Example 2

t / x	$V(t)$ / $f(x)$
1	20.1
2	22.5
3	25.6
4	28.9

$$\int_1^4 f(x)\,dx = ??$$

by simpson 3/8 Rule (because we have 4 points

$$\int_1^4 f(x)\,dx = \frac{3(1)}{8}\left(f(1)+3f(2)+3f(3)+f(4)\right)$$

or by trapzoidal

$$\int_1^4 f(x)\,dx = \frac{4}{2}\left(f(1)+f(4)\right)$$

Example

Derive trapzoidal error or Rule.

we use $P_1(x)$ and $\int_{x_0}^{x_1} f(x)\,dx \approx \int_{x_0}^{x_1} P_1(x)\,dx$

$$= \int_{x_0}^{x_1} \left(\frac{x-x_1}{x_0-x_1} y_0 + \frac{x-x_0}{x_1-x_0} y_1\right) dx \qquad x = x_0+ht \qquad dx = h\,dt$$

$$x_k = x_0 + kh$$

$$= \int_0^1 \left(\frac{h(t-1)}{-h} y_0 + \frac{ht}{h} y_1\right) h\,dt = -y_0 h \int_0^1 (t-1)\,dt + h y_1 \int_0^1 t\,dt$$

$$= \frac{y_0 h}{2} + \frac{h y_1}{2} = \frac{h}{2}(y_0+y_1)$$

$$\text{Error} = \int_{x_0}^{x_1} E_1(x) = \int_{x_0}^{x_1} \frac{(x-x_0)(x-x_1)}{2!} f^{(2)}(c)\,dx$$

$$= \int_0^1 h(t)\,h(t-1)\,\frac{f^{(2)}(c)}{2}\,h\,dt = \frac{f^{(2)}(c)}{2} \int_0^1 h(t)\,h(t-1)\,h\,dt$$

$$= \frac{h^3 f^{(2)}(c)}{2} \int_0^1 (t^2-t)\,dt = \frac{-h^3 f^{(2)}(c)}{12}$$

Def:-

The degree of precision or accuracy of a quadrature Formula is the largest positive integer is such that the Formula is exact For x^k, $k = 0, 1, 2, \dots$

Examples:-

Find the degree of accuracy of simpson's method:-

$\frac{h}{3}(f_0 + 4f_1 + f_2)$

$\underset{x_0}{0} \quad \underset{x_1}{1} \quad \underset{x_2}{2}$

F(x)	Formula	Exact	Error
$x^0 = 1$	$\frac{1}{3}(f(0) + 4f(1) + f(2))$ $\frac{1}{3}(1 + 4(1) + 1) = 2$	$\int_0^2 1\,dx = 2$	0
$x^1 = x$	$\frac{1}{3}(0 + 4(1) + 2) = 2$	$\int_0^2 x\,dx = \frac{x^2}{2}\Big\|_0^2 = 2$	0
x^2	$\frac{1}{3}(0 + 4(1) + 4) = \frac{8}{3}$	$\int_0^2 x^2\,dx = \frac{x^3}{3}\Big\|_0^2 = \frac{8}{3}$	0
x^3	$\frac{1}{3}(0 + 4(1) + 8) = 4$	$\int_0^2 x^3\,dx = \frac{x^4}{4}\Big\|_0^2 = 4$	0
x^4	$\frac{1}{3}(0 + 4(1) + 16) = \frac{20}{3}$	$\int_0^2 x^4\,dx = \frac{x^5}{5}\Big\|_0^2 = \frac{32}{5}$	$\frac{32}{5} - \frac{20}{3} \neq 0$

degree of accuracy of simpson's is 3

Note:-

degree of accurarcy of trapzoidal is 1

degree of accuracy of simpson $\frac{1}{3}$ is 3

degree of accuracy of simpson 3/8 is 3

Theorey:

