Chapter Two

Solving non linear equations $f(x)=0$

* In this chapter we will learn methods used to solve non linear equations numerically.

Exp ① solve $\sqrt{x}\, e^{\cos x} = 5 \Rightarrow \underbrace{\sqrt{x}\, e^{\cos x} - 5}_{f(x)} = 0$

② solve $e^{x} = x \Rightarrow \underbrace{e^{x} - x}_{f(x)} = 0$

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* Numerical Methods to solve $f(x)=0$:

1. Fixed Point Iteration (FPI)
2. Bisection Method
3. False Position Method (or Regula Falsi Method)
4. Newton - Raphson Method
5. Secant Method

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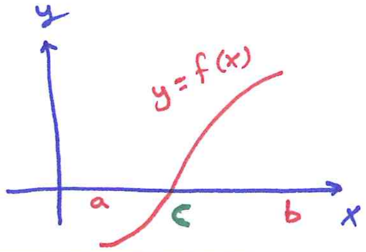
2 and 3 are also called Bracketing Methods for locating a roots

Ch2: Solution of Nonlinear Equations $f(x)=0$

- Let $f(x)=0$
- Assume $\exists\ c \in (a,b)$ s.t $f(c)=0$
- How to estimate c ?



2.1 Iteration for Solving $x=g(x)$

Def Iteration is a repeated process until an answer is achieved.

- Iteration is used to find roots of equations, solution of linear and nonlinear systems of equations, solutions of differential equations, ---
- To build an iteration $\{P_k\}$, we need a rule or function $g(x)$ and a starting point P_0.

Def (Fixed Point)

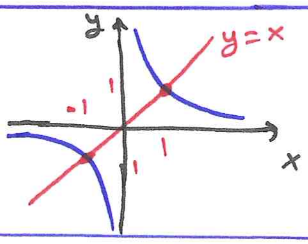
- A fixed point of a function $g(x)$ is a real number P s.t

$$P = g(P)$$

- That is, the fixed points of $y=g(x)$ are the points of intersection of $y=g(x)$ and $y=x$

Exp ① $g(x)=\frac{1}{x}$ $\Rightarrow$ fixed points are $1, -1$

since $P=g(P) \Leftrightarrow P=\frac{1}{P} \Leftrightarrow P^2=1$
$\Leftrightarrow P=\pm 1$



② $g(x)=x \Rightarrow$ all points are fixed points

③ $g(x)=x+1 \Rightarrow$ No fixed points

④ $g(x)=x^3 \Leftrightarrow x=x^3 \Leftrightarrow x(x^2-1)=0 \Leftrightarrow x=0,1,-1$

⑤ $g(x)=\cos x \Leftrightarrow x=\cos x$ "harder"

Def (Fixed Point Iteration)

- The iteration $p_{n+1} = g(p_n)$, $n = 0, 1, 2, \ldots$ is called fixed point iteration.
- That is,

$$\begin{aligned} &p_0 \quad \text{starting value} \\ p_1 &= g(p_0) \\ p_2 &= g(p_1) \\ &\vdots \\ p_{k+1} &= g(p_k) \\ &\vdots \end{aligned}$$

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* which types of functions $g(x)$ that produce convergent sequence $\{p_k\}$ is our main aim to study in this section.

Th* • Assume g is continuous function.

- Let $\{p_n\}_{n=0}^{\infty}$ be a fixed point iteration.
- If $\{p_n\}_{n=0}^{\infty}$ converges to p " $\lim_{n\to\infty} p_n = p$ ", then p is fixed point of $g(x)$

Proof • Since $\{p_n\}_{n=0}^{\infty}$ is fixed point iteration $\Rightarrow p_{n+1} = g(p_n)$

- If $\lim_{n\to\infty} p_n = p$, then $\lim_{n\to\infty} p_{n+1} = p$.
- Hence,

$$g(p) = g\left(\lim_{n\to\infty} p_n\right) \underset{(g \text{ is cont.})}{=} \lim_{n\to\infty} g(p_n) = \lim_{n\to\infty} p_{n+1} = p$$

Remark • Let $f(x) = x - g(x)$

- To solve $f(x) = 0$, we solve $x = g(x)$ for fixed points.
- That is, to find the roots of $f \Rightarrow$ we find the fixed points of $g(x)$.

Exp $x^2 + 3x - 4 = 0$

- The fixed points are $(x-1)(x+4) = 0 \Leftrightarrow x = 1, -4$
- $g(x)$ can have one of the following forms:

[1] $g_1(x) = \dfrac{4 - x^2}{3}$ • $\Rightarrow$ if $p_0 = 3$ then

3 digits

$p_1 = -1.67 \Rightarrow p_2 = 0.403 \Rightarrow p_3 = 1.28 \cdots \Rightarrow p_{13} = 1.005$

which converges to the fixed point 1

• $\Rightarrow$ if $p_0 = -6$ then

$p_1 = -10.7 \Rightarrow p_2 = -36.7 \Rightarrow p_3 = -445 \ldots$ diverges

• Hence, g_1 can find only the fixed point 1

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[2] $g_2(x) = -\sqrt{4 - 3x}$ • $\Rightarrow$ if $p_0 = -6$ then

$p_1 = -4.69 \Rightarrow p_2 = -4.25 \Rightarrow p_3 = -4.10 \Rightarrow \cdots \Rightarrow p_{10} = -4.0000964$

which converges to the fixed point -4

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[3] $g_3(x) = \dfrac{4}{x+3}$ • $\Rightarrow$ if $p_0 = 3$ then

$$\left\{ \begin{array}{l} x(x+3) = 4 \\ x = \dfrac{4}{x+3} = g(x) \end{array} \right.$$

$p_1 = 0.667 \Rightarrow p_2 = 1.09 \Rightarrow p_3 = 0.978 \cdots \Rightarrow p_8 = 1.00002$

which converges to the fixed point 1

• $\Rightarrow$ if $p_0 = -6$ then

$p_1 = -1.33 \Rightarrow p_2 = 2.40$ $p_3 = 0.741$ $p_4 = 1.07$ ... 1.004 which converges to 1

---

[4] $g_4(x) = x^2 + 4x - 4$

$$\left\{ \begin{array}{l} x^2 + 3x - 4 + x = x \\ x^2 + 4x - 4 = x = g(x) \end{array} \right.$$

it will diverge for both cases

Exp. Consider $f(x) = x^2 - 2x - 3$

- clearly the roots of $f(x) = 0 \Leftrightarrow (x-3)(x+1) = 0$ are $x = 3$ and $x = -1$

- Note that we can estimate the roots as follows:

1. $x^2 - 2x - 3 = 0 \Leftrightarrow x^2 = 2x - 3 \Leftrightarrow \boxed{x = \sqrt{2x+3} = g(x)}$

if $P_0 = 4$ then (If $P_0 = \frac{-3}{2}$ then $P_n \to 3$ and we could not find -1)

$P_1 = g(P_0) = g(4) = \sqrt{11} \approx 3.31662$

$P_2 = g(P_1) = g(3.31662) = \sqrt{9.63324} \approx 3.10375$

$P_3 = g(P_2) = g(3.10375) = \sqrt{9.2075} \approx 3.03439$

$P_4 = g(P_3) = g(3.03439) = \sqrt{9.06878} \approx 3.01144$

$\vdots$

$P_n \to 3$ (Note that 3 is fixed point of $g(x) = \sqrt{2x+3}$)

2. $x(x-2) = 3 \Rightarrow x = \frac{3}{x-2} = g(x)$

If $P_0 = 4$ then

$P_1 = g(P_0) = g(4) = \frac{3}{2} = 1.5$

$P_2 = g(P_1) = g(1.5) = \frac{3}{-0.5} = -6$

$P_3 = g(P_2) = g(-6) = \frac{3}{-8} = -0.375$

$P_4 = g(P_3) = g(-0.375) = \frac{3}{-2.375} = -1.26316$

$P_5 = g(P_4) = g(-1.26316) = \frac{3}{-3.26316} = -0.91935$

$P_6 = g(P_5) = g(-0.91935) = -1.02763$

$P_7 = g(P_6) = g(-1.02763) = -0.99087$

$\vdots$

$P_n \to -1$ (Note that -1 is fixed point of $g(x) = \frac{3}{x-2}$)

If $P_0 = -2$ then

$P_1 = \frac{3}{-4} = -0.75$

$P_2 = \frac{3}{-2.75} = -1.091$

$P_3 = \frac{3}{-3.091} = -0.971$

$P_4 = -1.01$

$P_5 = -0.997$

$P_6 = -1.001$

$P_7 = -1$

the fixed point

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3 Note that if we choose

$$2x = x^2 - 3 \Rightarrow x = \frac{x^2-3}{2} = g(x)$$ then we get a divergence sequence and so it will not work.

$p_0 = 4$

$p_1 = g(p_0) = g(4) = 6.5$

$p_2 = g(p_1) = g(6.5) = 19.625$

$p_3 = g(p_2) = g(19.625) = 191.0703$

$\vdots$

This is something related to the slope of the function

so it depends on how to write $g(x)$

Th (Fixed Point Theorem I) - FPTI

Assume $g \in C[a,b]$.

- If $g(x) \in [a,b]$ for all $x \in [a,b]$, then g has a fixed point in $[a,b]$.
- Furthermore, if $|g'(x)| \leq K < 1$ for all $x \in (a,b)$, then g has a unique fixed point in $[a,b]$.

Proof • If $g(a) = a$ or $g(b) = b$, then we are done.

Otherwise, $g(a) \in (a,b]$ and $g(b) \in [a,b)$.

Now let $f(x) = x - g(x) \Rightarrow f$ is continuous and

$$f(a) = a - g(a) < 0 \text{ and}$$
$$f(b) = b - g(b) > 0$$

Hence, by Belzano Theorem $\exists\ p \in (a,b)$ s.t

$$f(p) = 0$$
$$p - g(p) = 0$$
$$p = g(p) \quad \Rightarrow p \text{ is a fixed point.}$$

- (Uniqueness)

Suppose ℓ_1 and ℓ_2 are two fixed points of g

$\Rightarrow$ $g(\ell_1) = \ell_1$ and $g(\ell_2) = \ell_2$

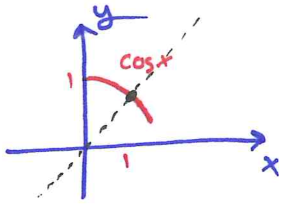
Apply Mean Value Theorem on (ℓ_1, ℓ_2) $\Rightarrow$

$\exists$ $c \in (\ell_1, \ell_2)$ s.t

$$g'(c) = \frac{g(\ell_2) - g(\ell_1)}{\ell_2 - \ell_1} = \frac{\ell_2 - \ell_1}{\ell_2 - \ell_1} = 1 \quad \times$$

Since $|g'(x)| \leq K < 1$. Hence, $\ell_1 = \ell_2$

Exp Show that $g(x) = \cos x$ has a unique fixed point in $[0,1]$



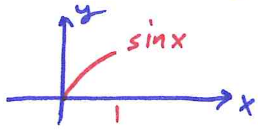
- g is continuous on $[0,1]$ $\Rightarrow$ $g \in C[0,1]$

  g is decreasing on $[0,1]$ with

  $$\cos 1 \leq \cos x \leq \cos 0$$

  Hence, $g(x) \in [\underset{0.5402}{\cos 1}, 1] \subseteq [0,1]$ for all $x \in [0,1]$

  Thus, g has a fixed point in $[0,1]$



- for all $x \in (0,1)$ $\Rightarrow$ $|g'(x)| = |-\sin x| = \sin x \leq \sin 1 < 1$

  Thus, $K = \sin 1 \approx 0.8415 < 1$ $\Rightarrow$ g has a unique fixed point in $[0,1]$.

