Solving $n \times n$ Linear and Nonlinear Systems

Linear System: $\underset{n\times n}{A}\ \underset{n\times 1}{X} = \underset{n\times 1}{b}$ which is equivalent to

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

Cost (Complexity): is the number of operations $+, -, \times, \div$ required to complete a certain calculation.

Exp¹ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Find the cost of finding $|A| = \det(A)$

$$|A| = a \times d - b \times c \quad \Rightarrow \quad \text{Cost} = 3$$

Exp² Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Find the cost of calculating $|A|$.

$$|A| = a \times \underbrace{\begin{vmatrix} e & f \\ h & i \end{vmatrix}}_{\text{cost}=3} - b \times \underbrace{\begin{vmatrix} d & f \\ g & i \end{vmatrix}}_{\text{cost}=3} + c \times \underbrace{\begin{vmatrix} d & e \\ g & h \end{vmatrix}}_{\text{cost}=3} \quad \text{by exp}^1$$

Cost $= 14$

Remark: My student proved that the cost of finding $|A_{n\times n}|$ for $n \geq 2$ is ① $\text{cost} = n! \sum_{k=2}^{n} \frac{2k-1}{k!}$ or

② $\text{cost} = [n!\, e - 2]$ where $[\ ]$ is the greatest integer function and $e \approx 2.718$

Exercise show that $n!\sum_{k=2}^{n}\frac{2k-1}{k!}=[n!\,e-2]$

where $[\;]$ is the greatest integer function

Proof:

$$n!\sum_{k=2}^{n}\frac{2k-1}{k!}=n!\sum_{k=2}^{n}\left(\frac{2k}{k!}-\frac{1}{k!}\right)$$

$k!=k(k-1)!$

$$=n!\left[2\sum_{k=2}^{n}\frac{1}{(k-1)!}-\sum_{k=2}^{n}\frac{1}{k!}\right]$$

$$=n!\left[2\left(1+\sum_{k=3}^{n}\frac{1}{(k-1)!}\right)-\sum_{k=2}^{n}\frac{1}{k!}+\sum_{k=2}^{n}\frac{1}{k!}-\sum_{k=2}^{n}\frac{1}{k!}\right]$$

$$=n!\left[2+\sum_{k=2}^{n}\frac{1}{k!}+2\left(\underbrace{\sum_{k=3}^{n}\frac{1}{(k-1)!}}_{\text{shifting index}}-\sum_{k=2}^{n}\frac{1}{k!}\right)\right]$$

Note that $\sum_{k=2}^{n-1}\frac{1}{k!}-\sum_{k=2}^{n}\frac{1}{k!}=0+0+\cdots+0-\frac{1}{n!}=\frac{-1}{n!}$

$$=n!\left[\underbrace{2+\sum_{k=2}^{n}\frac{1}{k!}}_{\sum_{k=0}^{n}\frac{1}{k!}}-\frac{2}{n!}\right]$$

$$=n!\left[\sum_{k=0}^{n}\frac{1}{k!}-\frac{2}{n!}\right]$$

$$=n!\sum_{k=0}^{n}\frac{1}{k!}-2$$

- But $e^x = \sum_{n=0}^{\infty} \frac{x^2}{n!}$ so $e = \sum_{k=0}^{\infty} \frac{1}{k!}$

- Hence, $n!\, e = n! \sum_{k=0}^{n} \frac{1}{k!} + \boxed{n! \sum_{k=n+1}^{\infty} \frac{1}{k!}}$

- That is, $n! \sum_{k=0}^{n} \frac{1}{k!} = n!\, e - n! \sum_{k=n+1}^{\infty} \frac{1}{k!}$

- Now $n! \sum_{k=2}^{n} \frac{2k-1}{k!} = n! \sum_{k=0}^{n} \frac{1}{k!} - 2$

$$= n!\, e - 2 - \underbrace{\boxed{n! \sum_{k=n+1}^{\infty} \frac{1}{k!}}}_{\text{Assume this value is } R}$$

- Note that R represents the error of calculating $n!\, e$ which is always less than one " since k starts at $n+1$ and we have $n \geq 2$ ".

- So we can get rid of R by taking the floor function or greatest integer number.

$\lfloor 2.9 \rfloor = 2$
$\lfloor 2.5 \rfloor = 2$
$\lfloor 2.1 \rfloor = 2$
$\lfloor -2.9 \rfloor = -3$
$\lfloor -2.1 \rfloor = -3$

- That is, $n! \sum_{k=2}^{n} \frac{2k-1}{k!} = \lfloor n!\, e - 2 \rfloor$

Exp³ Let A be 3×3 matrix.
Find the cost of calculating A^2.

$$A^2 = AA = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$= \begin{bmatrix} \underset{5}{a\times a + b\times d + c\times g} & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$$

Cost $= (9)(5) = 45$

Exp⁴ Let A and B be 3×3 matrices.
Find the cost of $A + |B|\,B$

$|B|$ requires cost $= 14$ by Exp²

$|B|\times B$ requires cost $= 9$

$A + |B|B$ requires cost $= 9$

Total Cost $= 32$

Result: If A is $n\times n$ matrix, then the cost of calculating A^2 is $2n^3 - n^2$

Check! See page 58

see Exp³ $\Rightarrow n = 3 \Rightarrow 2(3)^3 - (3)^2 = 2(27) - 9$
$= 54 - 9$
$= 45$

3.3 Backward Substitution

Backward substitution method used to solve a linear system of equations that has an upper-triangular coefficient matrix:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
\ddots \qquad &\vdots \\
a_{nn}x_n &= b_n
\end{aligned}
$$

$$
\left[\begin{array}{cccccc|c}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & & b_1 \\
0 & a_{22} & a_{23} & \cdots & a_{2n} & & b_2 \\
0 & 0 & a_{33} & \cdots & a_{3n} & & b_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & a_{nn} & & b_n
\end{array}\right] = [\underset{\uparrow}{U} | b]
$$

coefficient matrix

Exp Solve the following linear system using Backward Substitution and find the cost.

