})$

error $= 1 * 10^{-7}$ ثبتنا ٧ منازل

- Derivation

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) + \frac{h^5}{5!}f^{(5)}(c)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(3) + \frac{h^4}{4!}f^{(4)}(x) - \frac{h^5}{5!}f^{(5)}(c)$$

نطرحهم

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{2h^3}{3!}f'''(x) + \frac{2h^5}{5!}f^{(5)}(c)$$

$$-8\,(f(x+h) - f(x-h)) = 16hf'(x) + \frac{16h^3}{3!}f'''(x) + \frac{16h^5}{5!}f^{(5)}(c)$$

2) -- $f(x+2h) - f(x-2h) = 4hf'(x) + \frac{16h^3}{3!}f'''(x) + \frac{64h^5}{5!}f^{(5)}(c)$

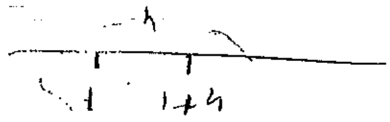
نطرح (١) من (٢)

$$-f(x+2h) + 8\,\{f(x+h) - 8f(x+h) + f(x-2h) = 12f'(x) - \frac{48h^5}{120}f(c)$$

$$\underbrace{\frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h}}_{F2} + \underbrace{\frac{1}{30}h^4 f^{(6)}(c)}_{E_{trac}(f,h)} = f'(x)$$

- $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$f'(x) \cong \frac{f(x+h) - f(x)}{h}$$



when h is smaller we get best estimation for $f'(x)$. ?

Example

$f(x) = e^x$

$f'(x) = e^x$

$f'(1) = e$

$f'(1) \cong \frac{f(1+h) - f(1)}{h} = \frac{e^{1+h} - e^h}{h} \xrightarrow{??} e$

هل هنا صحيح يقترب
من e عندما $h \to 0$

h	$D_n = e^{1+h} - e/h$
0.1	2.858841960
0.01	2.731918700
0.001	2.719642000
0.0001	2.718420000
10^{-5}	2.718300000 $\rightarrow$ the best h
10^{-6}	2.719000000
10^{-7}	$\vdots$
10^{-10}	0000000

- Notation

$$f(x+h) = y_1 + e_1$$
$$f(x-h) = y_{-1} + e_{-1}$$
$$\vdots$$
$$f(x+kh) = y_k + e_k$$

$$f(x+h) = \cos(0.81) = \underbrace{0.689498433}_{y_1} \text{ (is not exact (have error))}$$
$$= y_1 + e_1$$
$$|e_1| < 0.5 * 10^{-10}$$
$$< 0.5 * 10^{-9}$$

$$F_1 = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2 f^{(3)}(c)}{6}$$
$$= \frac{(y_1 + e_1) - (y_{-1} + e_{-1})}{2h} - \frac{h^2 f^{(3)}(c)}{6}$$
$$= \frac{y_1 - y_{-1}}{2h} + \underbrace{\frac{e_1 - e_{-1}}{2h}}_{\substack{\text{Round off error} \\ E_{round}(f,h)}} - \underbrace{\frac{h^2 f^{(3)}(c)}{6}}_{\substack{\text{trancation error} \\ E_{tranc}(f,h)}}$$

Total error $= E_{tot}(f,h) = E_{round}(f,h) + E_{tranc}(f,h)$

$$= \frac{e_1 - e_{-1}}{2h} - \frac{h^2 f^{(3)}(c)}{6}$$

$$|E_{tot}(f,h)| = \left|\frac{e_1 - e_{-1}}{2h}\right| + \left|\frac{h^2 f^{(3)}(c)}{6}\right| \quad \text{if } |e_k| < \epsilon$$

$$\leq \underbrace{\frac{2\epsilon}{2h} + \frac{h^2 M_3}{6}}_{g(n)} \qquad M_3 = \max |f^{(3)}(x)|$$

$$g(n) = \frac{\epsilon}{h} + \frac{h^2}{6} M$$

$$g'(h) = -\frac{\epsilon}{h^2} + \frac{h}{3} M = 0$$

$$\frac{h}{3} M = \frac{\epsilon}{h^2}$$

$$h^3 = \frac{3\epsilon}{M}$$

$$h = \left(\frac{3\epsilon}{M}\right)^{1/3} \quad \text{best } h$$

- $f(x) = \cos x$, $\epsilon = 0.5 * 10^{-9}$

$$h = \left(\frac{3 * 0.5 * 10^{-9}}{1}\right)^{1/3} = 0.001144714$$

max for $|f^{(3)}(x)|$

$h = 0.001$ best h

- Find best h for F2.