Question: When the fixed point Iteration (FPI)

$$p_{k+1} = g(p_k) \quad , \quad p_0 \quad , \quad k = 0, 1, 2, \ldots$$

produce a convergence or divergence sequence?

Note FPITII page $\boxed{21}$ does not apply when $g'(p) = 1$.

see Exp* page 24

Th (Fixed Point Iteration Theorem II) - FPIT II

Assume the following :

$g, g' \in C[a,b]$, K is positive constant, $p_0 \in (a,b)$ is the starting point, $g(x) \in [a,b]$ for all $x \in [a,b]$

*• If $|g'(x)| \leq K < 1$ for all $x \in [a,b]$, then the FPI converges to the unique fixed point $p \in [a,b]$.

(In this case p is called attractive fixed point)

• If $|g'(x)| > 1$ for all $x \in [a,b]$, then the FPI diverges.

(In this case p is called repelling fixed point)

Remark* If p is given, then we can replace the above two conditions by :

- If $|g'(p)| < 1$, then FPI converges and p is attracter
- If $|g'(p)| > 1$, then FPI diverges and p is repeller

Proof We will prove *• First we prove that the points $\{p_n\}_{n=0}^{\infty} \in (a,b)$.

• since $p_0 \in (a,b) \Rightarrow$ Apply MVT on (p_0, p)

① ... $|p - p_1| = |g(p) - g(p_0)| = |g'(c_0)(p - p_0)| = |g'(c_0)|\,|p - p_0|$

$\leq K|p - p_0| < |p - p_0|$

• Note that
↳ the assumptions above implies that $\exists_{\text{unique}}\, p \in [a,b]$ by FPTI page 19
↳ $K < 1$
↳ $c_0 \in (a,b)$

• We see from ① that p_1 is closer to p than $p_0 \Rightarrow p_1 \in (a,b)$

• Similar to ① we see that $|p - p_2| < |p - p_1|$ for some $c_1 \in (a,b)$ and so $p_2 \in (a,b)$

• In general, suppose that $p_{n-1} \in (a,b) \Rightarrow$

② ... $|p - p_n| = |g(p) - g(p_{n-1})| = |g'(c_{n-1})|\,|p - p_{n-1}| \leq K|p - p_{n-1}| < |p - p_{n-1}|$

Therefore, $p_n \in (a,b)$ and hence, by induction all the points $\{p_n\}_{n=0}^{\infty} \in (a,b)$.

- Now we need to prove $\lim_{n\to\infty} |p - p_n| = 0$

Claim $|p - p_n| \le K^n |p - p_0|$

Proof by induction when $n = 1 \Rightarrow |p - p_1| \le K |p - p_0|$ See ①

- Assume the claim holds for $n-1 \Rightarrow |p - p_{n-1}| \le K^{n-1} |p - p_0|$
- From ② we have

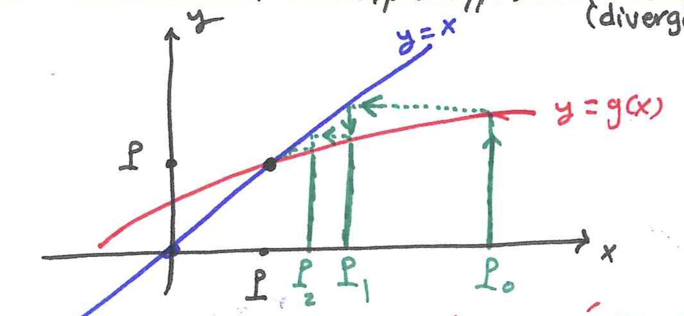
$$|p - p_n| \le K |p - p_{n-1}| \le K K^{n-1} |p - p_0| = K^n |p - p_0|$$

- Since $0 < K < 1 \Rightarrow \lim_{n\to\infty} K^n = 0$. Hence,

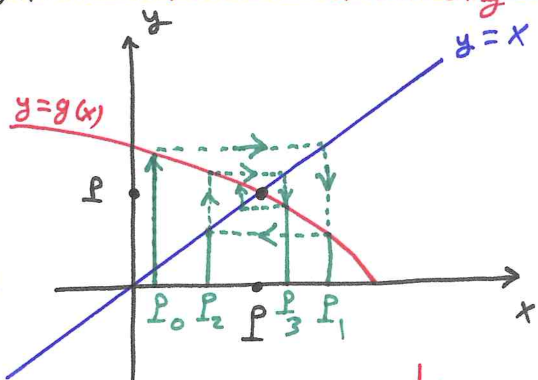
$$0 \le \lim_{n\to\infty} |p - p_n| \le \lim_{n\to\infty} K^n |p - p_0| = 0$$

- Thus, by Sandwich Theorem $\lim_{n\to\infty} |p - p_n| = 0$. So $\lim_{n\to\infty} p_n = p$. By Th* page 17 the iteration $p_n = g(p_{n-1})$ converges to the fixed point p (since it is unique).

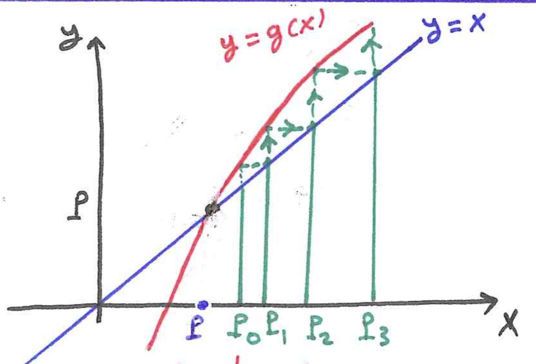
Two simple types of (convergent) (divergent) iteration: Monoton and oscillating.



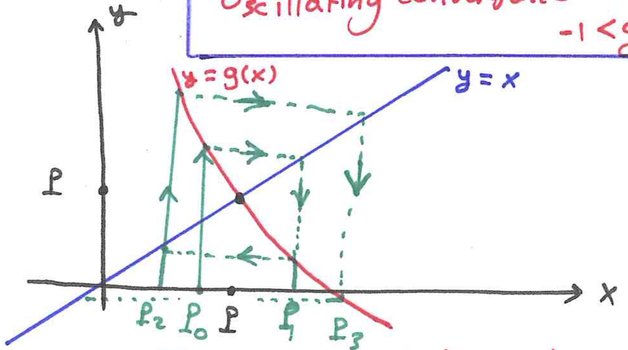
Monoton convergence when $0 < g'(p) < 1$



Oscillating convergence when $-1 < g'(p) < 0$.



Monoton divergence when $1 < g'(p)$



Divergence oscillation when $g'(p) < -1$

Exp Given $g(x) = \sqrt{3x-2}$.
Find the fixed points of $g(x)$ and determine the nature of these fixed points.

- $x = g(x) \iff x = \sqrt{3x-2} \iff x^2 = 3x-2$
  $\iff x^2 - 3x + 2 = 0 \iff (x-1)(x-2) = 0$
  $\iff x = 1$, $x = 2$ are fixed points

- $g'(x) = \dfrac{3}{2\sqrt{3x-2}}$ $\Rightarrow |g'(1)| = \frac{3}{2} > 1 \Rightarrow x = 1$ is repeller
  $\Rightarrow$ FPI diverges

  $\Rightarrow |g'(2)| = \frac{3}{4} < 1 \Rightarrow x = 2$ is attractor
  $\Rightarrow$ FPI converges to 2

Exp $x^2 + 3x - 4 = 0$ has fixed points $x = 1, -4$

1. $g_1(x) = \dfrac{4 - x^2}{3}$

$g_1'(x) = -\frac{2}{3}x \Rightarrow$

- $|g_1'(1)| = |-\frac{2}{3}| < 1 \Rightarrow x = 1$ is attractor

$\Rightarrow$ FPI converges to 1. That is we can use the FPI to find the solution of $x = g(x)$. (see page 17.1)

- $|g_1'(-4)| = |\frac{8}{3}| > 1 \Rightarrow x = -4$ is repeller

$\Rightarrow$ FPI diverges (see page 17.1)

2. $g_2(x) = -\sqrt{4 - 3x} \Rightarrow g_2'(x) = \dfrac{3}{2\sqrt{4-3x}}$

- $g_2'(1) = |\frac{3}{2}| > 1 \Rightarrow x = 1$ is repeller ($p_0 \neq 2$)
  $\Rightarrow$ FPI does not work (see page 17.1)

see also Expy page 17.1

- $g_2'(-4) = |\frac{3}{8}| < 1 \Rightarrow x = -4$ is attractor
  $\Rightarrow$ FPI Converges to -4 (see page 17.1)

Exp Test the convergence of FPI for 1. $g(x) = -4 + 4x - \frac{x^2}{2}$

- $g(x) = x \Leftrightarrow -4 + 3x - \frac{x^2}{2} = 0 \Leftrightarrow x^2 - 6x + 8 = 0 \Leftrightarrow x = 2, 4$ fixed points
- $g'(x) = 4 - x \Rightarrow g'(2) = |4-2| = 2 > 1 \Rightarrow x = 2$ is repeller $\Rightarrow$ FPI div.
  $\Rightarrow g'(4) = |4-4| = 0 < 1 \Rightarrow x = 4$ is attractor $\Rightarrow$ FPI conv.

page 24

2. $g(x) = 2\sqrt{x-1} \Leftrightarrow x = g(x) \Leftrightarrow x^2 = 4(x-1) \Leftrightarrow x^2 - 4x + 4 = 0 \Leftrightarrow$
$x = +2$ fixed point

$g'(x) = \dfrac{1}{\sqrt{x-1}} \Rightarrow g'(2) = |1| = 1$ Test Fails so we build FPI to check (see Exp* page 24).

Exp Let $g(x) = 1 + x - \frac{x^2}{4}$. Can we use the FPI to find the solution of the equation $x = g(x)$? Why?

Solution

• $x = g(x) \Leftrightarrow x = 1 + x - \frac{x^2}{4} \Leftrightarrow x^2 = 4 \Leftrightarrow \boxed{x = \pm 2}$ Fixed Points

• $\boxed{x=2}$ $\Rightarrow g'(x) = 1 - \frac{x}{2} \Rightarrow |g'(2)| = 0 < 1$ K for $x=2$

Hence, by Remark* page 21 $\Rightarrow$ the FPI converges to 2

• So we can use the FPI to find the solution of $x = g(x)$:

$p_0 = 1.6$

$p_1 = g(p_0) = g(1.6) = 1 + 1.6 - \frac{(1.6)^2}{4} = 1.96$

$p_2 = g(p_1) = g(1.96) = 1 + 1.96 - \frac{(1.96)^2}{4} = 1.9996$

$p_3 = g(p_2) = g(1.9996) = 1 + 1.9996 - \frac{(1.9996)^2}{4} = 2$

$\boxed{p_n \to 2}$ attracter fixed point

$p_0 = 2.5$

$p_1 = g(p_0) = g(2.5) = 1 + 2.5 - \frac{(2.5)^2}{4} = 1.9375$

$p_2 = g(p_1) = g(1.9375) = 1.999$

$p_3 = g(p_2) = g(1.999) = 2$

$\boxed{p_n \to 2}$

• $\boxed{x=-2}$ $\Rightarrow |g'(-2)| = \left|1 - \frac{-2}{2}\right| = 2 > 1$. Hence, by Remark*

FPI diverges and $p = -2$ is repeller (repuisive) fixed point.