$$
\begin{aligned}
4x_1 - x_2 + 2x_3 + 3x_4 &= 20 \\
-2x_2 + 7x_3 - 4x_4 &= -7 \\
6x_3 + 5x_4 &= 4 \\
3x_4 &= 6
\end{aligned}
$$

step 1 : $x_4 = \frac{6}{3} = 2$ $\Rightarrow$ one operation

$-,+$	0
$\div,\times$	1

step 2 : $x_3 = \frac{4 - 5\times 2}{6} = -1$ $\Rightarrow$ Three operations

$-,+$	1
$\div,\times$	2

step 3 : $x_2 = \frac{-7\times -1 + 4\times 2 - 7}{-2} = -4$ $\Rightarrow$ Five operations

$-,+$	2
$\div,\times$	3

step 4 : $x_1 = \frac{1\times -4 - 2\times -1 - 3\times 2 + 20}{4} = 3$ $\Rightarrow$ Seven operations

$-,+$	3
$\div,\times$	4

Hence, total cost $= 16 = (4)^2 = n^2$

Cost of Backward Substitution (B.S.) for solving $n\times n$ linear system

step	$+,-$	$\times,\div$
1	0	1
2	1	2
3	2	3
4	3	4
⋮	⋮	⋮
n	$n-1$	n

Total $+,-$ is $0 + 1 + 2 + 3 + \cdots + n-1 = \frac{n(n-1)}{2} = \frac{n^2-n}{2}$

Total $\times,\div$ is $1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$

Hence, total cost $= \frac{n^2-n}{2} + \frac{n^2+n}{2} = n^2$

Remark

$$1+2+3+\cdots+n=\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$1+2^2+3^2+\cdots+n^2=\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Five Methods to solve the linear system $AX=b$

1 Gaussian Elimination:

$$\underset{\text{Augmented matrix}}{(A|b)} \rightarrow \underset{\text{Upper matrix}}{(U|c)} \quad \text{solve by B.S.}$$

2 Gauss - Jordan Reduction:

$$(A|b) \longrightarrow (I|X)$$

3 Inverse Method:

- Find A^{-1}:

$$(A|I) \rightarrow (I|A^{-1})$$

- Then $X = A^{-1}b$

4 Cramer's Rule:

- Find $|A| \neq 0$
- Then $x_1 = \frac{|A_1|}{|A|}$, $x_2 = \frac{|A_2|}{|A|}$, $\ldots$, $x_n = \frac{|A_n|}{|A|}$

5 LU Factorization :

- Write $A = LU$ where L is lower triangle matrix
  U is upper triangle matrix

- Let $Y = UX \Rightarrow$

$$AX = b \quad \text{becomes} \quad LUX = b$$
$$LY = b$$

- Now solve $LY = b$ by F.S and find Y

- Then solve $UX = Y$ by B.S and find X

Remark: The speed of these methods is like this

$$\boxed{4} < \boxed{3} < \boxed{2} < \boxed{1} = \boxed{5}$$

Exp show that if A is $n \times n$ matrix, then the cost of finding A^2 is $2n^3 - n^2$

$$A^2 = AA = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$$

$$c_{11} = a_{11} \times a_{11} + a_{12} \times a_{12} + \cdots + a_{1n} \times a_{1n}$$

costs n for $\times$
$n-1$ for $+$
$2n-1$

Hence, c_{11} costs $2n-1$

But A^2 has n^2 elements and each one costs $2n-1$

Hence, total cost of calculating A^2 is

$$n^2(2n-1) = 2n^3 - n^2$$

[1] Gaussian Elimination (G.E.)

The Augmented matrix is denoted by

$$[A|b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right] \quad \cdots \; *$$

Th (Elementary Row Operations ERO)

The following operations applied to the augmented matrix * yield an equivalent linear system:

Row Operation I : Interchange two rows

Row Operation II : Multiply a row by a nonzero constant

Row operation III : The row R_K can be replaced by the sum of R_K and a nonzero multiple of any other row R_p. That is, $R_K = R_K - m_{kp} R_p$

where $m_{kp} = \dfrac{a_{kp}}{a_{pp}}$ is called kp^{th} multiplier

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Exp Consider the following linear system:

$$\begin{aligned} x_1 + 2x_2 + x_3 + 4x_4 &= 13 \\ 2x_1 \qquad + 4x_3 + 3x_4 &= 28 \\ 4x_1 + 2x_2 + 2x_3 + x_4 &= 20 \\ -3x_1 + x_2 + 3x_3 + 2x_4 &= 6 \end{aligned}$$

$n = 4$

$\boxed{i}$ Express this system in augmented matrix form.

Pivot row

Pivot

$$m_{21} = \frac{a_{21}}{a_{11}} = 2 \qquad \left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 2 & 0 & 4 & 3 & 28 \\ 4 & 2 & 2 & 1 & 20 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right) = (A \mid b)$$

$$m_{31} = \frac{a_{31}}{a_{11}} = 4$$

$$m_{41} = \frac{a_{41}}{a_{11}} = -3$$

are the multipliers

or $(12)(2) = 24 + 3$ multiplies

entry operations $+,-:1$ $\times:1$

$\boxed{ii}$ Find an equivalent upper-triagular system using G.E.

• Step 1 • Find multipliers m_{21}, m_{31}, m_{41} →

• Apply them

$$\begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \\ R_4 - 3R_1 \end{array} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & -6 & -2 & -15 & -32 \\ 0 & 7 & 6 & 14 & 45 \end{array}\right)$$

Costs

$\div : 3$

$+/- : 3(4)$

$\times : 3(4)$

• Step 2 • Find multipliers $m_{32} = \frac{-6}{-4} = 1.5$ and $m_{42} = \frac{7}{-4} = -1.75$

• Apply them

$$\begin{array}{l} \\ \\ R_3 - 1.5R_2 \\ R_4 + 1.75R_2 \end{array} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & 0 & -5 & -7.5 & -35 \\ 0 & 0 & 9.5 & 5.25 & 48.5 \end{array}\right)$$

$\div : 2$

$+/- : 2(3)$

$\times : 2(3)$

- step 3 • Find the multiplier $m_{43} = \dfrac{9.5}{-5} = -1.9$
  - Apply it

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & 0 & -5 & -7.5 & -35 \\ 0 & 0 & 0 & -9 & -18 \end{array}\right)$$

$\div$: 1

$+,-$: 1 (2)

$\times$: 1 (2)

---

iii Use B.S to solve the resulting system

$$x_4 = \frac{-18}{-9} = 2$$

$$x_3 = \frac{7.5(2) - 35}{-5} = 4$$

$$x_2 = \frac{-2(4) + 5(2) + 2}{-4} = -1$$

$$x_1 = 13 - 2(-1) - 4 - 4(2) = 3$$

---

iv Find the cost of solving this system using G.E.

