Ch 4

Interpolation and Polynomial Approximation

- Interpolation means polynomial approximation.
  - given a function $f(x)$ on $[a, b] = [x_0, x_n]$
  - given $n+1$ points

$$(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$$
$$\downarrow \qquad \downarrow \qquad \downarrow$$
$$f(x_0) \qquad f(x_1) \qquad f(x_n)$$

through the partition

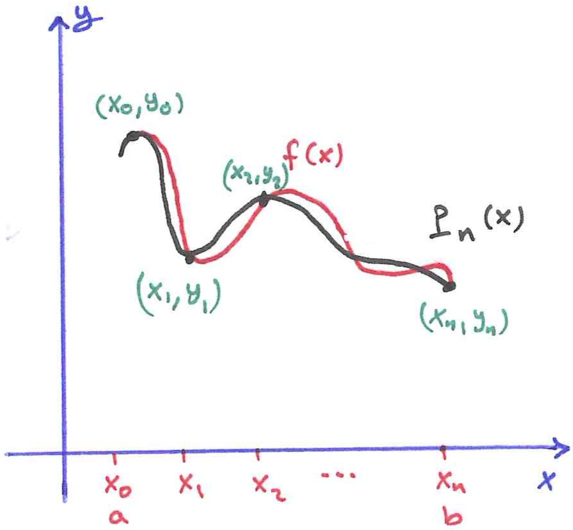
$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

- We need to approximate $f(x)$ by a polynomial of order at most n passing through these $n+1$ points "nodes" on $[a, b]$.

- That is, $f(x) \approx \underbrace{P_n(x)}_{\text{is called interpolation polynomial}} + \underbrace{E_n(x)}_{\text{Truncation error}}$ on $[a, b]$

- degree $(P_n(x)) \leq n$

- $P_n(x) \approx f(x)$ on $[x_0, x_n] = [a, b]$



Th Given $n+1$ points: $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$.
Then $\exists$ a unique polynomial $P_n(x)$ with degree $\leq n$ that passes through these points.

Exp Find a linear interpolating polynomial passes through (x_0, y_0) and (x_1, y_1)

- $P_1(x)$ is the interpolating polynomial of order 1 "linear" given by $P_1(x) = ax + b$

- To find a and b $\Rightarrow$ $P_1(x_0) = y_0 = ax_0 + b$

  $P_1(x_1) = y_1 = ax_1 + b$

- Hence, $a = \dfrac{y_1 - y_0}{x_1 - x_0}$ and $b = y_0 - \left(\dfrac{y_1 - y_0}{x_1 - x_0}\right) x_0$

- Thus, $P_1(x) = \left(\dfrac{y_1 - y_0}{x_1 - x_0}\right) x + y_0 - \left(\dfrac{y_1 - y_0}{x_1 - x_0}\right) x_0$

- Note that $f(x) = P_1(x)$ on $[x_0, x_1]$ with no errors.

- If $(x_0, y_0) = (1, 2)$ and $(x_1, y_1) = (3, 4)$ then

  $a = \dfrac{4-2}{3-1} = \dfrac{2}{2} = 1$ and $b = 2 - (1)(1) = 1$

  $\Rightarrow P_1(x) = x + 1$

- One can write $P_1(x) = m(x - x_0) + y_0$ where

  $m = \dfrac{y_1 - y_0}{x_1 - x_0}$ is the slope.

Exp* Given $(-1, 6)$, $(2, 9)$, $(0, 3)$.

1. Find the polynomial of degree ≤ 2 that passes through these points
2. Estimate $f(1)$
3. Estimate $f'(\frac{1}{4})$
4. Estimate $\int_1^2 f(x)\,dx$

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1 • $P_2(x) = ax^2 + bx + c$ $\qquad P_2(0) = 3 \Rightarrow \boxed{c = 3}$

$P_2(-1) = a - b + 3 = 6 \Rightarrow \boxed{a - b = 3} \rightarrow \boxed{a = 2}$

$P_2(2) = 4a + 2b + 3 = 9 \Rightarrow \boxed{2a + b = 3} \rightarrow \boxed{b = -1}$

• Hence, $\boxed{P_2(x) = 2x^2 - x + 3}$

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2 $f(1) \approx P_2(1) = 2 - 1 + 3 = 4$

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3 $f'(x) \approx P_2'(x) = 4x - 1 \Rightarrow f'(\frac{1}{4}) \approx P_2'(\frac{1}{4}) = 1 - 1 = 0$

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4 $\int_1^2 f(x)\,dx \approx \int_1^2 P_2(x)\,dx = \frac{2}{3}x^3 - \frac{x^2}{2} + 3x \Big|_1^2$