To see that:

$p_0 = -2.05$

$p_1 = g(p_0) = g(-2.05) = 1 - 2.05 - \frac{(-2.05)^2}{4} = -2.1$

$p_2 = g(p_1) = g(-2.1) = -2.2025$

$p_3 = g(p_2) = g(-2.2025) = -2.4153$

$\vdots$

p_n diverges

Exp* Consider the iteration $p_{n+1} = g(p_n)$ where $g(x) = 2\sqrt{x-1}$ $\boxed{24}$ for $x \geq 1$. Can FPI be used to find the solution of $x = g(x)$?

Fixed point

$$x = g(x) \iff x = 2\sqrt{x-1} \iff x^2 = 4(x-1)$$

$$\iff x^2 - 4x + 4 = 0 \iff (x-2)(x-2) = 0 \iff \boxed{x = 2}$$

FPI

$$g'(x) = \frac{1}{\sqrt{x-1}} \Rightarrow g'(2) = 1$$

$\Rightarrow$ FPIT II does not apply

There are two cases to consider:

$\boxed{\text{Case 1}}$ start with $p_0 = 1.5$

$p_1 = g(p_0) = g(1.5) = 2\sqrt{1.5-1} = 1.4142$

$p_2 = g(p_1) = g(1.4142) = 1.2872$

$p_3 = g(p_2) = g(1.2872) = 1.0718$

$p_4 = g(p_3) = g(1.0718) = 0.5359$ outside the domain of g

p_5 can not be computed

$\boxed{\text{case 2}}$ start with $p_0 = 2.5$

$p_1 = g(p_0) = g(2.5) = 2\sqrt{2.5-1} = 2.4495$

$p_2 = g(p_1) = g(2.4495) = 2.4079$

$p_3 = g(p_2) = g(2.4079) = 2.3731$

$p_4 = g(p_3) = g(2.3731) = 2.3436$

$\vdots$

$\lim_{n\to\infty} p_n = 2$ "slowly" $\Rightarrow p_{1000} = 2.004$

Hence, the FPI converges in this exp for every $p_0 > 2$ and diverges for every $1 \leq p_0 < 2$

Corollary Assume g satisfies conditions of FPIT II.

If p_n is used to approximate the fixed point p, then an upper bounds for the error are

$$|p - p_n| \leq K^n |p - p_0| \quad \text{for all } n \geq 1 \qquad \cdots \text{ (A)}$$

Error

and

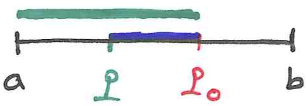
$$|p - p_n| \leq \frac{K^n}{1-K} |p_1 - p_0| \quad \text{for all } n \geq 1 \qquad \cdots \text{ (B)}$$

Proof (HW) see page 25.1

- (A) was the claim in the proof of FPIT II and has been proven. (K is the upper bound of $g'(p)$)

- To prove (B) we use (A) as follows:

$$|p - p_n| \leq K^n |p - p_0| \leq K^n \max\{p_0 - a, b - p_0\}$$



- For $n \geq 1$ we have

$$* \quad |p_{n+1} - p_n| = |g(p_n) - g(p_{n-1})| \leq K |p_n - p_{n-1}| \leq K^n |p_1 - p_0|$$

- Therefore, for $m > n \geq 1$ we have

$$|p_m - p_n| = |p_m - p_{m-1} + p_{m-1} + \cdots + p_{n+1} - p_n|$$

$$\leq |p_m - p_{m-1}| + |p_{m-1} - p_{m-2}| + \cdots + |p_{n+1} - p_n|$$

$$\leq K^{m-1} |p_1 - p_0| + K^{m-2} |p_1 - p_0| + \cdots + K^n |p_1 - p_0| \quad \text{by } *$$

$$= (1 + K + K^2 + \cdots + K^{m-n-1}) K^n |p_1 - p_0| = K^n |p_1 - p_0| \sum_{i=0}^{m-n-1} K^i$$

- Since $\lim_{m \to \infty} p_m = p$ it follows that

$$|p - p_n| = \lim_{m \to \infty} |p_m - p_n| \leq$$

$m > n \Rightarrow K^m < K^n$

since $K < 1$

$$K^n |p_1 - p_0| \sum_{i=0}^{\infty} (K)^i = \frac{K^n}{1-K} |p_1 - p_0|$$

Proof of B in different way :

- $$|p - p_0| = |p - p_1 + p_1 - p_0|$$
$$\leq |p - p_1| + |p_1 - p_0| \quad \text{by Triangle Inequality}$$
$$\leq K\,|p - p_0| + |p_1 - p_0| \quad \text{by } A$$

- Hence, $$|p - p_0| - K|p - p_0| \leq |p_1 - p_0|$$
$$(1-K)\,|p - p_0| \leq |p_1 - p_0|$$
$$|p - p_0| \leq \frac{1}{1-K}\,|p_1 - p_0| \quad \cdots \ast$$

- From A we have
$$|p - p_n| \leq K^n\,|p - p_0|$$
$$\leq K^n \frac{1}{1-K}\,|p_1 - p_0|$$
$$= \frac{K^n}{1-K}\,|p_1 - p_0|$$

Remark
- This Corollary provides stopping crecteria to the FPI
- That is, it tells us the number of iterations n for a given upper bound of the error.

Exp Let $x^3 - x - 5 = 0$

1. show that f(x) has a root on $[0,2]$

$f(x) = x^3 - x - 5$ $\Rightarrow$ $\left.\begin{array}{l} f(0) = -5 < 0 \\ f(2) = 1 > 0 \end{array}\right\}$ $\Rightarrow$ By Bolzano $\exists$ a root on $[1,2]$

f cont. on $[0,2]$

2. Consider $g(x) = \sqrt[3]{x+5}$. Use 5 decimal with $P_0 = 1.5$ to approximate the root of f(x) " FP of g(x)" by finding P_1, P_2, P_3

$$P_1 = g(P_0) = g(1.5000) = \sqrt[3]{6.5000} = 1.8663$$
$$P_2 = g(P_1) = g(1.8663) = \sqrt[3]{6.8663} = 1.9007$$
$$P_3 = g(P_2) = g(1.9007) = \sqrt[3]{6.9007} = 1.9038$$

FP

$P = 1.9042$

True

$P = -0.9521 \pm 1.3113\, i$

Complex roots

3. Find K

K is upper bound for $|g'(x)|$ on $[0,2]$

$$|g'(x)| = \left|\frac{1}{3\sqrt[3]{(x+5)^2}}\right| = \frac{1}{3\sqrt[3]{(x+5)^2}} \leq \frac{1}{3}\frac{1}{\sqrt[3]{25}}$$ since g' decreasing on $[0,2]$

$$|g'(x)| \leq \frac{1}{3}\frac{1}{\sqrt[3]{25}} \leq \frac{1}{3}\frac{1}{\sqrt[3]{8}} = \frac{1}{3}\,\frac{1}{2} = \frac{1}{6} < 1$$

K

$K_{max} = 0.114$

$$K_{min} = \frac{1}{3\sqrt[3]{49}} = 0.0911$$

4. show that g(x) has a unique fixed point in $[0,2]$

- g(x) is cont. on $[0,2]$

g(x) is increasing on $[0,2]$ $\Rightarrow$ $0 \leq \sqrt[3]{5} = g(0) \leq g(x) \leq g(2) = \sqrt[3]{7} \leq \sqrt[3]{8}$

$0 \leq g(x) \leq 2 \quad \forall x \in [0,2]$

Hence, by FPT I g has a fixed point in $[0,2]$

- $|g'(x)| = \dfrac{1}{3\sqrt[3]{(x+5)^2}} \leq K = \dfrac{1}{6} < 1$ since g' is decreasing

g has a unique fixed point in $[0,2]$

Exp Consider $g(x) = \frac{1}{x^3} + 2$ on $[2,3]$

[1] show that $g(x)$ has fixed points in $[2,3]$

- Note that g is continuous on $[2,3]$
- Note also g is decreasing on $[2,3]$

$$2 < 2.037 = g(3) \leq g(x) \leq g(2) = 2.125 < 3 \quad \forall x \in [2,3]$$

- Hence, $g(x) \in [2,3] \quad \forall x \in [2,3]$
- Therefore by FPTI page 19 g has a fixed point in $[2,3]$

[2] Is it unique?

- $|g'(x)| = \left|-\frac{3}{x^4}\right| = \frac{3}{x^4}$ decreasing on $(2,3)$
- $|g'(x)| \leq \frac{3}{(2)^4} = 0.1875 = K_{max} < 1$ for all $x \in (2,3)$
- Hence, g has a unique fixed point in $[2,3]$

[3] show that the FPI converges for every $P_0 \in (2,3)$

- Note that $g, g' \in C[2,3]$ and $g(x) \in [2,3] \ \forall x \in [2,3]$
- Note also $|g'(x)| \leq 0.1875 = K < 1 \quad \forall x \in [2,3]$
- Hence, by FPIT II page 21 the FPI converges $\forall P_0 \in (2,3)$

[4] Using 4 digits rounding, estimate the fixed point of $g(x)$ using $P_0 = 1.5$ and with error less than 0.001.

n	p_n	Upper bound of the error
0	1.500	—
1	2.296	0.1838 > 0.001
2	2.083	0.03445 > 0.001
3	2.111	0.006458 > 0.001
4	2.106	0.001211 > 0.001
5	2.107	0.0002270 < 0.001

$$p_{n+1} = \frac{1}{p_n^3} + 2$$

$$|p - p_n| \le \frac{k^n}{1-k} |p_1 - p_0|$$

we can use the idea of $\boxed{5}$ below here too, by finding n first

see page 36

we can use $|p_n - p_{n-1}|$ as stopping criteria

- $p_1 = \frac{1}{p_0^3} + 2 = \frac{1}{(1.5)^3} + 2 = 2.296$

$p_2 = \frac{1}{p_1^3} + 2 = \frac{1}{(2.296)^3} + 2 = 2.083$

$\vdots$

- $E_1 \le \frac{k}{1-k} |p_1 - p_0| = \frac{0.1875}{1 - 0.1875} |2.296 - 1.500| = 0.1838$

$E_2 \le \frac{k^2}{1-k} |p_1 - p_0| = \frac{(0.1875)^2}{1 - 0.1875} |2.296 - 1.500| = 0.03445$

$\vdots$

$\boxed{5}$ Find number of iterations required to estimate the fixed point of $g(x)$ with accuracy of 10^{-5}.