$$\text{Error} = k f^{(n+1)}(\xi)$$, k is the degree of accuracy

Example

For simpson's $\frac{1}{3}$ method

$$\text{Error} = k f^{(4)}(\xi)$$

$f(x) = x^4$

$f'(x) = 4x^3$

$f''(x) = 12x^2$

$f'''(x) = 24x$

$f^{(4)}(x) = 24$

$$\text{Error} = k f^{(4)}(\xi)$$

$$\frac{32}{5} - \frac{20}{3} = k(24)$$

$$\frac{96-100}{15} = 24k$$

$$k = \frac{-\cancel{4}^{1}}{15 \times \cancel{24}_{6}} = -\frac{1}{90}$$

- if $f(x) = (x-x_0)^4$

$$\int_{x_0}^{x_2} f(x)\,dx$$

$$\text{Error} = \text{Exact} - \text{Formula}$$

$$\text{Exact} = \int_{x_0}^{x_2} F(x)\,dx = \int_{x_0}^{x_2} (x-x_0)^4\,dx = \left.\frac{(x-x_0)^5}{5}\right|_{x_0}^{x_2} = \frac{32h^5}{5}$$

$$\text{Formula} = \frac{h}{3}\left[f(x_0) + 4f(x+h) + f(x_2)\right] = \frac{20}{3}h^5$$

$$\text{Error} = -\frac{1}{90}h^5$$

$\frac{7}{6.1}$

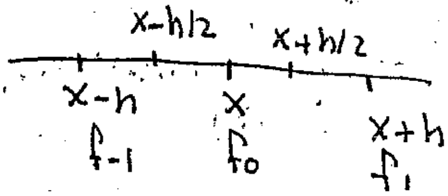
$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{2h}$$

$$f_x(x,y) = \frac{f(x+h,y) - f(x-h,y)}{2h}$$

$$f_y(x,y) = \frac{f(x,y+h) - f(x,y-h)}{2h}$$

$\frac{10}{6.2}$

$$f'\left(x+\frac{h}{2}\right) = \frac{f_1 - f_0}{h}$$



$$f'\left(x-\frac{h}{2}\right) = \frac{f_0 - f_{-1}}{h}$$

$$f''(x) = \left(f'(x)\right)' = \frac{f'(x+h/2) - f'(x-h/2)}{2(h/2)}$$

$$= \frac{\frac{f_1 - f_0}{h} - \frac{f_0 - f_{-1}}{h}}{h}$$

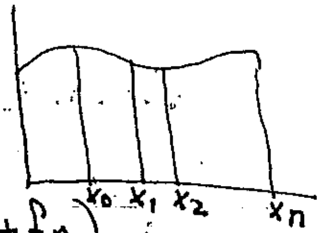
$$f''(x) = \frac{f_1 - 2f_0 + f_{-1}}{h^2}$$

$$f'''(x) = \left(f'(x)\right)'' = \left(f''(x)\right)'$$

7.2 Composite Rules

1. Composite trapozoidal Rule

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$



$$= \frac{h_1}{2}(f_0+f_1) + \frac{h_2}{2}(f_1+f_2) + \cdots + \frac{h_n}{2}(f_{n-1}+f_n)$$

$h_k = h$

$$= \frac{h}{2}(f_0 + 2f_1 + 2f_2 + \cdots + 2f_{n-1} + f_n)$$

$$= \frac{h}{2}\sum_{k=1}^{n}(f_{k-1}+f_k) = T(f,h)$$

EXAMPLE

t	v(t)
0	10
1	12
2	13
3	15

$$D(3) = \frac{1}{2}\left(f(0) + 2f(1) + 2f(2) + f(3)\right)$$

EXAMPLE

t	v(t)
1	10
3	15
4	20
5	21

$$D(5) = \frac{2}{2}\left(f(1)+f(3)\right) + \frac{1}{2}\left(f(3)+f(4)\right) + \frac{1}{2}\left(f(4)+f(5)\right).$$

- Error For Composit trapozoidal.