$= \left(\frac{16}{3} - 2 + 6\right) - \left(\frac{2}{3} - \frac{1}{2} + 3\right)$

$= \frac{37}{6} \approx 6.17$

Remark: If $f(x)$ is given and analytic at x_0 " has continuous derivatives of all orders and can be represented as Taylor series in an interval about x_0 ", then we can use Taylor Polynomial Approximation to estimate $f(x)$ by a Taylor Polynomial

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Th (Taylor Polynomial Approximation)

- Assume $f \in C^{n+1}[a,b]$ and $x_0 \in [a,b]$ is fixed
- If $x \in [a,b]$, then $f(x) \simeq P_n(x) + E_n(x)$

where $P_n(x)$ is the Taylor polynomial of degree n that estimates $f(x)$ on $[a,b]$ given by

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

and $E_n(x)$ is the truncation error given by

$$E_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

However, f is usually not known or hard to compute. So How to find $P_n(x)$

1. Lagrange Interpolation $\Rightarrow P_n(x)$ is the Lagrange Polynomial
2. Newton Interpolation $\Rightarrow P_n(x)$ is the Newton Polynomial

1 Lagrange's Polynomial

* linear interpolation uses a line segment that passes throw two points.

Exp Find Lagrange polynomial through the points (x_0, y_0) and (x_1, y_1).

$$P_1(x) \approx f(x) = y = y_0 + m(x - x_0) \quad \text{where } m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$$

$$P_1(x) = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

$$= y_0 + \frac{y_1}{x_1 - x_0}(x - x_0) - \frac{y_0}{x_1 - x_0}(x - x_0)$$

$$= y_0\left[1 - \frac{x - x_0}{x_1 - x_0}\right] + y_1 \frac{x - x_0}{x_1 - x_0}$$

$$= y_0 \frac{x_1 - x}{x_1 - x_0} + y_1 \frac{x - x_0}{x_1 - x_0}$$

Hence, the Lagrange polynomial of order 1 is

$$P_1(x) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}$$

$$= y_0 L_{1,0}(x) + y_1 L_{1,1}(x)$$

Lagrange coefficient polynomials

$$= \sum_{k=0}^{1} y_k L_{1,k}(x)$$

• Note that $P_1(x_0) = y_0$ and $P_1(x_1) = y_1$

Remark: In general, the Lagrange Polynomial of degree at most n passes through $n+1$ points: $(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$ has the form:

$$P_n(x) = \sum_{k=0}^{n} y_k \, L_{n,k}(x)$$

where the Lagrange coefficient polynomial based on these points:

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

Exp • Given $(x_0, y_0), (x_1, y_1), (x_2, y_2)$

• Lagrange polynomial of order of degree ≤ 2 is

$$P_2(x) = \sum_{k=0}^{2} y_k \, L_{2,k}(x)$$

$$= y_0 \, L_{2,0}(x) + y_1 \, L_{2,1}(x) + y_2 \, L_{2,2}(x)$$

$$= y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

Remark Note that $L_{n,k}(x) = \dfrac{\prod_{j=0,\, j\neq k}^{n} (x-x_j)}{\prod_{j=0,\, j\neq k}^{n} (x_k-x_j)}$

Exp* Given $\overset{x_0}{(-1}, \overset{y_0}{6)}, \overset{x_1}{(2}, \overset{y_1}{9)}, \overset{x_2}{(0}, \overset{y_2}{3)}$. $\boxed{n=2}$

Find Lagrange polynomial and estimate $f(1)$

- $$P_2(x) = y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$= (6)\frac{(x-2)(x-0)}{(-1-2)(-1-0)} + (9)\frac{(x-{}^-1)(x-0)}{(2-{}^-1)(2-0)} + (3)\frac{(x-{}^-1)(x-2)}{(0-{}^-1)(0-2)}$$

$$= 2x(x-2) + \frac{3}{2}x(x+1) - \frac{3}{2}(x+1)(x-2)$$

$$= 2x^2 - 4x + \cancel{\frac{3}{2}x^2} + \frac{3}{2}x - \cancel{\frac{3}{2}x^2} + \frac{3}{2}x + 3$$

$$\boxed{P_2(x) = 2x^2 - x + 3}$$

"quadratic interpolation polynomial"

- To estimate $f(1)$, we use Lagrange polynomial:

$$f(1) \approx P_2(1) = 2 - 1 + 3 = 4$$

Exp Given $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3) \Rightarrow P_3(x)$ is the Lagrange polynomial of degree ≤ 3 given by

$$P_3(x) = y_0 L_{3,0}(x) + y_1 L_{3,1}(x) + y_2 L_{3,2}(x) + y_3 L_{3,3}(x)$$

$$= y_0 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + y_1 \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} +$$

$$y_2 \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

"cubic interpolation polynomial"