- Cost 1 : $(A|b) \longrightarrow (U|c)$ for this 4×4 system

Step	$+,-$	$\div, \times$
1	3 (4)	3, 3(4)
2	2(3)	2, 2(3)
3	1 (2)	1, 1(2)
Total	20	26

cost 1 = 46

- Cost 2 : Using B.S. $\Rightarrow$ cost 2 $= n^2 = 4^2 = 16$

Hence, total cost for 4×4 system = cost1 + cost2 = 62

Result ① The cost of reducing $n\times n$ linear system to an upper-triangular system using G.E. is

$$\frac{4n^3+3n^2-7n}{6}$$

② The cost of solving $n\times n$ linear system using G.E is

$$\frac{4n^3+9n^2-7n}{6}$$

Proof ① Cost of $(A|b) \longrightarrow (U|c)$ as we have seen in $\boxed{\text{Exp - iv}}$ page 61 is as follow:

step	$+,-$	$\div$	$\times$
1	$(n-1)\,n$	$n-1$	$(n-1)\,n$
2	$(n-2)(n-1)$	$n-2$	$(n-2)(n-1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
K	$(n-K)(n-K+1)$	$n-K$	$(n-K)(n-K+1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
Total	$\sum_{K=1}^{n-1}(n-K)(n-K+1)$	$\sum_{K=1}^{n-1}(n-K)$	$\sum_{K=1}^{n-1}(n-K)(n-K+1)$

Hence, total cost is $\sum_{K=1}^{n-1}\left[2(n-K)(n-K+1)+n-K\right]$

$$=\sum_{m=1}^{n-1}\left[2m(m+1)+m\right]=\sum_{m=1}^{n-1}2m^2+3m$$

$\boxed{m=n-K}$

$K=1 \Rightarrow m=n-1$

$K=n-1 \Rightarrow m=1$

$$=2\,\frac{n(n-1)(2n-1)}{6}+3\,\frac{n(n-1)}{2}=\frac{4n^3+3n^2-7n}{6}$$

② Cost of any $n\times n$ system using G.E is cost of $(A|b)\rightarrow(U|c)$ + cost of B.S.

$$=\frac{4n^3+3n^2-7n}{6}+n^2=\frac{4n^3+9n^2-7n}{6}$$

Gaussian Elimination and Pivoting

- If the pivot element $a_{pp} = 0$ in row p, then row p can not be used to eliminate the elements in column p below the main diagonal.

- The process of finding a row k with nonzero pivote element $a_{kp} \neq 0$, $k > p$ and interchang row p by row k is called pivoting.

- Pivoting is two types:

  1. Trivial pivoting:
     - if $a_{pp} \neq 0$, then do not switch rows
     - if $a_{pp} = 0$, then switch row p by the first row k below p s.t $a_{kp} \neq 0$.

Exp

$$\left[\begin{array}{ccc|c} 0 & -1 & 2 & 3 \\ 4 & 3 & -1 & 2 \\ 7 & 0 & -3 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 4 & 3 & -1 & 2 \\ 0 & -1 & 2 & 3 \\ 7 & 0 & -3 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & -1 & 2 & 3 \\ 0 & 2 & 4 & 7 \\ 1 & 2 & 3 & 4 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 7 \\ 0 & -1 & 2 & 3 \end{array}\right]$$

2 Partial Pivoting:

- if $a_{pp} = 0$ or $a_{pp} \neq 0$, choose the pivotal row k whose pivot a_{kp}, $k > p$ satisfy $|a_{kp}| = \max\{|a_{pp}|, |a_{p+1,p}|, \dots, |a_{np}|\}$
- This will make all multipliers m_{rp}, $r = p+1, \dots, n$ less than or equal to 1 in absolute value.
- And hence, reducing the error being propagated when using a finite-digit arithmetic

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Exp Consider the following linear system

$$1.133\, x_1 + 5.281\, x_2 = 6.414$$
$$24.14\, x_1 - 1.210\, x_2 = 22.93$$

whose solution is $(x_1, x_2) = (1, 1)$.

1. Use G.E. with trivial pivoting and use four-digit arithmetic to solve this system.

$$\begin{array}{l} \text{pivot} \rightarrow \\ R_2 - m_{21} R_1 \end{array} \left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 24.14 & -1.210 & 22.93 \end{array}\right]$$

since pivot $a_{11} \neq 0$ so we do not switch rows

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{24.14}{1.133} = 21.31$$

$$\left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 0 & -113.7 & -113.8 \end{array}\right]$$

$$x_2 = \frac{-113.8}{-113.7} = 1.001$$

$$x_1 = \frac{6.414 - 5.281\,(1.001)}{1.133} = 0.9956$$

Note that the error in the solution is due to the magnitude of the multiplier m_{21} which is $\gg 1$.

$\boxed{2}$ Use G.E. with partial pivoting and use four-digit arithmetic to solve this system.

pivot
$$\left[\begin{array}{cc|c} 24.14 & -1.210 & 22.93 \\ 1.133 & 5.281 & 6.414 \end{array}\right]$$

- $\max\{1.133, 24.14\} = 24.14$
- Hence, pivot $= 24.14$

$$m_{21} = \frac{1.133}{24.14} = 0.04693 < 1$$

$$\left[\begin{array}{cc|c} 24.14 & -1.210 & 22.93 \\ 0 & 5.338 & 5.338 \end{array}\right]$$

$$x_2 = \frac{5.338}{5.338} = 1$$

$$x_1 = \frac{22.93 + 1.210\,(1)}{24.14} = 1$$

No Error

☺

2 LU - Factorization

To solve the linear system $AX = b$:

- Write $A = LU$ where L is lower triangular matrix and U is upper triangular matrix

- Let $Y = UX \Rightarrow$

$AX = b$ becomes $LUX = b$

$$LY = b$$

- Now solve $LY = b$ by F.S and find Y

- Then solve $UX = Y$ by B.S and find X

---

* We will see that :

cost of LU = Cost of G.E.