F2

$$F'(x) = \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h} + \frac{h^4 f^{(5)}(c)}{30}$$

$$E_{tot} = \frac{-e_2 + 8e_1 - 8e_{-1} + e_{-2}}{12h} + \frac{h^4 f^{(5)}(c)}{30}$$

$$|E(f,h)| \leq \frac{|e_2| + 8|e_1| + 8|e_{-1}| + |e_{-2}|}{12h} + \frac{h^4 M}{30} \qquad M = \max_{a \leq x \leq b} |f^{(5)}(x)|$$

$$\leq \frac{18\epsilon}{12h} + \frac{h^4 M}{30} = \frac{3\epsilon}{2h} + \frac{h^4 M}{30} = g(n) \qquad |e_n| < \epsilon$$

$$\therefore g'(h) = \frac{-3\epsilon}{2h^2} + \frac{4h^3 M}{30} = 0$$

$$\frac{2}{15} h^3 M = \frac{3\epsilon}{2h^2}$$

$$h^5 = \frac{45\epsilon}{4M}$$

$$\text{Optimal } h = \left(\frac{45\epsilon}{4M}\right)^{1/5}$$

$- f(x) = \cos x$

$\epsilon = 0.5 * 10^{-9}$ $\quad\quad M = 1$

$$h = \left(\frac{45 * 0.5 * 10^{-9}}{4 * 1}\right) = 0.022 \ldots$$

$$\text{Optimal } h = 0.01$$

Section 6.2

High order derivations

- $O(h^2)$

1. $f''(x) \cong \dfrac{f_1 - 2f_0 + f_{-1}}{h^2}$ $\qquad f_k = f(x+kh)$

2. $f'''(x) \cong \dfrac{f_2 - 2f_1 + 2f_{-1} - f_{-2}}{2h^3}$

$\vdots$

- $O(h^4)$

1. $f''(x) \cong \dfrac{-f_2 + 16f_1 - 30f_0 + 16f_{-1} - f_{-2}}{12h^2}$
2. $f'''(x) \cong \ldots$
3. $f''''(x) \cong \ldots$

حسب المعلومات نحدد ماذا نستخدم.

$$f_1 = f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(c)$$

$$f_{-1} = f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(c)$$

$$f_1 + f_{-1} = 2f_0 + h^2 f''(x) + \frac{h^4 f^{(4)}(c)}{12} \qquad \text{where } f_0 = f(x)$$

$$\underbrace{\frac{f_1 - 2f_0 + f_{-1}}{h^2}}_{\text{Formula}} - \underbrace{\frac{h^2 f^{(4)}(c)}{12}}_{\text{trancation error}} = f''(x)$$

Best h:-

$$E_{tot}(f,h) = E_{round}(f,h) + E_{tranc}(f,h)$$

$$E_{tot}(f,h) = \frac{e_1 - 2e_0 + e_{-1}}{h^2} - \frac{h^2 f^{(4)}(c)}{12}$$

if $|e_n| < \epsilon$, and $M = \max_{a \leq x \leq b} |f^{(4)}(x)|$

then

$$|E_{tot}| \leq \frac{4\epsilon}{h^2} + \frac{h^2 M}{12} = g(h)$$

$$g'(h) = \frac{-8\epsilon}{h^3} + \frac{hM}{6} = 0$$

$$\frac{hM}{6} = \frac{8\epsilon}{h^3}$$

$$h^4 = \frac{48\epsilon}{M}$$

$$h = \left(\frac{48\epsilon}{M}\right)^{1/4}$$

- Example

$f(x) = \cos x$

$f''(0.8)$ using $h = 0.01$.