Def (Uniform Partition)

- The partition of $[a,b]=[x_0,x_n]$ is uniform if the nodes $x_0, x_1, x_2, \dots, x_n$ are equally spaced.
- That is, $x_k = x_0 + hk$ for $k=0,1,2,\dots,n$

Exp Consider $y=f(x)=\cos x$ on $[0, 1.2]$

① Find Lagrange Polynomial of order 2 using equally spaced nodes. (use 3 digits)

- Nodes: $x_0=0$, $x_1=0.6$, $x_2=1.2$ which is $n+1=2+1=3$
- Points: (x_0,y_0), (x_1,y_1), (x_2,y_2)
  $(0,1)$, $(0.6, 0.825)$, $(1.2, 0.362)$
- $$P_2(x) = y_0\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$=(1)\frac{(x-0.6)(x-1.2)}{(0-0.6)(0-1.2)} + (0.825)\frac{(x-0)(x-1.2)}{(0.6-0)(0.6-1.2)} + (0.362)\frac{(x-0)(x-0.6)}{(1.2-0)(1.2-0.6)}$$

$$= 1.39(x-0.6)(x-1.2) - 2.29x(x-1.2) + 0.503x(x-0.6)$$

$$= -0.397x^2 + 0.246x + 0.698$$

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② Find Lagrange Polynomial of order 3 using uniform partition.

- Nodes: $x_0=0$, $x_1=0.4$, $x_2=0.8$, $x_3=1.2$ which is $n+1=3+1=4$
- Points: (x_0,y_0), (x_1,y_1), (x_2,y_2), (x_3,y_3)
  $(0,1)$, $(0.4, 0.921)$, $(0.8, 0.697)$, $(1.2, 0.362)$

$$P_3(x) = y_0 L_{3,0}(x) + y_1 L_{3,1}(x) + y_2 L_{3,2}(x) + y_3 L_{3,3}(x)$$

$$= (1)\frac{(x-0.4)(x-0.8)(x-1.2)}{(0-0.4)(0-0.8)(0-1.2)} + (0.921)\frac{(x-0)(x-0.8)(x-1.2)}{(0.4-0)(0.4-0.8)(0.4-1.2)} +$$

$$(0.697)\frac{(x-0)(x-0.4)(x-1.2)}{(0.8-0)(0.8-0.4)(0.8-1.2)} + (0.362)\frac{(x-0)(x-0.4)(x-0.8)}{(1.2-0)(1.2-0.4)(1.2-0.8)}$$

$$= -2.60(x-0.4)(x-0.8)(x-1.2) + 7.20x(x-0.8)(x-1.2)$$

$$-5.44x(x-0.4)(x-1.2) + 0.944x(x-0.4)(x-0.8)$$

$$= 0.100x^3 - 0.580x^2 + 0.0300x + 0.998$$

Exp Let $f(x) = e^x$ on $[1, 4]$ (use 3 digits)

H.W

1. Find Lagrange Polynomial of order 1 using equally space nodes.
2. Find Lagrange Polynomial of order 2 using uniform partition.
3. Find Lagrange Polynomial of order 3 using uniform partition.

Remark Now we study the second method "Newton Polynomial" then we study the error term since the error of these two interpolations is the same.

Newton's Polynomial

Th (Newton's Polynomial)

- Given $x_0, x_1, x_2, \dots, x_n$ $n+1$ distinct numbers in $[a,b]$.
- Then, $\exists$ a unique polynomial $P_n(x)$ " Called Newton's Polynomial " of degree at most n s.t $f(x_i) = P_n(x_i)$ for $i=1,2,\dots,n$.
- Furthermore, Newton's Polynomial is given by

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \dots$$
$$+ a_n(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1})$$

where the coefficients of Newton's Polynomial are given by the divided differences: $a_k = f[x_0, x_1, \dots, x_k]$ for $k=0,1,\dots,n$.

- That is:

$a_0 = f[x_0] = f(x_0) = y_0$: zero divided differences

$$a_1 = f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$ First divided differences

$$= \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

$$a_2 = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$ 2^{nd} D.D

$$= \frac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0}$$

$\vdots$

Divided Difference Table for $y=f(x)$

x_k	$f[x_k]=y_k$	1^{st} D.D	2^{nd} D.D	3^{rd} D.D
x_0	$y_0=a_0$			
x_1	y_1	$f[x_0,x_1]=a_1$		
x_2	y_2	$f[x_1,x_2]$	$f[x_0,x_1,x_2]=a_2$	
x_3	y_3	$f[x_2,x_3]$	$f[x_1,x_2,x_3]$	$f[x_0,x_1,x_2,x_3]=a_3$
x_4	y_4	$f[x_3,x_4]$	$f[x_2,x_3,x_4]$	$f[x_1,x_2,x_3,x_4]$

Exp* Given $(\overset{x_0}{-1},\overset{y_0}{6}),(\overset{x_1}{2},\overset{y_1}{9}),(\overset{x_2}{0},\overset{y_2}{3})$. Find Newton Polynomial.