Exp ① Use LU factorization to solve the linear system: $\boxed{67}$

$$4x_1 + 3x_2 - x_3 = 1$$
$$-2x_1 - 4x_2 + 5x_3 = 6$$
$$x_1 + 2x_2 + 6x_3 = 14$$

② find the cost of this method

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①
$$A = \begin{pmatrix} 4 & 3 & -1 \\ -2 & -4 & 5 \\ 1 & 2 & 6 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 6 \\ 14 \end{pmatrix}$$

- We need to write $A = LU$
- First we find U using row operations:

step 1

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{-2}{4} = -0.5 \Rightarrow R_2 + 0.5R_1$$

$$m_{31} = \frac{a_{31}}{a_{11}} = \frac{1}{4} = 0.25 \Rightarrow R_3 - 0.25R_1$$

$$\begin{pmatrix} 4 & 3 & -1 \\ 0 & -2.5 & 4.5 \\ 0 & 1.25 & 6.25 \end{pmatrix}$$

cost
$\div$: 2
$+,-$: 2(2)
$\times$: 2(2)

step 2

$$m_{32} = \frac{a_{32}}{a_{22}} = \frac{1.25}{-2.5} = -0.5 \Rightarrow R_3 + 0.5R_2$$

$$U = \begin{pmatrix} 4 & 3 & -1 \\ 0 & -2.5 & 4.5 \\ 0 & 0 & 8.5 \end{pmatrix}$$

cost
$\div$: 1
$+,-$: 1(1)
$\times$: 1(1)

$$L = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0.25 & -0.5 & 1 \end{pmatrix}$$

Note that cost of $A = LU$ is $13 = \left.\dfrac{4n^3 - 3n^2 - n}{6}\right|_{n=3}$

Now solve $LY = b$ using F.S. where $Y = UX$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ -0.5 & 1 & 0 & 6 \\ 0.25 & -0.5 & 1 & 14 \end{array}\right)$$

$y_1 = 1$

$y_2 = 6 + 0.5(1) = 6.5$

$y_3 = 14 - 0.25(1) + 0.5(6.5) = 17$

cost
$\div$: 0
$+,-$: 3
$\times$: 3

Note that cost of F.S. is $n^2 - n = 3^2 - 3 = 9 - 3 = 6$

Now solve $UX = Y$ using B.S.

$$\left(\begin{array}{ccc|c} 4 & 3 & -1 & 1 \\ 0 & -2.5 & 4.5 & 6.5 \\ 0 & 0 & 8.5 & 17 \end{array}\right)$$

$x_3 = \dfrac{17}{8.5} = 2$

$x_2 = \dfrac{6.5 - 4.5(2)}{-2.5} = \dfrac{-2.5}{-2.5} = 1$

$x_1 = \dfrac{1 + (2)(+1) - 3(1)}{4} = \dfrac{0}{4} = 0$

cost
$\div$: 3
$+,-$: 3
$\times$: 3

Note that cost of B.S. is $n^2 = 3^2 = 9$

(2) Hence, total cost is $13+6+9=28$

Remark The total cost of solving any linear $n\times n$ system $AX=b$ using LU factorization is cost of LU + cost of F.S. + cost of B.S.

$$= \frac{4n^3-3n^2-n}{6} + n^2-n + n^2$$

$$= \frac{4n^3+9n^2-7n}{6}$$

$=$ cost of G.E.

Exp Let A be 3×3 matrix and b_1, b_2 are 3×1 vectors. Find the cost of solving the linear systems $AX=b_1$ and $AX=b_2$ using LU factorization.

cost of $A=LU$ is $\left.\frac{4n^3-3n^2-n}{6}\right|_{n=3} = 13$

cost of solving $Ax=b_1 \Rightarrow LY=b_1$ costs $\left. n^2-n\right|_{n=3} = 6$ by FS

$UX=Y$ costs $\left. n^2\right|_{n=3} = 9$ by BS

cost of solving $Ax=b_2 \Rightarrow LY=b_2$ costs 6 using FS

$UX=Y$ costs 9 using BS

Total cost $= 13+6+9+6+9$

$= 43$

3 Cramer's Rule

To solve the linear $n \times n$ system $AX = b$:

- Find $|A|$ « it should not be zero »
- The solution $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is obtained as follow:

$$x_i = \frac{|A_i|}{|A|}$$ where A_i is obtained by replacing the i^{th} column of A by the column b for all $i = 1, 2, \dots, n$

Exp Solve the following linear system using Cramer's Rule

$$2x_1 - 3x_2 = 8$$
$$x_1 + 5x_2 = -9$$

Then find the cost.

- $A = \begin{pmatrix} 2 & -3 \\ 1 & 5 \end{pmatrix} \Rightarrow |A| = (2)(5) - (1)(-3) = 13$ with cost:

$+,-$: 1
$\times$: 2
3

- $x_1 = \dfrac{|A_1|}{|A|} = \dfrac{\begin{vmatrix} 8 & -3 \\ -9 & 5 \end{vmatrix}}{13} = \dfrac{13}{13} = 1$ with cost $= 3 + 1 = 4$

$x_2 = \dfrac{|A_2|}{|A|} = \dfrac{\begin{vmatrix} 2 & 8 \\ 1 & -9 \end{vmatrix}}{13} = \dfrac{-26}{13} = -2$ with cost $= 3 + 1 = 4$

Total cost $= 3\,(3) + 2 = 11$

number of determinants

cost of each one

division

Exp Assume $Ax = b$ is 3×3 linear system.
Find the cost of solving this system using Cramer's Rule.

$$x_1 = \frac{|A_1|}{|A|} \quad , \quad x_2 = \frac{|A_2|}{|A|} \quad , \quad x_3 = \frac{|A_3|}{|A|}$$

$$\text{Total cost} = \underbrace{4}_{\text{number of determinants}}\underbrace{(14)}_{\text{cost of each one}} + \underbrace{3}_{\text{number of divisions}} = 59$$