$$f''(0.8) \approx \frac{\cos(0.81) - 2\cos(0.8) + \cos(0.79)}{(0.01)^2} \cong -0.69669000\epsilon$$

$$\text{Exact} = -\cos(0.8) = -0.697067.$$

EXAMple

t	d
0.0	0.989992
0.1	0.999135
0.2	0.998295
0.3	0.987480

$V(0) = ??$

$V(0.1) \cong ✓$

$V(0.2) = ✓$

$V(0.3) = ??$

$a(0) = ??$

$$a(0.1) = \frac{d(0.2) - 2d(0.1) + d(0.0)}{(0.1)^2}$$

$$= \frac{0.998295 - 2(0.999135) + 0.989992}{0.01}$$

$$a(0.2) = ✓ = \frac{d(0.3) - 2d(0.2) + d(0.1)}{(0.1)^2}$$

- FoRWard differance Formula's of $O(h^2)$

$$f'(x) \cong \frac{-3f_0 + 4f_1 - f_2}{2h}$$

$$f''(x) \cong \frac{2f_0 - 5f_1 + 4f_2 - f_3}{h^2}$$

- Backward differance Formula's of $O(h^2)$

$$f'(x_0) \cong \frac{3f_0 + 4f_{-1} + f_{-2}}{2h}$$

$$f''(x_0) \cong \frac{2f_0 - 5f_{-1} + 4f_{-2} - f_{-3}}{h^2}$$

$$f'(x_2) = \frac{3f_2 + 4f_1 + f_0}{2h}$$

$$f''(x_2) \cong \frac{2f_2 - 5f_1 + 4f_0 - f_{-1}}{h^2}$$

Example

$f(x) = \cos x$

$h = 0.01$

- Forward

$$f'(0.8) = \frac{-3\cos(0.8) + 4\cos(0.81) - \cos(0.82)}{2(0.01)}$$

- Backward

$$f'(0.8) = \frac{3\cos(0.8) - 4\cos(0.79) + \cos(0.78)}{2(0.01)}$$

- Forward

$$f''(0.8) = \frac{2\cos(0.8) - 5\cos(0.81) + 4\cos(0.82) - \cos(0.83)}{(0.01)^2}$$

- Backward

$$f''(0.8) = \frac{2\cos(0.8) - 5\cos(0.79) + 4\cos(0.78) - \cos(0.77)}{(0.01)^2}$$

- Using the table

t	d
0.0	0.989992
0.1	0.999135
0.2	0.998295
0.3	0.998...

$$V(0) = \frac{-3d(0) + 4d(0.1) - d(0.2)}{2(0.1)}$$

يمكن استخدام Forward + central

$$V(0.1) = \frac{-3d(0.1) + 4d(0.2) - d(0.3)}{2(0.1)}$$

$V(0.2)$ يمكن استخدام central Forward

$$V(0.3) = \frac{3d(0.3) - 4d(0.2) + d(0.1)}{2(0.1)}$$ backward

$$a(0) \cong \frac{2d(0) - 5d(0.1) + 4d(0.2) - d(0.3)}{(0.1)^2}$$

$a(0.1) =$ Central

$a(0.2) =$ Central

$$a(0.3) = \frac{2d(0.3) - 5d(0.2) + 4d(0.1) - d(0)}{(0.1)^2}$$

- derive $f'(x_2) = \frac{3f_2 - 4f_1 + f_0}{2h}$ $\quad O(h^2)$

$$f_1 = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(c)$$

$$f_2 = f(x) + 2hf'(x) + 2h^2 f''(x) + \frac{8}{6} h^3 f'''(c)$$

$$3f_2 = 3f(x) + 6hf'(x) + 6h^2 f''(x) + 4h^3 f'''(c)$$

$$4f_1 = 4f(x) + 4hf'(x) + 2h^2 f''(x) + \frac{2}{3} h^3 f'''(c)$$

$$3f_2 - 4f_1 = -f(x) + 2hf'(x)$$

$$-f_{-1} = f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3 f'''(c)}{3!}$$

$$f_{-2} = f(x-2h) = f(x) - 2h f'(x) + 2h^2 f''(x) - \frac{8h^3 f'''(c)}{6} \qquad f(x) = f_0$$

$$-4f_{-1} = -4f_0 + 4hf'(x) - 2h^2 f''(x) + \frac{4h^3 f'''(c)}{6}$$

$$-4f_{-1} + f_{-2} = -3f_0 + 2hf'(x) + 0 - \frac{4}{6} h^3 f'''(c)$$

$$\underbrace{\frac{3f_0 - 4f_{-1} + f_{-2}}{2h}}_{\text{Formula}} + \underbrace{\frac{2}{3} h^2 f'''(c)}_{\text{Error}} = f'(x)$$