$$|p - p_n| \le \frac{k^n}{1-k} |p_1 - p_0| < 10^{-5}$$

$$\frac{(0.1875)^n}{1 - 0.1875} |2.296 - 1.500| < 10^{-5}$$

$$\frac{(0.1875)^n}{0.8125} (0.7960) < 10^{-5}$$

$$(0.1875)^n (0.9797) < 10^{-5}$$

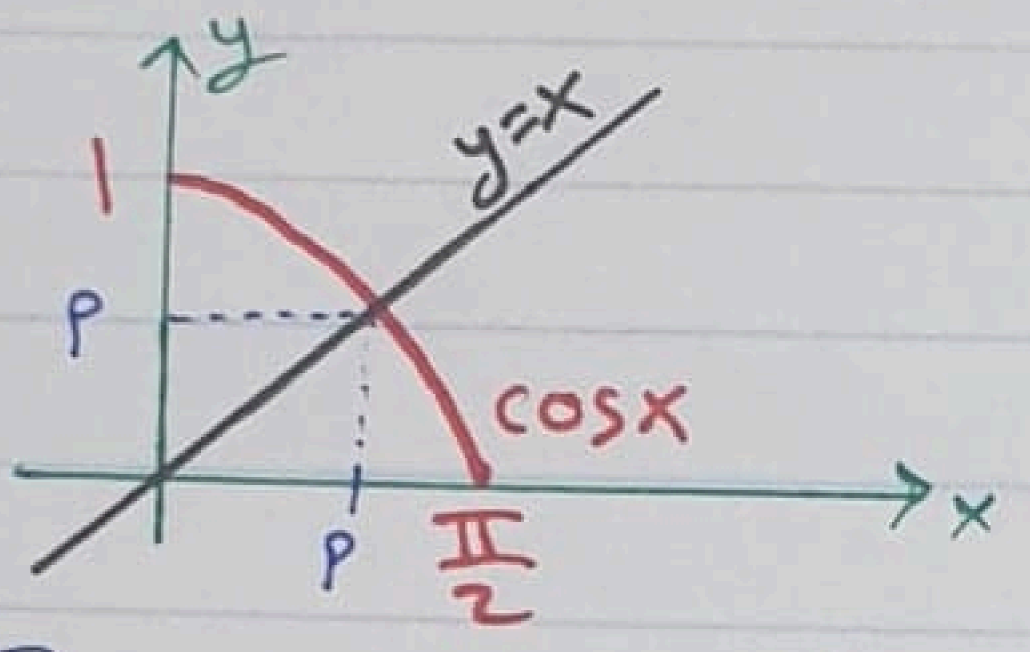
$$(0.1875)^n < 1.021 \times 10^{-5} = 0.00001021 \iff \boxed{n = 7}$$

Exp Let $g(x) = \cos x$ on $[0,1]$. Use 4 digits to

(1) show that g has a unique fixed point in $[0,1]$

- g is cont. on $[0,1]$
- g is decreasing on $[0,1]$ so $g(1) \le g(x) \le g(0)$

$0 \le 0.5403 \le g(x) \le 1$



Hence, $g(x) \in [0,1] \;\forall\, x \in [0,1]$
Therefore, by FPT I g has a fixed point in $[0,1]$

- To prove the fixed point is unique $\Rightarrow$

$$|g'(x)| = |-\sin x| = \sin x \le \sin 1 = 0.8415 = K < 1 \quad \forall\, x \in (0,1)$$

since $\sin x$ is increasing on $[0,1]$

Hence, g has a unique FP in $[0,1]$

---

(2) Find the number of iterations needed to estimate this FP of g with error of magnitude less than 10^{-3} (take $P_0 = 0.5$)

- $P_1 = g(P_0) = \cos(0.5) = 0.8776 \quad \Rightarrow |P_1 - P_0| = 0.3776$

- $1 - K = 1 - 0.8415 = 0.1585$

$$|P - P_n| \le \frac{K^n}{1-K} |P_1 - P_0| < 10^{-3}$$

$$\frac{(0.8415)^n}{0.1585}(0.3776) < 10^{-3}$$

$$(0.8415)^n < \frac{10^{-3}}{2.382} = 0.0004198$$

$$(0.8415)^n < 0.0004198$$

$$n \ge 46$$

2.2 Bracketing Methods for Locating a Root

Def. Assume $f(x)$ is continuous function.

- The number r is called a root of the equation $f(x)=0$ iff $f(r)=0$.
  or zero

Exp $2x^2+3x-2=0$ has two roots $r_1=\frac{1}{2}$ and $r_2=-2$

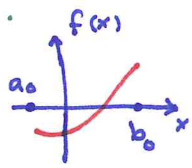
$x^2+\frac{3}{2}x-1=0 \iff (x-\frac{1}{2})(x+2)=0$

In this section we will learn two Bracketing methods for finding a zero of a continuous function $f(x)=0$:

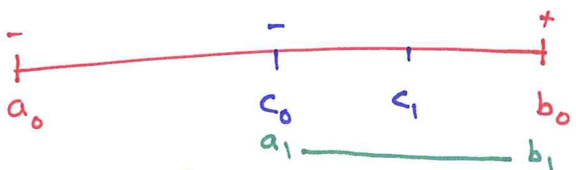
1. Bisection Method of Bolzano
2. False Position Method

Bisection Method of Bolzano

- This method used to solve $f(x)=0$.
- This method depends on Bolzano Theorem.
- Conditions required: • $f \in C[a,b]$ and
  • $f(a)\,f(b)<0$



- Let $[a,b]=[a_0,b_0]$



- First Iteration $c_0=\frac{a_0+b_0}{2}$ is the midpoint
- Now find $f(c_0)$. We have three cases

c_1 is 2^{nd} iteration

If $f(c_0)=0$, then c_0 is the root and we are done.

If $f(c_0)\,f(a_0)<0$, then $[a_1,b_1]=[a_0,c_0]$ and $c_1=\frac{a_1+b_1}{2}=\frac{a_0+c_0}{2}$

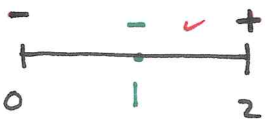
If $f(c_0)\,f(b_0)<0$, then $[a_1,b_1]=[c_0,b_0]$ and $c_1=\frac{a_1+b_1}{2}=\frac{c_0+b_0}{2}$

- In general, the $n+1^{th}$ iteration is $\boxed{c_n = \dfrac{a_n + b_n}{2}}$ in Bisection Method where $n = 0, 1, 2, \dots$

Exp Use the Bisection Method to find the first five iterations c_0, c_1, c_2, c_3, c_4 that estimate the root of $x\sin x - 1 = 0$ on $[0, 2]$. "Use 4 digits"

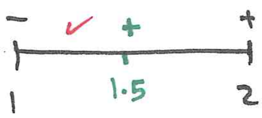
- $f(x) = x\sin x - 1$ $\qquad [a_0, b_0] = [0, 2]$ 1^{st} interval
- $f(0) = -1 < 0$ and $f(2) = 0.8180 > 0$

1^{st} iteration $c_0 = \dfrac{a_0 + b_0}{2} = \dfrac{0+2}{2} = 1$



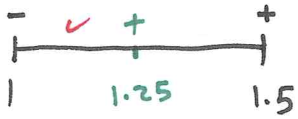
- $f(c_0) = f(1) = -0.1585 < 0 \Rightarrow [a_1, b_1] = [1, 2]$ 2^{nd} interval

2^{nd} iteration $c_1 = \dfrac{a_1 + b_1}{2} = \dfrac{1+2}{2} = 1.5$



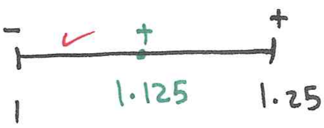
- $f(c_1) = f(1.5) = 0.4962 > 0 \Rightarrow [a_2, b_2] = [1, 1.5]$ 3^{rd} interval

3^{rd} iteration $c_2 = \dfrac{a_2 + b_2}{2} = \dfrac{1 + 1.5}{2} = 1.25$



- $f(c_2) = f(1.25) = 0.1862 > 0 \Rightarrow [a_3, b_3] = [1, 1.25]$ 4^{th} interval

4^{th} iteration $c_3 = \dfrac{a_3 + b_3}{2} = \dfrac{1 + 1.25}{2} = 1.125$



- $f(c_3) = f(1.125) = 0.01505 > 0 \Rightarrow [a_4, b_4] = [1, 1.125]$ 5^{th} interval

5^{th} iteration $c_4 = \dfrac{a_4 + b_4}{2} = \dfrac{1 + 1.125}{2} = 1.063$

- $f(c_4) = f(1.063) = -0.07183$

Note $\sin x$ is evaluated in radians $\Rightarrow$

$f(2) = 2\sin(114.6) - 1 = 2(0.9092) - 1$

$= 1.818 - 1 = 0.8180$

$\pi \rightarrow 3.14$

$? \rightarrow 2$

$? = 0.6369\pi$ ← 180

$= 114.6$

Th (Bisection Theorem)

- Assume that $f \in C[a,b]$,

  $\exists$ a number $r \in [a,b]$ s.t $f(r)=0$,

  $f(a)f(b)<0$ and $\{c_n\}_{n=0}^{\infty}$ represents the sequence of midpoints generated by the bisection method.

- Then an upper bound of the error is

$$|r-c_n| \le \frac{b-a}{2^{n+1}} \quad \text{for } n=0,1,2,\cdots$$

- Furthermore, the sequence $\{c_n\}_{n=0}^{\infty}$ converges to the zero r. That is, $\lim_{n\to\infty} c_n = r$.

Proof

- Let $[a_0, b_0] = [a,b]$

- Note that $b_1 - a_1 = \dfrac{b_0-a_0}{2} = \dfrac{b-a}{2}$

$$b_2 - a_2 = \frac{b_1-a_1}{2} = \frac{b-a}{2^2}$$

see Exp in page 31

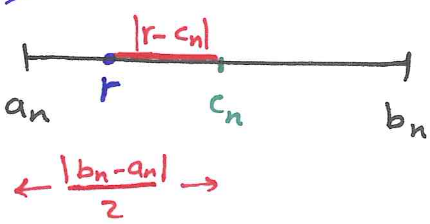
$$b_3 - a_3 = \frac{b_2-a_2}{2} = \frac{b-a}{2^3}$$

$$\vdots$$

$$\boxed{b_n - a_n = \frac{b-a}{2^n}} \longrightarrow *$$

- Now since both the zero r and the midpoint $c_n = \frac{b_n-a_n}{2}$ lie in the interval $[a_n, b_n]$ $\Rightarrow$

$$|r-c_n| \le \frac{b_n-a_n}{2} \quad \text{for all } n$$



- Using $*$ we obtain

$$|r-c_n| \le \frac{b-a}{2^{n+1}} \quad \text{for all } n$$

• To prove the second part, note that

$$0 \le |r - c_n| \le \frac{b-a}{2^{n+1}}$$

and $\lim_{n\to\infty} \frac{b-a}{2^{n+1}} = 0$. Hence, by Sandwich Theorem

$$\lim_{n\to\infty} |r - c_n| = 0 \iff \lim_{n\to\infty} (r - c_n) = 0$$

$$\iff r = \lim_{n\to\infty} c_n$$

Remark: The Bisection Theorem provides a strategy to find the number of iteration for a given accuracy δ:

$$\frac{b-a}{2^{n+1}} < \delta \implies \ln(b-a) - \ln 2^{n+1} < \ln \delta$$

$$\implies \ln(b-a) - \ln\delta < (n+1)\ln 2$$

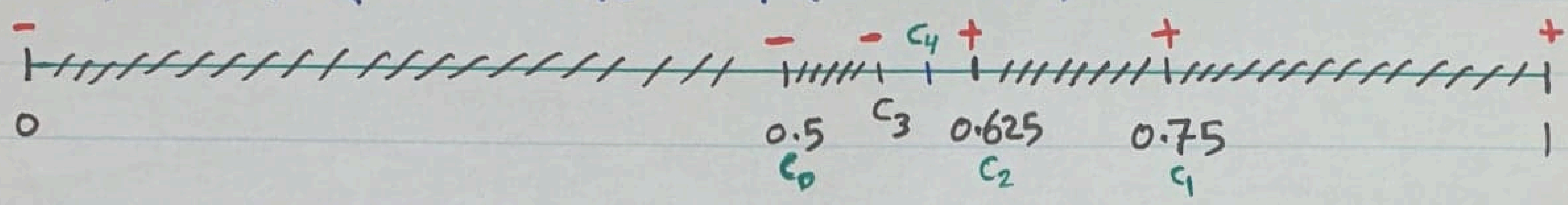
$$\implies \frac{\ln(b-a) - \ln\delta}{\ln 2} < n+1$$

$$\implies \boxed{n > \frac{\ln\left(\frac{b-a}{\delta}\right)}{\ln 2} - 1}$$

Exp Use Bisection method to find c_4 as an estimate to the root of this equation
$e^x - \cos x = 1$ on $[0,1]$