Remark To solve $n\times n$ linear system by Cramer's Rule, the cost will be

$$(n+1)\,D_n + n$$

where D_n is the cost of $|A|$

Exp : For 4×4 system $\Rightarrow$ the cost is

- $5\,D_4 + 4$ where

D_4 is the cost of $|A| = 4\,D_3 + \overset{\times}{4} + \overset{+}{3}$

$= 4(14) + 7$

$= 63$

- Hence, $\text{cost} = 5\,D_4 + 4$

$= 5(63) + 4$

$= 319$

4 Gauss - Jordan Reduction (G.J.R)

To solve the linear system $AX = b$:

$$(A|b) \longrightarrow (I|X)$$

Exp Find the cost of solving the 4x4 linear system

$$2x_1 + 2x_2 + 2x_3 + 2x_4 = 4$$
$$2x_1 - x_2 + 3x_3 - 5x_4 = -7$$
$$3x_1 - 2x_2 - x_3 - 4x_4 = -2$$
$$-x_1 + 3x_2 - 2x_3 + 2x_4 = 0$$

using G.J.R

$$(A|b) = \left(\begin{array}{cccc|c} 2 & 2 & 2 & 2 & 4 \\ 2 & -1 & 3 & -5 & -7 \\ 3 & -2 & -1 & -4 & -2 \\ -1 & 3 & -2 & 2 & 0 \end{array}\right) \Rightarrow (I|X) = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

step	$+, -$	$\div$, $\times$
1	3(4)	4 , 3(4)
2	3(3)	3 , 3(3)
3	3(2)	2 , 3(2)
4	3(1)	1 , 3(1)

$$\text{Total cost} = 70 = \left.\frac{2n^3 + n^2 - n}{2}\right|_{n=4}$$

72.1

step 1

$$\left[\begin{array}{cccc|c} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{array}\right]$$

cost of $\times$ = 3×4 = 12
cost of $\pm$ = 3×4 = 12

$$\begin{array}{r} 12 \\ 12 \\ \hline 24 \end{array}$$

step 2

$$\left[\begin{array}{cccc|c} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \end{array}\right]$$

cost of $\times$ = 3×3 = 9
cost of $\pm$ = 3×3 = 9

$$\begin{array}{r} 9 \\ 9 \\ \hline 18 \end{array}$$

step 3

$$\left[\begin{array}{cccc|c} * & 0 & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{array}\right]$$

cost of $\times$ = 3×2 = 6
cost of $\pm$ = 3×2 = 6

$$\begin{array}{r} 6 \\ 6 \\ \hline 12 \end{array}$$

step 4

$$\left[\begin{array}{cccc|c} * & 0 & 0 & * & * \\ 0 & * & 0 & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \end{array}\right]$$

cost of $\times$ = 3×1 = 3
cost of $\pm$ = 3×1 = 3

$$\begin{array}{r} 3 \\ 3 \\ \hline 6 \end{array}$$

$$\left[\begin{array}{cccc|c} * & 0 & 0 & 0 & * \\ 0 & * & 0 & 0 & * \\ 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & * & * \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & x_1 \\ 0 & 1 & 0 & 0 & x_2 \\ 0 & 0 & 1 & 0 & x_3 \\ 0 & 0 & 0 & 1 & x_4 \end{array}\right]$$

cost of $\div$ =

$$\begin{array}{r} 4 \\ 3 \\ 2 \\ 1 \\ \hline 10 \end{array}$$

Total cost =

$$\begin{array}{r} 24 + \\ 18 + \\ 12 + \\ 6 + \\ 10 \\ \hline 70 \end{array}$$

Exp Show that the cost of solving $n \times n$ linear system using G.J.R is $\frac{2n^3+n^2-n}{2}$

step	$+ , -$	$\times , \div$
1	$(n-1)\,n$	$n + (n-1)\,n$
2	$(n-1)(n-1)$	$(n-1) + (n-1)(n-1)$
3	$(n-1)(n-2)$	$(n-2) + (n-1)(n-2)$
$\vdots$	$\vdots$	$\vdots$
k	$(n-1)(n-k+1)$	$(n-k+1) + (n-1)(n-k+1)$
$\vdots$	$\vdots$	$\vdots$
n	$(n-1)(1)$	$1 + (n-1)(1)$

$$\text{Total cost} = \sum_{k=1}^{n}\left[(n-1)(n-k+1) + (n-k+1) + (n-1)(n-k+1)\right]$$

$$= \sum_{k=1}^{n}(2n-1)(n-k+1)$$

$$= (2n-1)\left[n\sum_{k=1}^{n}1 - \sum_{k=1}^{n}k + \sum_{k=1}^{n}1\right]$$

$$= (2n-1)\left[n^2 - \frac{n(n+1)}{2} + n\right]$$

$$= (2n-1)\left(\frac{n^2+n}{2}\right)$$

$$= \frac{2n^3+n^2-n}{2}$$

[redacted]

5 Inverse Method

To solve the linear system $AX=b$:

- Find A^{-1}: $(A\,|\,I) \rightarrow (I\,|\,A^{-1})$
- Then $X = A^{-1}b$

Exp Find the cost of solving the following linear system using inverse method:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

- $$(A\,|\,I) = \left(\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{31} & a_{33} & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 1 & \end{array}\right)$$

- Cost of $A^{-1}_{3\times3}$:

Step	$+,-$	$\div$	$\times$
1	2(5)	5	2(5)
2	2(4)	4	2(4)
3	2(3)	3	2(3)
	24	12	24

Total cost of $A^{-1}_{3\times3} = 60 = \frac{n}{2}(2n-1)(3n-1)$

$= \frac{n(6n^2-5n+1)}{2}$ at $n=3$

- Cost of $X = A^{-1}b$

$$= \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$ is $(3+3+3)+(2+2+2) = 15 = n(2n-1)$

$= 2n^2 - n$ at $n=3$

- Hence, total cost of 3x3 linear system using inverse method is $60+15=75$

Exp show that the cost of finding $A^{-1}_{n\times n}$ is

$\frac{n}{2}(2n-1)(3n-1)$

Step	$+,-$	$\div$, $\times$
1	$(n-1)(2n-1)$	$(2n-1)$, $(n-1)(2n-1)$
2	$(n-1)(2n-2)$	$(2n-2)$, $(n-1)(2n-2)$
3	$(n-1)(2n-3)$	$(2n-3)$, $(n-1)(2n-3)$
$\vdots$	$\vdots$	$\vdots$
K	$(n-1)(2n-K)$	$(2n-K)$, $(n-1)(2n-K)$
$\vdots$	$\vdots$	$\vdots$
n	$(n-1)n$	n , $(n-1)n$