$$\text{Error} = \frac{-h^3 f^{(2)}(c_1)}{12} - \frac{h^3 f^{(2)}(c_2)}{12} + \cdots - \frac{h^3 f^{(2)}(c_n)}{12}$$

$$= \frac{-h^3}{12}\left(f^{(2)}(c_1) - f^{(2)}(c_2) - \cdots - f^{(2)}(c_n)\right)$$

$$= \frac{-h^3}{12}\left(n f^{(2)}(c)\right) \qquad h = \frac{b-a}{n}$$

$$= -\frac{h^3}{12}\left(\frac{b-a}{n} f^{(2)}(c)\right)$$

$$E_T(f,h) = \frac{-(b-a)\, f^{(2)}(c)\, h^2}{12} \approx O(h^2)$$

- EXAMPLE

Find the number m at step size h so that $|E_T(f,h)| \leq 5 * 10^{-9}$ of the approximation $\int_2^7 \frac{dx}{x} = T(f,h)$

where m is the number of trapozoidal composite

$m = n$

$$|E_T(f,h)| \leq 5 * 10^{-9}$$

$$\frac{(b-a)\, f^{(2)}(c) \left(\frac{b-a}{n}\right)^2}{12} \leq 5 * 10^{-9}$$

- $f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = -\frac{1}{x^2}$

$f''(x) = \frac{2}{x^3}$

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2 لان
الاقتران
هنا قمت اعلى
قيمة له عند 2

$\max |f''(x)| = \frac{2}{8} = \frac{1}{4}$
$2 \leq x \leq 7$

$$\frac{(b-a)\, f^{(2)}(c) \left(\frac{b-a}{h}\right)^2}{12} \leq 5*10^{-9}$$

$$\frac{5\,(0.25)\left(\frac{5}{n}\right)^2}{12} \leq 5*10^{-9}$$

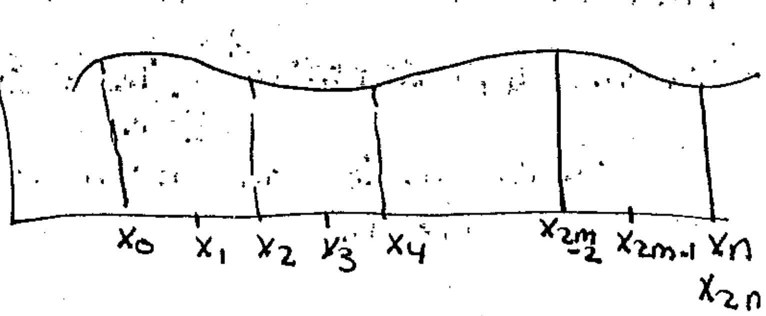
$$n \geq \sqrt{\frac{5*0.25*25}{12*5*10^{-9}}} = 22821.77$$

$$n = 22822$$

$$h = \frac{b-a}{n} = \frac{5}{22822} = 0.000219$$

2. Composite Simpson's $\frac{1}{3}$ Rule

$$\int_{x_0}^{x_n = x_{2m}} f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \int_{x_4}^{x_6} f(x)\,dx + \dots + \int_{x_{2m-2}}^{x_{2m}} f(x)\,dx$$



$n = 2m$

$m = \frac{n}{2}$

$$= \frac{h_1}{3}(f_0 + 4f_1 + f_2) + \frac{h_2}{3}(f_2 + 4f_3 + f_4) + \dots + \frac{h_n}{3}(f_{2m-2} + 4f_{2m-1} \dots$$

$h_k = h$

$$= \frac{h}{3}\left(f_0 + 4f_1 + f_2 + f_2 + 4f_3 + 2f_4 + \dots + 2f_{2m-2} + 4f_{2m-1} + f_{2m}\right)$$

$$= \frac{h}{3} \sum_{k=1}^{m} \left(f_{2k-2} + 4f_{2k-1} + f_{2k}\right) = S(f,h)$$