$e^x - \cos x - 1 = 0 \Rightarrow \boxed{f(x) = e^x - \cos x - 1}$ on $[0,1]$ ($a_0 = 0$, $b_0 = 1$)

$f(0) = -1 < 0$ and $f(1) = 1.18 > 0$



$c_0 = \frac{0+1}{2} = 0.5 \Rightarrow f(0.5) = -0.229 \Rightarrow [a_1, b_1] = [0.5, 1]$

$c_1 = \frac{0.5+1}{2} = 0.75 \Rightarrow f(0.75) = 0.385 > 0 \Rightarrow [a_2, b_2] = [0.5, 0.75]$

$c_2 = \frac{0.5+0.75}{2} = 0.625 \Rightarrow f(0.625) = 0.0573 > 0 \Rightarrow [a_3, b_3] = [0.5, 0.625]$

$c_3 = \frac{0.5+0.625}{2} = 0.5625 \Rightarrow f(0.5625) = -0.0909 < 0 \Rightarrow [a_4, b_4] = [0.5625, 0.625]$

speed slow

$c_4 = \frac{0.5625+0.625}{2} = 0.59375 \Rightarrow f(0.59375) = -0.0178$

Exp Suppose the Bisection method is used to find a zero of $f(x)$ on $[2,7]$. How many times this interval must be bisected to guarantee that the approximation c_n has an accuracy of 5×10^{-9}

$a = 2$, $b = 7$, $\delta = 5 \times 10^{-9}$

$n > \frac{\ln\left(\frac{b-a}{\delta}\right)}{\ln 2} - 1 \Rightarrow n > \frac{\ln\left(\frac{7-2}{5 \times 10^{-9}}\right)}{\ln 2} - 1$

$n > \frac{9 \ln 10}{\ln 2} - 1 \Rightarrow n > \frac{20.72}{0.6931} - 1$

$n > 29.89 - 1 \Rightarrow n > 28.89 \Rightarrow n \in \{29, 30, 31, \dots\}$

False Position Method - FPM
(Regula Falsi Method)

- Bisection method converges at slow speed
- However FPM converges faster.
- Conditions to solve $f(x)=0$: $f \in C[a,b]$ and
  $f(a)\, f(b) < 0$

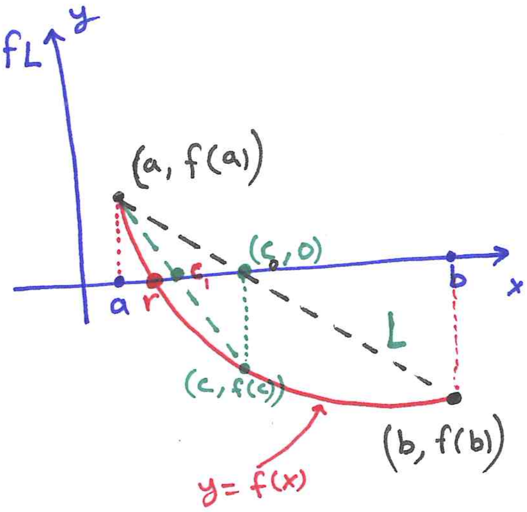
- The Bisection method uses the midpoint of $[a,b]$ as next iterate. A better approximation is to use the point $(c,0)$ where the secant L crosses the x-axis.

- To find c we equalize the slopes of L

$$\frac{f(b)-f(a)}{b-a} = \frac{0-f(b)}{c-b}$$

which gives the next iterat as

$$\boxed{c_0 = b_0 - \left(\frac{b_0 - a_0}{f(b_0)-f(a_0)}\right) f(b_0)}$$



- As in Bisection Method, we have three cases:
  1. If $f(a_0)\, f(c_0) < 0$, then the zero $r \in [a_0, c_0]$
  2. If $f(c_0)\, f(b_0) < 0$, then the zero $r \in [c_0, b_0]$
  3. If $f(c_0) = 0$, then the zero $r = c_0$.

- In general

$$c_n = b_n - \left(\frac{b_n - a_n}{f(b_n)-f(a_n)}\right) f(b_n)$$

$n = 0, 1, 2, 3, \dots$

Exp Use False Position Method to find the root of $x \sin x - 1 = 0$ that is located in the interval $[0,2]$

- $f(x) = x \sin x - 1$ with $[a_0, b_0] = [0,2]$
- $f(0) = -1 < 0$ and $f(2) = 0.81859485$

$$c_0 = b_0 - \left(\frac{b_0 - a_0}{f(b_0) - f(a_0)}\right) f(b_0) = 2 - \left(\frac{2-0}{0.81859485 - -1}\right) 0.81859485$$

$$= 1.09975017$$

- $f(c_0) = -0.02001921 \Rightarrow [a_1, b_1] = [c_0, b_0] = [1.0997507, 2]$

$$c_1 = b_1 - \left(\frac{b_1 - a_1}{f(b_1) - f(a_1)}\right) f(b_1) = 2 - \frac{(2 - 1.0997507)(0.81859485)}{0.81859485 - -0.02001921}$$

$$= 1.12124074$$

- $f(c_1) = 0.00983461 \Rightarrow [a_2, b_2] = [1.0997507, 1.12124074]$

$\vdots$

n	a_n	c_n	b_n	$f(c_n)$
0	0	1.09975017	2	-0.02001921
1	1.09975017	1.12124074	2	0.00983461
2	1.09975017	1.11416120	1.12124074	0.00000563
3	1.09975017	1.11415714	1.11416120	$\boxed{0.00000000}$

To see the speed: $f(c_n) \to 0$ or $\{a_n - b_n\} \to 0$

Remark. The termination criterion in Bisection Method is not useful for the False Position method and may result in infinite loop.

This is in section 2.3

- Stopping Criterias (in general)

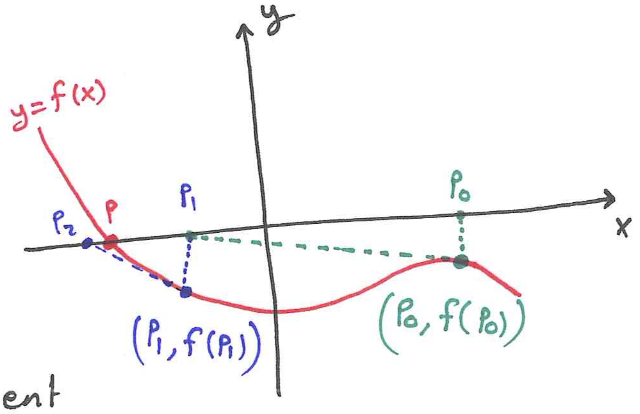
① $|f(c_n)| < \epsilon$ for given tolerance value تفاوت ϵ (given) "mostly used"

② $|c_n - c_{n-1}| < \delta$ this is estimate for the absolute error

③ $2\dfrac{|c_n - c_{n-1}|}{|c_n| + |c_{n-1}|} < \delta$ this is estimate for the relative error

$\boxed{2.4}$ Newton – Raphson Method

- or simply " Newton's Method "
- This method develops algorithm that produces a sequence $\{p_n\}$ converges to the root p faster than Bisection and False Position methods (The best known method).
- Conditions to find the root of $f(x)=0$: $f \in C^2[a,b]$ with $f'(x) \neq 0$
- Assume the initial approximation p_0 is near the root p



- Next approximation p_1 is the point intersection between the x-axis and the line tangent to the curve at $(p_0, f(p_0))$:

$$f'(p_0) = m = \frac{0 - f(p_0)}{p_1 - p_0}$$

Hence,

$$\boxed{p_1 = p_0 - \frac{f(p_0)}{f'(p_0)}}$$

- The process is repeated to obtain a sequence $\{p_n\}$ that converges to p. That is Newton's method iteration:

$$\boxed{p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}} \qquad n = 0, 1, 2, \dots$$

Exp Use Newton's Method to solve $x^2 = \sin x + 1$ using $P_0 = 1.5$ with accuracy 10^{-3}.

- $f(x) = x^2 - \sin x - 1 \Rightarrow f'(x) = 2x - \cos x$
- $$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)} = P_n - \frac{P_n^2 - \sin P_n - 1}{2P_n - \cos P_n}$$

n	P_n	$\lvert P_n - P_{n-1} \rvert$
0	1.5	—
1	1.413799126	$0.0862 > 0.001$
2	1.409633752	$0.00416 > 0.001$
3	1.409624004	$0.00001 < 0.001$

Stope

180 → 3.14
85.987 → 1.5
for sin and cos
⋮

Exp Use Newton's Method with $P_0 = 1$ estimate the root of $f(x) = e^x - \cos x - 1$ with error $< 10^{-4}$

- $P_0 = 1 \Rightarrow P_{n+1} = P_n - \dfrac{e^{P_n} - \cos P_n - 1}{e^{P_n} + \sin P_n}$
- $P_1 = 0.66908389 8 \Rightarrow |P_1 - P_0| > 10^{-4}$
- $P_2 = 0.603760843 \Rightarrow |P_2 - P_1| > 10^{-4}$
- $P_3 = 0.601349991 \Rightarrow |P_3 - P_2| > 10^{-4}$
- $P_4 = 0.601346767 \Rightarrow |P_4 - P_3| < 10^{-4}$ stope

180 → 3.14
57.325 → 1
for sin and cos ...