Total cost of A^{-1} $= \sum_{K=1}^{n}\left[(n-1)(2n-K)+(2n-K)+(n-1)(2n-K)\right]$

$$= \sum_{K=1}^{n}(2n-1)(2n-K)$$

$$= 2n^2(2n-1)-(2n-1)\sum_{K=1}^{n}K$$

$$= 2n^2(2n-1)-(2n-1)\frac{n(n+1)}{2}$$

$$= (2n-1)\left(2n^2-\frac{n^2+n}{2}\right)$$

$$= \frac{n}{2}(2n-1)(3n-1)$$

Hence total cost of solving $n\times n$ linear system by inverse method is

$$\frac{n}{2}(2n-1)(3n-1)+n(2n-1)=\frac{n}{2}(2n-1)(3n+1)$$

Solving Nonlinear Systems of Equations

We will study the following three Methods:

1. Newton's Method → for 2x2 nonlinear system
2. Fixed Point Iteration
3. Gauss-Seidel Iteration

(2 and 3) → for 2x2 or 3x3 nonlinear systems

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1 Newton's Method

- Given 2x2 non linear system

$$f(x,y) = 0$$
$$g(x,y) = 0$$

with initial point (x_0, y_0)

- This method find a sequence of points

$(x_1, y_1), (x_2, y_2), \ldots$ that approximates (x, y)

- We will only find (x_1, y_1) as follow:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - J^{-1}(x_0, y_0) \begin{pmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{pmatrix} \quad \cdots\cdots \ *$$

where J is the Jacobian matrix given by

$$J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

Exp Solve the following nonlinear system using Newton's method with initial guess $(1, \frac{1}{2})$

$$x^2 + y^2 = 10$$
$$xy = 5$$

use three significant digits and find the first iteration

- $x^2 + y^2 - 10 = 0 \rightarrow f(x,y) = x^2 + y^2 - 10$
  $xy - 5 = 0 \rightarrow g(x,y) = xy - 5$

- $(x_0, y_0) = (1, \frac{1}{2}) \rightarrow f(1, \frac{1}{2}) = 1 + 0.25 - 10 = -8.75$
  $g(1, \frac{1}{2}) = 1 - 5 = -4.50$

- $J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 2x & 2y \\ y & x \end{pmatrix} \rightarrow J(1, \frac{1}{2}) = \begin{pmatrix} 2 & 1 \\ 0.5 & 1 \end{pmatrix}$

- $J^{-1}(1, \frac{1}{2}) = \frac{1}{1.5} \begin{pmatrix} 1 & -1 \\ -0.5 & 2 \end{pmatrix} = \begin{pmatrix} 0.667 & -0.667 \\ -0.334 & 1.33 \end{pmatrix}$

- $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - J^{-1}(x_0, y_0) \begin{pmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{pmatrix}$

$$= \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} - \begin{pmatrix} 0.667 & -0.667 \\ -0.334 & 1.33 \end{pmatrix} \begin{pmatrix} -8.75 \\ -4.50 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} - \begin{pmatrix} -2.84 \\ -3.07 \end{pmatrix} = \begin{pmatrix} 3.84 \\ 3.57 \end{pmatrix}$$

To find $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - J^{-1}(x_1, y_1) \begin{pmatrix} f(x_1, y_1) \\ g(x_1, y_1) \end{pmatrix}$

Exp Find the first iteration that approximates the solution of the following non linear system

$$x^2 - 2x - y + 0.5 = 0$$
$$x^2 + 4y^2 - 4 = 0$$

using Newton's method and starting with $(2, 0.25)$.

- $f(x,y) = x^2 - 2x - y + 0.5 \Rightarrow f(2, 0.25) = 0.25$

  $g(x,y) = x^2 + 4y^2 - 4 \Rightarrow g(2, 0.25) = 0.25$

- $(x_0, y_0) = (2, 0.25)$

- $J(x,y) = \begin{pmatrix} 2x-2 & -1 \\ 2x & 8y \end{pmatrix} \Rightarrow J(2, 0.25) = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}$

- $J^{-1}(2, 0.25) = \frac{1}{8}\begin{pmatrix} 2 & 1 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} 0.25 & 0.125 \\ -0.5 & 0.25 \end{pmatrix}$

- $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0.25 \end{pmatrix} - \begin{pmatrix} 0.25 & 0.125 \\ -0.5 & 0.25 \end{pmatrix}\begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix}$

  $= \begin{pmatrix} 2 \\ 0.25 \end{pmatrix} - \begin{pmatrix} 0.0938 \\ -0.0625 \end{pmatrix}$

  $= \begin{pmatrix} 1.91 \\ 0.313 \end{pmatrix}$

Exp show the formula * page 76

- Given the following nonlinear system

$$f(x,y)=0 \qquad g(x,y)=0 \qquad \text{starting at } (x_0,y_0)$$

- The Taylor series expansions of f and g at (x_0,y_0) are

$$f(x,y) \approx f(x_0,y_0) + f_x(x-x_0) + f_y(y-y_0)$$

$$g(x,y) \approx g(x_0,y_0) + g_x(x-x_0) + g_y(y-y_0)$$

- But $f(x,y)=g(x,y)=0 \Rightarrow$

$$\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}f(x_0,y_0)\\g(x_0,y_0)\end{pmatrix} + \begin{pmatrix}f_x & f_y\\g_x & g_y\end{pmatrix}\begin{pmatrix}x-x_0\\y-y_0\end{pmatrix}$$

- Multiply both sides by $J^{-1}_{(x_0,y_0)} \Rightarrow$

$$\begin{pmatrix}0\\0\end{pmatrix} = J^{-1}\begin{pmatrix}f(x_0,y_0)\\g(x_0,y_0)\end{pmatrix} + \begin{pmatrix}x\\y\end{pmatrix} - \begin{pmatrix}x_0\\y_0\end{pmatrix}$$