- Error For Composit Simpson

$$E_S(f,h) = -\frac{h^5 f^{(4)}(c_1)}{90} - \frac{h^5 f^{(4)}(c_2)}{90} - \dots - \frac{h^5 f^{(4)}(c_m)}{90}$$

$$= -\frac{h^5}{90}\left(f^{(4)}(c_1) + f^{(4)}(c_2) + \dots + f^{(4)}(c_m)\right)$$

$$= -\frac{h^5}{90}\left(m f^{(4)}(c)\right)$$

$$= \frac{-h^5}{90}\left(\frac{b-a}{2h} f^{(4)}(c)\right)$$

$$M = \frac{b-a}{2h}$$

$$= \frac{-(b-a)h^4 f^{(4)}(c)}{180}$$

$$\approx Ch^4$$

EXAMPLE

Find the number $\overset{n}{\underline{m}}$ and step size h that $|E_S(f,h)| \leq 5 * 10^{-9}$ of the approximation $\int_2^7 \frac{dx}{x} = S(f,h)$

$$|E_S(f,h)| \leq 5 * 10^{-9}$$

$$\left|\frac{b-a\left(\frac{b-a}{2m}\right)^4 f^{(4)}(c)}{180}\right| < 5*10^{-9}$$

$$\frac{5\cdot\left(\frac{5}{2m}\right)^4 0.75}{180} < 5*10^{-9}$$

$$m > \sqrt[4]{\frac{5*5^4*0.75}{2^4*180*5*10^{-9}}} = 112.9$$

$f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = \frac{-1}{x^2}$

$f''(x) = \frac{2}{x^3}$

$f'''(x) = \frac{-6}{x^4}$

$f^{(4)}(x) = \frac{24}{x^5}$

$\max_{2 \leq x \leq 7} |f^{(4)}(x)| = \left|\frac{24}{x^5}\right|$

$= \frac{24}{32} = 0.$

7.5 Gauss - legendre Formulas

2 points Formula

$$\int_a^b F(x)\,dx \cong \omega_1 f(x_1) + \omega_2 f(x_2)$$

عندنا ٤ مجاهيل بدنا ٤ معادلات .

we assume degree of presecion 3

1. $E(\phi)=0 \rightarrow \int_{-1}^{1} 1\,dx = 2 \rightarrow$ Formula $= \omega_1(1)+\omega_2(1) \rightarrow \omega_1+\omega_2=2$
2. $E(x)=0 \rightarrow \int_{-1}^{1} x\,dx = 0 \rightarrow$ Formula $= \omega_1 x_1 + \omega_2 x_2 \rightarrow \omega_1 x_1 + \omega_2 x_2 = 0$
3. $E(x^2)=0 \rightarrow \int_{-1}^{1} x^2\,dx = 2/3 \rightarrow$ Formula $= \omega_1 x_1^2 + \omega_2 x_2^2 \rightarrow \omega_1 x_1^2 + \omega_2 x_2^2 = 2/3$
4. $E(x^3)=0 \rightarrow \int_{-1}^{1} x^3\,dx = 0 \rightarrow$ Formula $= \omega_1 x_1^3 + \omega_2 x_2^3 \rightarrow \omega_1 x_1^3 + \omega_2 x_2^3 = 0$

Exact $= \int_{-1}^{1} f(x)\,dx$

Formula $= \omega_1 f(x_1) + \omega_2 f(x_2)$

Exact $=$ Formula

حل المعادلات

$$\omega_1 x_1^3 = -\omega_2 x_2^3$$
$$\omega_1 x_1 = -\omega_2 x_2$$
$$x_1^2 = x_2^2$$

مرفوضة $x_1 = x_2$ or $\boxed{x_1 = -x_2}$

$$\omega_1 x_1 + \omega_2(-x_1) = 0$$
$$x_1(\omega_1 - \omega_2) = 0$$
$$\omega_1 - \omega_2 = 0$$
$$\boxed{\omega_1 = \omega_2}$$

$$\omega_1 + \omega_2 = 2$$
$$2\omega_1 = 2$$
$$\boxed{\omega_1 = 1}, \boxed{\omega_2 = 1}$$

$$1(x_1)^2 + 1(x_1)^2 = \frac{2}{3}$$
$$2x_1^2 = \frac{2}{3}$$
$$x_1^2 = \frac{1}{3} \rightarrow \boxed{x_1 = \pm\frac{1}{\sqrt{3}}}$$

$$\int_{-1}^{1} f(x)\,dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$
$$= G_2(f)$$