Exp Use Newton's method to estimate $\sqrt{5}$ starting with $P_0 = 2$

- Let $x = \sqrt{5} \Rightarrow x^2 = 5 \Rightarrow x^2 - 5 = 0 \Rightarrow f(x) = x^2 - 5$
  $\Rightarrow f'(x) = 2x$

- Hence,

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$$

$$= P_n - \frac{P_n^2 - 5}{2P_n}$$

$$= \frac{P_n - \frac{5}{P_n}}{2}$$

- $P_1 = \dfrac{P_0 - \frac{5}{P_0}}{2} = \dfrac{2 + \frac{5}{2}}{2} = 2.25$

- $P_2 = \dfrac{2.25 + \frac{5}{2.25}}{2} = 2.236111111$

- $P_3 = \dfrac{2.236111111 + 5/2.236111111}{2} = 2.236067978$

- $P_4 = \dfrac{2.236067978 + 5/2.236067978}{2} = 2.236067978$

- Note that all $\{P_n\}$ with $n > 4$ will give same result as in P_4, so we see the convergence accurate to 9 decimal places.

Exp Estimate $\sqrt[3]{15}$ $\Rightarrow x = \sqrt[3]{15} \Rightarrow x^3 - 15 = 0 \Rightarrow P_{n+1} = P_n - \dfrac{P_n^3 - 15}{3P_n^2}$

- $P_0 = 2$

  $P_1 = 2.\overset{5}{\cancel{8}}\bar{3}$

  $P_2 = 2.471441785$

  $P_3 = 2.466223133$

  $P_4 = 2.466212074$ as in calculator

طريقة نيوتن الأفضل والأسرع

Th (Newton-Raphson Theorem)

- Assume $f \in C^2[a,b]$ and $\exists$ a number $p \in [a,b]$ s.t $f(p)=0$.
- If $f'(p) \neq 0$, then $\exists$ $\delta > 0$ s.t the sequence $\{p_k\}_{k=0}^{\infty}$ defined by

$$* \quad \boxed{p_{k+1} = g(p_k) = p_k - \frac{f(p_k)}{f'(p_k)} \quad \text{for } k=0,1,2,\dots}$$

will converge to p for any initial approximation $p_0 \in [p-\delta, p+\delta]$, where $g(x) = x - \dfrac{f(x)}{f'(x)}$

Proof. Taylor polynomial of degree 1 about p_0 is

$$f(x) = f(p_0) + f'(p_0)(x-p_0)$$

- Substitute $x=p$ and note that $f(p)=0 \Rightarrow$

$$0 = f(p_0) + f'(p_0)(p-p_0)$$

- Solve for $p \Rightarrow p = p_0 - \dfrac{f(p_0)}{f'(p_0)} \approx p_1$
- This is used to define the next approximation p_1 and so $*$ is established.
- To prove the convergence: Note that $g(p) = p - \dfrac{f(p)}{f'(p)} = p$ so p is fixed point of g.
- $g'(x) = 1 - \dfrac{f'f' - ff''}{(f'(x))^2} = \dfrac{ff''}{(f')^2} \Rightarrow g'(p) = 0 < 1$ and $g'(x)$ is continuous

Hence, $\exists$ $\delta > 0$ s.t $|g'(x)| < 1$ on $(p-\delta, p+\delta)$ by Th FPIT II page 21.

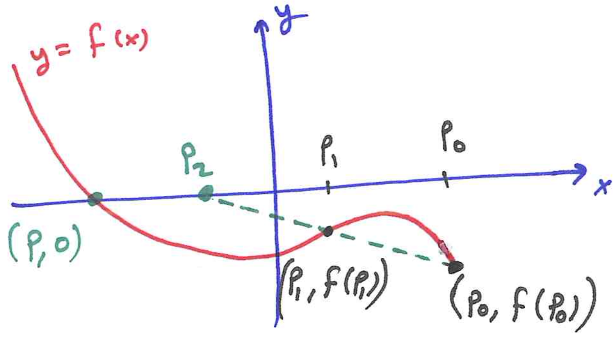
Secant Method

- Recall that in Newton-Raphson method, it is required the evaluation of $f(P_n)$ and $f'(P_n)$ per iteration since

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)} \quad \text{for } n = 0, 1, 2, \dots$$

- It is desirable to have a method "secant method" that converges almost as fast as Newton's method and involves only evaluations of f and not f'.

$\Rightarrow$ Given $(P_0, f(P_0))$
$(P_1, f(P_1))$



$\Rightarrow$ To find P_2 :

$$\frac{f(P_1) - f(P_0)}{P_1 - P_0} = m = \frac{0 - f(P_1)}{P_2 - P_1}$$

$\Rightarrow$ solve for P_2 $\Rightarrow$ $P_2 = P_1 - \left(\frac{P_1 - P_0}{f(P_1) - f(P_0)}\right) f(P_1)$

$\Rightarrow$ In general : $\boxed{P_{n+2} = P_{n+1} - \left(\frac{P_{n+1} - P_n}{f(P_{n+1}) - f(P_n)}\right) f(P_{n+1})}$

Exp Consider the equation $x = \cos x$. Take $P_0 = 0.5$ and $P_1 = \frac{\pi}{4}$ to Find the next iteration "P_2" using secant method to approximate the solution of $x = \cos x$.

$3.14 \to \pi = 180$
$\frac{1}{2} \to ? = 28.7$

- $f(x) = x - \cos x$, $P_1 = \frac{\pi}{4} = 0.785$, $f(\frac{\pi}{4}) = \frac{\pi}{4} - \cos\frac{\pi}{4} = 0.785 - 0.707 = 0.078$

$f(\frac{1}{2}) = \frac{1}{2} - \cos(28.7) = {}^{-}0.377$

- $P_2 = P_1 - \left(\frac{P_1 - P_0}{f(P_1) - f(P_0)}\right) f(P_1) = \frac{\pi}{4} - \left(\frac{\frac{\pi}{4} - \frac{1}{2}}{f(\frac{\pi}{4}) - f(\frac{1}{2})}\right) f(\frac{\pi}{4}) = 0.73638414$

Def (Multiplicity of Roots)

- Assume $f, f', \dots, f^{(M)}$ are defined and continuous on interval about the root p, where $M \in \mathbb{Z}^+$.
- We say $f(x) = 0$ has a root of order M at $x = p$ (or p has multiplicity M) iff

$$f(p) = 0,\ f'(p) = 0,\ f''(p) = 0,\ \dots,\ f^{(M-1)}(p) = 0,\ f^{(M)}(p) \neq 0$$

Def
- A root p of order $M = 1$ is called simple root.
- A root p of order $M > 1$ is called multiple root.
    - ↳ if $M = 2$, then p is called double root.
    - ↳ if $M = 3$, then p is called cubic root.
    - ⋮

Exp Find the roots of $f(x)$ and their multiplicity

1) $f(x) = x^3 - 3x + 2$

- one can write $f(x) = (x+2)(x-1)^2$ so $p = -2$, $p = 1$ roots
- $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$

$\boxed{p = 1}$ $\Rightarrow f(1) = 0,\ f'(1) = 0,\ f''(1) = 6 \neq 0$ so $M = 2$ and $p = 1$ is double root.

$\boxed{p = -2}$ $\Rightarrow f(-2) = 0,\ f'(-2) = 9 \neq 0$ so $M = 1$ and $p = -2$ is simple root.

---

② $f(x) = (x-1)\ln x$

- $p = 1$ is the only root.
- $f'(x) = \dfrac{x-1}{x} + \ln x$ and $f''(x) = \dfrac{1}{x^2} + \dfrac{1}{x}$
- $f(1) = 0$, $f'(1) = 0$, $f''(1) = 2 \neq 0$

so The multiplicity of $p=1$ is 2 and its double root.

Lemma If $f(x)=0$ has a root p and $\exists$ a continuous function $h(x)$ s.t

$f(x) = (x-p)^M h(x)$ where $h(p) \neq 0$ then the root p has multiplicity M

Remark • In Exp ① page 41 $\Rightarrow$ $p_1 = 1$ has $M_1 = 2$ and $p_2 = -2$ has $M_2 = 1$ so

$f(x) = (x+2)(x-1)^2$ with $h_1(x) = x+2$, $h_1(1) \neq 0$

$h_2(x) = (x-1)^2$, $h_2(-2) \neq 0$

• In Exp ② page 42 $\Rightarrow$ $p = 1$ has $M = 2$

$f(x) = (x-1)\ln x$ but $\ln x = (x-1) - \dfrac{(x-1)^2}{2} + \dfrac{(x-1)^3}{3} - \cdots$

$= \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}(x-1)^{n+1}}{n}$ $\quad = \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}(x-1)^n}{n}$

with $h(1) = 0$

$= (x-1)^2 - \dfrac{(x-1)^3}{2} + \dfrac{(x-1)^4}{3} + \cdots$

---

③ $f(x) = x^{101} - x^{100} + x^{30} - 1$ $\Rightarrow p=1$ is root

$f'(x) = 101x^{100} - 100x^{99} + 30x^{29}$ $\Rightarrow f'(1) = 31 \neq 0$

$\Rightarrow p = 1$ has $M = 1$

Def (Speed of Convergence)

- Assume that the sequence $\{p_n\}_{n=0}^{\infty}$ converges to p.
- If there exist two positive constants $A \neq 0$ and $R > 0$ s.t

$$\lim_{n\to\infty} \frac{|p - p_{n+1}|}{|p - p_n|^R} = \lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^R} = A,$$

Then we say that $\{p_n\}$ converges to p with order of convergence R.

Remarks ① We use the order of convergence R to measure the speed of convergence of any method:

- if $R=1$, then the convergence of $\{p_n\}$ is linear
- if $R=\frac{3}{2}$, then the convergence of $\{p_n\}$ is super linear
- if $R=2$, then the convergence of $\{p_n\}$ is quadratic
- if $R=3$, then the convergence of $\{p_n\}$ is cubic

② when $R \uparrow \Rightarrow$ speed $\uparrow \Rightarrow$ error $\downarrow$

③ A is called the asymptotic error constant

This is because • as n gets large $\Rightarrow$

$$|E_{n+1}| \approx A\,|E_n|^R$$

- if $|E_n| = 0.01$ then for

$$R=1 \Rightarrow |E_{n+1}| \approx A|E_n| = A\,(0.01)$$

$$R=2 \Rightarrow |E_{n+1}| \approx A|E_n|^2 = A(0.01)^2 = A(0.0001)$$

Exp Find A and R for the following sequences:

1. $\left\{\frac{1}{10^n}\right\}_{n=0}^{\infty} = 1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots$

- $\lim_{n\to\infty} \frac{1}{10^n} = 0 = p$

- $$\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^R} = \lim_{n\to\infty} \frac{|p - p_{n+1}|}{|p - p_n|^R} = \lim_{n\to\infty} \frac{\left|0 - \frac{1}{10^{n+1}}\right|}{\left|0 - \frac{1}{10^n}\right|^R}$$

$$= \lim_{n\to\infty} \frac{10^{nR}}{10^{n+1}} = \begin{cases} \boxed{\frac{1}{10} \quad \text{if } R=1} \\ \infty \quad \text{if } R>1 \quad \times \\ 0 \quad \text{if } R<1 \quad \times \end{cases}$$

$$= \frac{1}{10} \lim_{n\to\infty} 10^{n(R-1)}$$

- Hence, by definition $\Rightarrow$ $A = \frac{1}{10}$ and $R = 1$

$\Rightarrow$ The convergence is linear

---

2. $p_n = \left\{\frac{1}{2^n}\right\}_{n=0}^{\infty}$ $\Rightarrow \lim_{n\to\infty} \frac{1}{2^n} = 0 = p$