- Rearrange the last equation $\Rightarrow$

$$\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x_0\\y_0\end{pmatrix} - J^{-1}(x_0,y_0)\begin{pmatrix}f(x_0,y_0)\\g(x_0,y_0)\end{pmatrix}$$

2 Fixed Point Iteration (F.P.I)

- This method can be used to solve 2x2 or 3x3 nonlinear systems:

$$f_1(x,y)=0 \qquad f_2(x,y)=0$$

or

$$f_1(x,y,z)=0 \qquad f_2(x,y,z)=0 \qquad f_3(x,y,z)=0$$

- For 2x2 system:

→ write
$$x = g_1(x,y)$$
$$y = g_2(x,y) \qquad (1)$$

→ Given $(x_0,y_0)=(p_0,q_0)$

→ The FPI is

$$p_{n+1} = g_1(p_n,q_n)$$
$$q_{n+1} = g_2(p_n,q_n)$$

$n=0,1,2,\dots$

- For 3x3 system:

→ write
$$x = g_1(x,y,z)$$
$$y = g_2(x,y,z) \qquad (2)$$
$$z = g_3(x,y,z)$$

→ Given $(x_0,y_0,z_0)=(p_0,q_0,r_0)$

→ The FPI is

$$p_{n+1} = g_1(p_n,q_n,r_n)$$
$$q_{n+1} = g_2(p_n,q_n,r_n)$$
$$r_{n+1} = g_3(p_n,q_n,r_n)$$

Def

- The point (p,q) is fixed point of the system (1) if

$$p = g_1(p,q) \quad \text{and}$$
$$q = g_2(p,q).$$

- The point (p,q,r) is fixed point of the system (2) if

$$p = g_1(p,q,r) \quad \text{and}$$
$$q = g_2(p,q,r) \quad \text{and}$$
$$r = g_3(p,q,r).$$

Exp Find the fixed points of the following system

$$x - \sin y = 0$$

$$x^2 + \cos^2 y = \frac{y}{\pi} + \frac{1}{2}$$

- $x = g_1(x,y) \iff x = \sin y$

  $y = g_2(x,y) \iff y = \left(x^2 + \cos^2 y - \frac{1}{2}\right)\pi$

- $y = \left(\sin^2 y + \cos^2 y - \frac{1}{2}\right)\pi = \left(1 - \frac{1}{2}\right)\pi = \frac{\pi}{2}$

  $x = \sin y = \sin \frac{\pi}{2} = 1$

  Hence, $(p,q) = (x,y) = \left(1, \frac{\pi}{2}\right)$ is fixed point

Exp* Consider the following nonlinear system:

$$x^2 + y^2 - x = 0$$

$$e^x + y^2 - y = 0$$

Use initial approximation $(p_0, q_0) = (0.5, 0.4)$ to find the next three approximation using the FPI. 3-digits

- $x = g_1(x,y) = x^2 + y^2 \Rightarrow p_{n+1} = p_n^2 + q_n^2$

  $y = g_2(x,y) = e^x + y^2 \Rightarrow q_{n+1} = e^{p_n} + q_n^2$

- $p_1 = g_1(p_0, q_0) = g_1(0.5, 0.4) = 0.25 + 0.16 = 0.41$

  $q_1 = g_2(p_0, q_0) = g_2(0.5, 0.4) = 1.65 + 0.16 = 1.81$

- $p_2 = g_1(p_1, q_1) = g_1(0.41, 1.81) = 0.168 + 3.28 = 3.45$

  $q_2 = g_2(p_1, q_1) = g_2(0.41, 1.81) = 1.51 + 3.28 = 4.79$

- $p_3 = g_1(p_2, q_2) = g_1(3.45, 4.79) = 11.9 + 22.9 = 34.8$

$q_3 = g_2(p_2, q_2) = g_2(3.45, 4.79) = 31.5 + 22.9 = 54.4$

- Note that the FPI here diverges (see $\boxed{2}$ in Remark below).

Th* (Convergence of FPI - Two dimensions)

- Assume (p, q) is fixed point of $x = g_1(x,y)$ and $y = g_2(x,y)$.
- If (p_0, q_0) is sufficiently close to (p,q) and if

$$\left|\frac{\partial g_1}{\partial x}(p,q)\right| + \left|\frac{\partial g_1}{\partial y}(p,q)\right| < 1 \quad \text{and}$$

$$\left|\frac{\partial g_2}{\partial x}(p,q)\right| + \left|\frac{\partial g_2}{\partial y}(p,q)\right| < 1$$

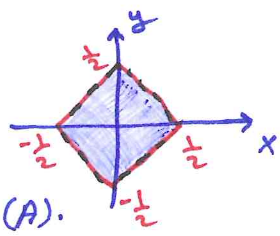
then the FPI converges to the fixed point (p,q)

Remarks: $\boxed{1}$ Converges of FPI for three dimensions follows similarly to Th above by adding z-component.

$\boxed{2}$ In Exp* page 81 $\Rightarrow$ note that

(A) ... $\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| = 2|x| + 2|y| < 1 \iff |x| + |y| < \frac{1}{2}$

(B) ... $\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| = e^{x} + 2|y| < 1$



but $(p_0, q_0) = (0.5, 0.4)$ does not satisfy (A).
That is why the FPI in this Exp* diverges from the fixed point.

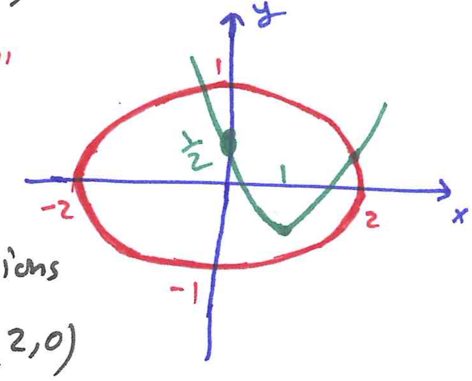
Exercise $\boxed{1}$ Find the fixed point in Exp*

$\boxed{2}$ Solve Exp* again using $(p_0, q_0) = (-0.45, 0.04)$ and show the FPI still diverges.