Gauss - lagendre 2 points Formula

EXAMPLE

Estimate $\int_{-1}^{1} \frac{1}{x+2}\,dx$ using

$-\frac{1}{(x+2)} = \cancel{\frac{1}{x}}$

$f\ (x+2)^{-1}$

1. $T(f,2) = \frac{2}{2}\left(f(-1)+f(1)\right) = 1.3333$ $\quad f' = -(x+2)^{-2}$

2. $S(f,2) = \frac{1}{3}\left(f(-1)+4f(0)+f(1)\right) = 1.11111$ $\quad f'' = 2(x+2)^{-3}$

3. $G_2(f) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$ $\quad f''' = -6(x+3)^{-4}$

$f^{(4)} = 24(x+3)^{-5}$

$= 1.09091$

$E = \text{Exact} - \text{Formula} = \frac{24}{(x+2)^5}$

Exact $= 1.09861$

$= 1.09091 - 1.09861$

The Error For $G_2(F) = \frac{f^{(4)}(c)}{135}$ $\quad = K f^{(4)}(c)$

$\text{Error} = k f^{(4)}(c)$

$f(x) = x^4$

- Gauss legendre 3 points Formula should have 5 degree of accuracy

$$\int_{-1}^{1} f(x)\,dx = w_1 f(x_1) + w_2 f(x_2) + w_2 f(x_3)$$

$$G_3(f) = \frac{5}{9} f\left(-\sqrt{\tfrac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\tfrac{3}{5}}\right)$$

$$\text{Error} = \frac{f^{(6)}(c)}{15.750}$$

- Gauss - legendre Formulas

- Gauss legendre two pts Formula

$G_2(F) = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$ with error $= \frac{1}{135} f^{(4)}(c)$

has 3 - degree of acuracy

- Gauss legender three pts Formula

$$G_3(F) = \frac{5}{9} f\left(-\sqrt{\tfrac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\tfrac{3}{5}}\right) \text{ with error} = \frac{1}{15750} f^{(6)}$$

has 5 - degree of accuracy

$$G_n(F) = w_1 f(x_1) + w_2 f(x_2) + \cdots + w_n f(x_n)$$

has $2n-1$ degree of accuracy

- $G_8(F) \rightarrow \text{error} = \dfrac{f^{(16)}(c)\, 2^{17} (8!)^4}{(16!)^3\, 17!}$ Very accurate formula

- Theorem

if x_n's are the pts of Gauss legnder Formula, and w_n's are the weights in $[-1,1]$ to apply the Formula on $[a,b]$ we use the transformation.

$$t = \frac{a+b}{2} + \frac{b-a}{2} x \quad : \quad [-1,1] \longrightarrow [a,b]$$

$$dt = \frac{b-a}{2} dx$$

$$\int_a^b f(t)\, dt = \frac{b-a}{2} \sum_{k=1}^{n} w_k f\left(\frac{a+b}{2} + \frac{b-a}{2} x\right)$$

$$= \frac{b-a}{2} \left(w_1 f\left(\frac{a+b}{2} + \frac{b-a}{2} x_1\right) + w_2 f\left(\frac{a+b}{2} + \frac{b-a}{2} x_2\right) + \cdots \right.$$

EXAMPLE

Use $G_3(F)$ to estimate $\int_1^5 \frac{1}{t}\,dt$

$$G_3(F) = 2\left[\frac{5}{9} f\left(3+2\left(-\sqrt{\tfrac{3}{5}}\right)\right) + \frac{8}{9} f\left(3+2(0)\right) + \frac{5}{9} f\left(3+2\sqrt{\tfrac{3}{5}}\right)\right]$$

$$= 1.602694$$

- Gauss legendre Formula are very accurate.