- $$\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^R} = \lim_{n\to\infty} \frac{\left|0 - \frac{1}{2^{n+1}}\right|}{\left|0 - \frac{1}{2^n}\right|^R} = \lim_{n\to\infty} \frac{2^{nR}}{2^{n+1}}$$

$$= \frac{1}{2} \lim_{n\to\infty} 2^{n(R-1)} = \begin{cases} \boxed{\frac{1}{2} \quad \text{if } R=1} \\ \infty \quad \text{if } R>1 \\ 0 \quad \text{if } R<1 \end{cases}$$

- Hence, $A = \frac{1}{2}$ and $R = 1$

and the convergence is linear

Th (Speed of Newton's Method)

Assume Newton-Raphson iteration

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}, \quad \text{given } p_0, \quad n = 0, 1, 2, \cdots$$

produces a sequence $\{p_n\}$ that converges to the root p of the function $f(x)$. Then,

1. if p is simple root $(M=1)$, then Newton's iteration $\{p_n\}$ converges to p quadratically $(R=2)$ with

$$A = \left| \frac{f''(p)}{2f'(p)} \right| \quad \text{and} \quad \lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^2} \approx A$$

2. if p is a multiple root (of order $M > 1$), then Newton's iteration $\{p_n\}$ converges to p linearly $(R=1)$ with

$$A = \frac{M-1}{M} \quad \text{and} \quad \lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|} \approx \frac{M-1}{M}$$

Exp Let $f(x) = x^3 - 3x + 2$

1. Find the order of convergence R and the asymptotic error constant A when Newton-Raphson iteration is used to find the roots of $f(x) = 0$

- Recall the roots of $f(x) = 0 \Rightarrow p = -2, \; p = 1$

- Note that $p = -2$ is simple root $(M=1) \Rightarrow$ so $R = 2$ by Th above and hence,

$$\boxed{A = \left| \frac{f''(-2)}{2f'(-2)} \right| = \left| \frac{-12}{2(9)} \right| = \frac{2}{3}}$$

- Note that $p = 1$ is multiple root $(M=2) \Rightarrow$ so $R = 1$ by Th above and hence,

$$\boxed{A = \frac{M-1}{M} = \frac{2-1}{2} = \frac{1}{2}}$$

$\boxed{2}$ start with $P_0 = -2.4$ and use Newton's – Raphson iteration to find the root $p = -2$. (Prove the quadratic convergence at simple root in $\boxed{1}$). 46

$P = -2$, $P_0 = -2.4$

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)} = P_n - \frac{P_n^3 - 3P_n + 2}{3P_n^2 - 3}$$

$$= \frac{2P_n^3 - 2}{3P_n^2 - 3}$$

no need

n	P_n	$\lvert P_{n+1} - P_n \rvert$	$E_n = \lvert P - P_n \rvert$	$\lvert E_{n+1}\rvert / \lvert E_n \rvert^2$
0	-2.4	0.323809524	0.4	0.476190475
1	-2.076190476	0.072594465	0.076190476	0.619469086
2	-2.003596011	0.003587422	0.003596011	0.664202613
3	-2.000008589	0.000008589	0.000008589	$\approx \frac{2}{3}$
4	-2	0	0	

- Note that $|E_{n+1}| \approx A\,|E_n|^2$ for larg n
- To check this $\Rightarrow$

$|E_3| = |P - P_3| = 0.0000085\,89$

$|E_2| = |P - P_2| = 0.00359\,6011 \Rightarrow |E_2|^2 = 0.000012931$

- Now it is easy to see that

$$|E_3| \approx A\,|E_2|^2 \Leftrightarrow 0.00000\,8589 \approx \frac{2}{3}(0.000012931)$$
$$= 0.00000\,8621$$

3 Start with $p_0 = 1.2$ and use Newton's Method to prove the linear convergence at the double root $p = 1$. **47**

$p = 1$, $p_0 = 1.2$

$$p_{n+1} = \frac{2p_n^3 - 2}{3p_n^2 - 3}$$

n	p_n	$E_n = \|p - p_n\|$	$\|E_{n+1}\| / \|E_n\|$
0	1.2	0.2	0.515151515
1	1.103030303	0.103030303	0.508165253
2	1.052356420	0.052356420	0.496751115
3	1.026400811	0.026400811	0.509753688
4	1.013257730	0.013257730	0.501097775
5	1.006643419	0.006643419	0.500550093 ≈ 0.5
⋮	1.003325375	0.003325375	
20	1.000000409		

- Note that $|E_{n+1}| \approx A\,|E_n|$ for larg n
- To check this $\Rightarrow$

$$|E_5| = |p - p_5| = 0.006643419$$

$$|E_4| = |p - p_4| = 0.13257730$$

- Now it is easy to see that

$$|E_5| \approx A\,|E_4| \iff 0.006643419 \approx (0.5)(0.13257730)$$

$$= 0.06628865$$

Remark. • In the previous Exp the root p was known.

• However, sometimes p is unknown (see next Exp).

Exp (P is unknown)

Consider the equation $x^2 - \sin x - 1 = 0$

① Use Newton's method with $P_0 = 1.5$ to estimate the solution of this equation with error less than 10^{-3}.

$3.14 \longrightarrow 180$
$1.5 \longrightarrow 85.987$

n	P_n	$\lvert P_{n+1} - P_n \rvert$
0	1.5	
1	1.413799126	0.086
2	1.409633752	0.004
3	1.409624004	0.000009748

$f(x) = x^2 - \sin x - 1$

$f'(x) = 2x - \cos x$

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$$

$P = 1.409624004$

② Find the order of convergence and the asymptotic error constant

• We find the multiplicity of the root p

• $f'(p) = 2(1.409624004) - \cos(80.81)$
$= 2.819248008 - 0.1597088975$
$= 2.6595391033 \neq 0$

$\Rightarrow M = 1$ and $p = 1.409624004$ is simple root.

• Hence, by Th above $R = 2$ and $A = \left| \dfrac{f''(p)}{2f'(p)} \right| = 0.56173286$

3 Prove part 2 Numerically

n	P_n	$E_n = \lvert P - P_n \rvert$	$\lvert E_{n+1} \rvert / \lvert E_n \rvert^2$
0	1.5	0.090375	0.5111162
1	1.413799126	0.004175	0.559212 $\approx A$
2	1.409624004	0.0000009748	

- $|E_2| = 0.0000009748$

  $|E_1| = 0.004175 \Rightarrow |E_1|^2 = 0.0000174306$

- Note that $|E_2| \approx A\,|E_1|^2 \Leftrightarrow$

$$0.0000009748 \approx (0.56173286)(0.0000174306)$$

$$= 0.00000097913$$

Proof of Th (Speed of Newton's Method) $\boxed{49.1}$
(page 45)

Recall Newton-Raphson iteration

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)} \quad , \text{ given } P_0 \; , \; n = 0, 1, 2, \dots$$

Proof Part $\boxed{I}$: we need to show if p is simple root ($M = 1$) then Newton's iteration $P_n \longrightarrow P$ quadratically ($R = 2$) with $A = \left| \frac{f''(P)}{2 f'(P)} \right|$

- Define $\boxed{g(x) = x - \frac{f(x)}{f'(x)}} \longrightarrow *$
- since p is root of $f(x) \Rightarrow f(P) = 0 \Rightarrow$
  p is fixed point of $g \Rightarrow g(P) = P - \frac{f(P)}{f'(P)} = P$
- The Taylor expansion of $g(x)$ about the fixed point P is

$$g(x) = g(P) + g'(P)(x - P) + \frac{g''(c)}{2!}(x - P)^2 \quad , \quad c \in (x, P)$$

$$g(P_n) = g(P) + g'(P)(P_n - P) + \frac{g''(c)}{2!}(P_n - P)^2 \quad , \quad c \in (P_n, P) \quad \text{(A)}$$

- Find g' and g'' $\Rightarrow$ Use $*$

$$g'(x) = 1 - \frac{f'(x) f'(x) - f(x) f''(x)}{(f'(x))^2} = \frac{f(x) f''(x)}{[f'(x)]^2}$$

$$g''(x) = \frac{[f'(x)]^2 \left[f(x) f'''(x) + f''(x) f'(x) \right]}{[f'(x)]^4} = \frac{f(x) f'''(x) + f''(x) f'(x)}{[f'(x)]^2}$$

since P is simple root $\Rightarrow f(P) = 0$ and $f'(P) \neq 0$

- Note that $g'(P) = 0$ since $f(P) = 0$ and $f'(P) \neq 0$
  $g''(P) = \frac{f''(P)}{f'(P)} \neq 0$ since $f'(P) \neq 0$

substitute $g'(P) = 0$ and $g''(P) = \frac{f''(P)}{f'(P)}$ (B) in (A) $\Rightarrow$
$g(P) = P$

$$g(P_n) = P + 0(P_n - P) + \frac{g''(c)}{2}(P_n - P)^2$$

$$P_{n+1} = P + \frac{g''(c)}{2}(P_n - P)^2$$

$c \in (P_n, P)$

$P_n < c < P$

$$P_{n+1} - P = \frac{g''(c)}{2}(P_n - P)^2$$

since $P_n \to P$ as $n \to \infty$

$$|P_{n+1} - P| = \left|\frac{g''(c)}{2}\right| |P_n - P|^2$$

as $n \to \infty \Rightarrow$ $c \to P$

$$\frac{|P_{n+1} - P|}{|P_n - P|^2} = \frac{1}{2}|g''(c)|$$

$$\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^2} = \boxed{\frac{1}{2}\lim_{n\to\infty}|g''(c)| = \frac{1}{2}|g''(P)|}$$

using (B)

$$\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^2} = \frac{1}{2}\left|\frac{f''(P)}{f'(P)}\right|$$

Hence, $A = \left|\frac{f''(P)}{2f'(P)}\right|$ and $R = 2$

Proof Part 2 : we need to show if p is multiple root $(M>1)$ then Newton Iteration converges to p linearly $(R=1)$ with $A = \frac{M-1}{M}$

- we can prove this part same way as in part A, or as follows (this method works also for $M=1$)

Since the root p is multiple $(M>1)$ $\Rightarrow$ $f(x) = (x-p)^M h(x)$ where $h(x)$ is cont. s.t $h(p) \neq 0$ (see page 42)

we know that $P_{n+1} = g(P_n) = P_n - \frac{f(P_n)}{f'(P_n)}$

Define $g(x) = x - \frac{f(x)}{f'(x)}$

$$= x - \frac{(x-p)^M h(x)}{(x-p)^M h'(x) + M(x-p)^{M-1} h(x)} \cdot \frac{(x-p)^{1-M}}{(x-p)^{1-M}}$$

$$\boxed{g(x) = x - \frac{(x-p)\, h(x)}{M\, h(x) + (x-p)\, h'(x)}}$$

Note that $g(P) = P$

- Now expand $g(x)$ about P using Taylor $\Rightarrow$

$g(x) = g(P) + g'(c)(x-P)$ $\quad c \in (x, P)$

$g(P_n) = g(P) + g'(c)(P_n - P)$ $\quad c \in (P_n, P)$

$P_{n+1} = P + g'(c)(P_n - P)$

$P_{n+1} - P = g'(c)(P_n - P)$

$$|P_{n+1} - P| = |g'(c)|\,|P_n - P| \qquad P_n < c < P$$

since $P_n \to P$ as $n \to \infty$

$$|E_{n+1}| = |g'(c)|\,|E_n|$$

$\downarrow$

$c \to P$ as $n \to \infty$

$$\frac{|E_{n+1}|}{|E_n|^1} = |g'(c)| \qquad R=1$$

$$\lim_{n\to\infty} \left|\frac{E_{n+1}}{E_n}\right| = \boxed{\lim_{n\to\infty} |g'(c)| = |g'(P)|}$$

Hence,

$$A = |g'(P)|$$

$$Mh'(x) + (x-P)h''(x) + h'(x)$$

$$\text{But } g'(x) = 1 - \frac{[Mh(x) + (x-P)h'(x)][(x-P)h'(x) + h(x)] - (x-P)h(x)[\quad]}{[Mh(x) + (x-P)^{M-1}h(x)]^2}$$

$$g'(P) = 1 - \frac{Mh^2(P)}{M^2h^2(P)}$$

$$= 1 - \frac{1}{M}$$

$$= \frac{M-1}{M}$$

Hence, $A = |g'(P)| = \dfrac{M-1}{M}$ since P is multiple $\Rightarrow M > 1$

with $R = 1$

Th (Accelerated Newton-Raphson Iteration)

- Assume Newton - Raphson iteration

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}, \text{ given } P_0, \ n=0,1,2,\cdots$$

produces a sequence $\{P_n\}$ that converges to the root p.