Exp Consider the following nonlinear system:

$$y = x^2 - 2x + 0.5 \quad \text{"parabola"}$$

$$x^2 + 4y^2 = 4 \quad \text{"Ellipse"}$$



Use the FPI to approximate the solutions using ① $(p_0, q_0) = (0,1)$ ② $(p_0, q_0) = (2,0)$

- The system is equivalent to

$$x = g_1(x,y) = \frac{x^2 - y + 0.5}{2}$$

$* \cdots$

$$y = g_2(x,y) = \frac{-x^2 - 4y^2 + 8y + 4}{8}$$

add to each side $-8y$

- This system has two solutions (or fixed points of $*$):

$$(p,q) \in \{ (-0.2, 1), (1.9, 0.3) \}$$

- To find the first solution $(p,q) = (-0.2, 1)$ we apply formula $*$ as follows:

$$p_{n+1} = \frac{p_n^2 - q_n + 0.5}{2} = g_1(p_n, q_n)$$

$$\cdots (1)$$

$$q_{n+1} = \frac{-p_n^2 - 4q_n^2 + 8q_n + 4}{8} = g_2(p_n, q_n)$$

① $(p_0, q_0) = (0, 1)$

n	p_n	q_n
1	-0.25	1
2	-0.21875	0.9921875
3	-0.2221680	0.9939880
4	-0.2223147	0.9938121
5	-0.2221941	0.9938029
6	-0.2222163	0.9938095

This FPI converges to the first solution

② $(p_0, q_0) = (2, 0)$

n	p_n	q_n
1	2.25	0
2	2.78125	-0.1328125
3	4.184082	-0.6085510
4	9.307547	-2.4820360
5	44.80623	-15.891091
6	1 011.995	-392.60426

This FPI diverges

- Note that Theorem* in page 82 can be used to show that iteration (1) converges to the fixed point near $(-0.2, 1)$:

(2)
$$\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| = |x| + 0.5 < 1 \quad \Leftrightarrow \quad |x| < 0.5$$

$$\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| = \frac{|x|}{4} + |1-y| < \frac{0.5}{4} + |1-y| < 1 \quad \Leftrightarrow$$

$$0.125 < y < 1.875$$

- The fixed point $(p, q) = (-0.2, 1)$ satisfy (2) and so Th implies that the FPI converges to $(p, q) = (-0.2, 1)$.

- However, the fixed point $(p, q) = (1.9, 0.3)$ does not satisfy (2):

$$\left|\frac{\partial g_1}{\partial x}(1.9, 0.3)\right| + \left|\frac{\partial g_1}{\partial y}(1.9, 0.3)\right| = 2.4 > 1$$

$$\left|\frac{\partial g_2}{\partial x}(1.9, 0.3)\right| + \left|\frac{\partial g_2}{\partial y}(1.9, 0.3)\right| = 1.16 > 1$$

So the FPI diverges from $(1.9, 0.3)$ if we use (1).

- Hence, the iteration (1) can not be used to find the second solution $(1.9, 0.3)$.

- To find this solution, we need a different formula for this iteration (1).

- If we add $-2x$ to the first equation and $-11y$ to the second equation, we get

$$x^2 - 4x - y + 0.5 = -2x$$

$$x^2 + 4y^2 - 11y - 4 = -11y$$

- The iteration now is

$$p_{n+1} = g_1(p_n, q_n) = \frac{-p_n^2 + 4p_n + q_n - 0.5}{2}$$

$$q_{n+1} = g_2(p_n, q_n) = \frac{-p_n^2 - 4q_n^2 + 11q_n + 4}{11} \quad \cdots (2)$$

- starting from same point $(p_0, q_0) = (2, 0) \Rightarrow$

n	p_n	q_n
1	1.75	0
2	1.71875	0.0852273
3	1.753063	0.1776676
4	1.808345	0.2504410
8	1.903595	0.3160782
12	1.900924	0.3112267
16	1.900652	0.3111994
20	1.900677	0.3112196

The FPI converges to the second solution using formula (2)

3 Seidel Iteration (Improvement of FPI)

- This method can be used to solve 2x2 or 3x3 non linear systems:

$$f_1(x,y)=0 \qquad\qquad f_1(x,y,z)=0$$
$$f_2(x,y)=0 \qquad \text{or} \qquad f_2(x,y,z)=0$$
$$f_3(x,y,z)=0$$

- For 2x2 system:

→ Write $x=g_1(x,y)$

$y=g_2(x,y)$

→ Given $(x_0,y_0)=(p_0,q_0)$

→ The Seidel Iteration is

$$p_{n+1}=g_1(p_n,q_n)$$

$$q_{n+1}=g_2(p_{n+1},q_n)$$

- For 3x3 system:

→ Write $x=g_1(x,y,z)$

$y=g_2(x,y,z)$

$z=g_3(x,y,z)$

→ Given $(x_0,y_0,z_0)=(p_0,q_0,r_0)$

→ The Seidel Iteration is

$$p_{n+1}=g_1(p_n,q_n,r_n)$$

$$q_{n+1}=g_2(p_{n+1},q_n,r_n)$$

$$r_{n+1}=g_3(p_{n+1},q_{n+1},r_n)$$

Exp • Consider the following non linear system:

$$x=e^{x}+zy$$

$$y=xy-x^{2}z+4$$

$$z=x^2-yz$$

- Use Seidal iteration to find the next two approximations if the initial approximation of the solution is $(1,-1,2)$.
- Use 4-significant digits.

- $x = g_1(x,y,z) = e^x + zy$

  $y = g_2(x,y,z) = xy - x^2 z + 4$

  $z = g_3(x,y,z) = x^2 - yz$

- $(x_0, y_0, z_0) = (p_0, q_0, r_0) = (1, -1, 2)$

- $p_1 = g_1(1,-1,2) = e - 2 = 0.7180$

  $q_1 = g_2(0.7180, -1, 2) = -0.7180 - (0.5155)(2) + 4 = 2.251$

  $r_1 = g_3(0.7180, 2.251, 2) = 0.5155 - 4.502 = -3.987$

- $p_2 = g_1(p_1, q_1, r_1) = g_1(0.7180, 2.251, -3.987) = -6.925$

  $q_2 = g_2(-6.925, 2.251, -3.987) = -202.8$

  $r_2 = g_3(-6.925, -202.8, -3.987) = -760.6$