- Assume p is a multiple root (of order $M>1$). Then

$\boxed{1}$ Done " by Th page 45 $\Rightarrow$ Newton's iteration $\{P_n\}$ converges linearly $(R=1)$ with $A = \frac{M-1}{M}$ and $\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|} = A$ "

$\boxed{2}$ the modification of Newton's iteration

$$P_{n+1} = P_n - \frac{M\, f(P_n)}{f'(P_n)}, \text{ given } P_0, \ n=0,1,2,\cdots$$

converges quadratically $(R=2)$ to p and $A = \lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^2}$

Exp $f(x) = x^3 - 3x + 2$. Estimate $p=1$ using accelerated newton method with $P_0 = 1.2$

- Note that $p=1$ has multiplicity $M=2$ since $f(1) = f'(1) = 0$ but $f''(1) = 6 \neq 0$
- Acceleration formular : $P_{n+1} = P_n - \frac{2 f(P_n)}{f'(P_n)} = \frac{P_n^3 + 3P_n - 4}{3P_n^2 - 3}$

n	P_n	$E_n = \|p - P_n\|$	$\|E_{n+1}\| / \|E_n\|^2$
0	1.2	0.2	0.151515150
1	1.006060606	0.006060606	0.165718578 $\approx A$
2	1.000006087	0.000006087	
3	1	0	

Proof of Th (Accelerated Newton-Raphson Iteration) $\boxed{50.1}$

page 50

we need to show if p is multiple root $(M>1)$, then the accelerated iteration

$$P_{n+1} = P_n - \frac{M f(P_n)}{f'(P_n)} \quad , \text{ given } P_0 , \ n = 0,1,2,\ldots$$

converges quadratically $(R=2)$ to the root P.

- Since p is multiple $(M>1) \Rightarrow f(x) = (x-p)^M h(x)$ where $h(x)$ is cont. s.t $h(p) \neq 0$.

- Define $$\boxed{g(x) = x - \frac{M f(x)}{f'(x)} = x - \frac{M (x-p)\, h(x)}{M h(x) + (x-p) h'(x)}}$$

see page 49.3

$$g'(P) = 1 - \frac{M^2 h^2(P)}{M^2 h^2(P)}$$ see page 49.4

$$= 1 - 1$$

$$g'(P) = 0$$

- Expand g about P using Taylor expansion

$$g(x) = g(P) + g'(P)(x-P) + \frac{g''(c)}{2!}(x-P)^2 , \quad c \in (x,P)$$

$$g(P_n) = P + 0\,(P_n - P) + \frac{g''(c)}{2!}(P_n - P)^2 , \quad c \in (P_n, P)$$

$$P_{n+1} = P + \frac{g''(c)}{2}(P_n - P)^2$$

$P_n < c < P$

$$P_{n+1} - P = \frac{g''(c)}{2}(P_n - P)^2$$

since $P_n \to P$ as $n \to \infty$

$\Rightarrow c \to P$ as $n \to \infty$

$$E_{n+1} = \frac{g''(c)}{2} E_n^2$$

$$\frac{E_{n+1}}{E_n^2} = \frac{g''(c)}{2}$$

$$\frac{|E_{n+1}|}{|E_n|^2} = \frac{1}{2} |g''(c)|$$

$$\lim_{n \to \infty} \frac{|E_{n+1}|}{|E_n|^2} = \frac{1}{2} \lim_{n \to \infty} |g''(c)|$$

$$A = \frac{1}{2} |g''(p)|$$

$R = 2$ since $g''(p) \neq 0$

Remark: Comparing the Exp page 47 using Newton Iteration with same Exp page 50 we see that using Accelerated Newton Iteration we need only 4 iterations to reach the exact root.

Exercises: solve questions 17, 18, 21, 23 page 86, 87, 88

Speed of Convergence for FPI

50.3

Exp show that if $g(P)=P$ and $g'(P)=g''(P)=0$, then the fixed point iteration converges to P with R at least 3 and $A=\frac{1}{6}|g'''(P)|$

- Recall the fixed Point Iteration $P_{n+1}=g(P_n)$
- Expand g about the root p

$$g(x)=g(P)+g'(P)(x-P)+\frac{g''(P)}{2!}(x-P)^2+\frac{g'''(c)}{3!}(x-P)^3$$

$$g(P_n)=P+0+0+\frac{g'''(c)}{6}(P_n-P)^3$$

$$P_{n+1}=P+\frac{g'''(c)}{6}(P_n-P)^3$$

$$P_{n+1}-P=\frac{g'''(c)}{6}(P_n-P)^3$$

$$E_{n+1}=\frac{|g'''(c)|}{6}E_n^3$$

$x<c<P$

$P_n<c<P$

$c\to P$ as $n\to\infty$

since $P_n\to P$ as $n\to\infty$

$$\frac{|E_{n+1}|}{|E_n|^3}=\frac{1}{6}|g'''(c)|$$

$$\lim_{n\to\infty}\frac{|E_{n+1}|}{|E_n|^3}=\frac{1}{6}\lim_{n\to\infty}|g'''(c)|=\frac{1}{6}|g'''(P)|$$

$$A=\frac{1}{6}|g'''(P)|$$, R at least 3 since

if $g'''(P)\neq 0 \Rightarrow R=3$

if $g'''(P)=0 \Rightarrow R>3$

Exp Let p be a fixed point of $g(x)$. Show that if $g'(p) = g''(p) = \cdots = g^{(k-1)}(p) = 0$ and $g^{(k)}(p) \neq 0$, then the fixed point iteration of $g(x)$ will converge to p with

$$R = k \quad \text{and} \quad A = \left| \frac{g^{(k)}(p)}{k!} \right|$$

Apply Taylor Series of $g(x)$ about p $\Rightarrow$

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(p)}{2!}(x-p)^2 + \cdots + \frac{g^{(k-1)}(p)}{(k-1)!}(x-p)^{k-1} + \frac{g^{(k)}(c)}{k!}(x-p)^k$$

$$g(p_n) = p + 0 + 0 + \cdots + 0 + \frac{g^{(k)}(c)}{k!}(p_n - p)^k$$

$$p_{n+1} = p + \frac{g^{(k)}(c)}{k!}(p_n - p)^k \qquad \text{since } p_{n+1} = g(p_n) \text{ is FPI}$$

$$p_{n+1} - p = \frac{g^{(k)}(c)}{k!}(p_n - p)^k$$

$$E_{n+1} = \frac{\left|g^{(k)}(c)\right|}{k!} E_n^k$$

$$\frac{E_{n+1}}{E_n^k} = \frac{1}{k!}\left|g^{(k)}(c)\right|$$

$x < c < p$

$p_n < c < p$

$c \to p$ as $n \to \infty$ since $p_n \to p$ as $n \to \infty$

$$\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^k} = \frac{1}{k!} \lim_{n\to\infty} \left|g^{(k)}(c)\right| = \frac{1}{k!}\left|g^{(k)}(p)\right|$$

$$A = \left| \frac{g^{(k)}(p)}{k!} \right| \quad \text{with} \quad R = k$$

Remark: Newton's iteration is special case of FPI with $g(x) = x - \frac{f(x)}{f'(x)}$. compare page 49.1 with 50.4

Speed of Convergence for Secant Method

$$p_{n+2} = p_{n+1} - \left(\frac{p_{n+1} - p_n}{f(p_{n+1}) - f(p_n)} \right) f(p_{n+1}) \qquad \text{given } p_0, p_1$$

1. if p is simple root $(M=1)$, then secant's iteration $\{p_n\}$ converges to p with

$$R = 1.618 \quad \text{and} \quad \lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^{1.618}} = A = \left| \frac{f''(p)}{2f'(p)} \right|^{0.618}$$

2. if p is multiple root (of order $M>1$), then secant's iteration converges to p with

$$R = 1 \quad \text{and } A \text{ depends on } f(x)$$

Exp start with $p_0 = -2.6$ and $p_1 = -2.4$ and use the secant method to

1. find the root $p=-2$ of $f(x) = x^3 - 3x + 2$
2. find the order of convergence R for $p=-2$
3. find the asymptotic error constant A for $p=-2$
4. Prove part $\boxed{3}$ numerically.

$\boxed{2}$ Recall that $p=-2$ is simple root since $f(-2)=0$ but $f'(-2) = 9 \neq 0$. Hence, $R = 1.618$

$\boxed{3}$ $A = \left| \frac{f''(-2)}{2f'(-2)} \right|^{0.618} = \left(\frac{2}{3}\right)^{0.618} = 0.778351205$

n	p_n	$E_n = \|p - p_n\|$	$\|E_{n+1}\| / \|E_n\|^{1.618}$
0	-2.6	0.6	0.914152831
1	-2.4	0.4	0.469497765
2	-2.106598985	0.106598985	0.847290012
3	-2.022641412	0.022641412	0.693608922
4	-2.001511098	0.001511098	0.825841116
5	-2.000022537	0.000022537	0.727100987 $\approx A$
6	-2.000000022	0.000000022	
7	-2	0	

$1.618 \approx \frac{1+\sqrt{5}}{2}$

- This exp shows the convergence of the secant method at simple root $p = -2$

- Note that $E_5 = |p - p_5| = 0.000022537$

$$E_4^{1.618} = |p - p_4|^{1.618} = (0.001511098)^{1.618} = 0.000027296$$

- It is easy to check that $|E_5| \approx A\, |E_4|^{1.618}$ $\Leftarrow$

$$0.000022537 \approx (0.778351205)(0.000027296)$$

$$= 0.0000212459$$

- Speed of Convergence for Bisection Method: $R = 1$ and $A = \frac{1}{2}$

$$\frac{|E_{n+1}|}{|E_n|} \approx \frac{1}{2}$$

- Speed of Convergence for False Position Method:

$R = 1$ and

A depends on $f(x)$ $\Rightarrow \frac{|E_{n+1}|}{|E_n|} \approx A$