- Newton Polynomial is $P_2(x)=a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)$

$$= 6 + a_1(x+1)+a_2(x+1)(x-2)$$

- $$a_1=\frac{y_1-y_0}{x_1-x_0}=\frac{9-6}{2+1}=1$$

$$a_2=\frac{\frac{y_2-y_1}{x_2-x_1}-\frac{y_1-y_0}{x_1-x_0}}{x_2-x_0}=\frac{\frac{3-9}{0-2}-\frac{9-6}{2+1}}{0+1}=3-1=2$$

- Hence, $P_2(x)=6+x+1+2(x+1)(x-2)$

$$=2x^2-x+3$$

Exp Given $(\overset{x_0}{-2}, \overset{y_0}{-12}), (\overset{x_1}{-1}, \overset{y_1}{-4}), (\overset{x_2}{1}, \overset{y_2}{0}), (\overset{x_3}{2}, \overset{y_3}{8})$

- Construct the Divided Difference table
- Find Newton's Polynomial
- Estimate $f(0)$.

x_k	y_k	1^{st} D.D	2^{nd} D.D	3^{rd} D.D
-2	$a_0 = -12$			
-1	-4	$a_1 = 8$		
1	0	$f[x_1, x_2] = 2$	$a_2 = -2$	
2	8	$f[x_2, x_3] = 8$	$f[x_1, x_2, x_3] = 2$	$a_3 = 1$

$$f[x_2, x_3] = \frac{y_3 - y_2}{x_3 - x_2} = \frac{8-0}{2-1} = 8$$

- $a_0 = f[x_0] = y_0 = -12$

- $a_1 = f[x_0, x_1] = f[-2, -1] = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{-4+12}{-1+2} = 8$

- $a_2 = f[x_0, x_1, x_2] = f[-2, -1, 1] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \dfrac{\frac{0+4}{1+1} - 8}{1+2}$
$= -2$

- $a_3 = f[x_0, x_1, x_2, x_3]$

$$= \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = \frac{\frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} + 2}{2+2}$$

$$= \frac{\frac{8-2}{2+1} + 2}{4} = \frac{2+2}{4} = 1$$

- Newton's Polynomial is

$$P_3(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

$$= -12 + 8(x+2) - 2(x+2)(x+1) + (x+2)(x+1)(x-1)$$

$$= x^3 + x - 2$$

- $f(0) \approx P_3(0) = 0 + 0 - 2 = -2$

Error Terms and Error Bounds

- **Error Term $E_n(x)$:** • Given $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$.
  - Given $P_n(x)$ interpolating Polynomial.
    → $P_n(x)$ can be Lagrange or Newton Polynomial that approximates $f(x)$

  - The Error term $E_n(x)$ is the same for Lagrange and Newton approximation $P_n(x)$
  - And $E_n(x)$ is similar to the error term for Taylor polynomial except the factor $(x-x_0)^{n+1}$ is replaced by the product $(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_n)$
  - This is because the interpolation $P_n(x)$ is exact "$P_n(x) = f(x)$" at each $n+1$ nodes x_k $\Rightarrow$

$$E_n(x_k) = f(x_k) - P_n(x_k) = y_k - y_k = 0$$

Th (Error Term)

- Assume $f \in C^{n+1}[a,b]$ and $x_0, x_1, \dots, x_n \in [a,b]$ are $n+1$ nodes.
- Then $f(x) = P_n(x) + E_n(x)$ where $E_n(x)$ is the error term given by $E_n(x) = \dfrac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_n)\, f^{(n+1)}(c)}{(n+1)!}$ for some $c = c(x)$ lies in $[a,b]$.

$P_n(x) = y_0 L_{n,0}(x) + y_1 L_{n,1}(x) + \cdots + y_n L_{n,n}(x)$ Lagrange Polynomial or

$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$ Newton Polynomial

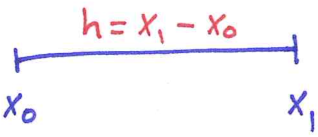
How to find an upper bound for the Error Term $E_n(x)$?

- That is, we need to find some constant s.t $|E_n(x)| \le$ constant.
- Finding the upper bound depends on whether the nodes are equally spaced (Uniform Partition) or not (not uniform partition).

Th (Upper Bound of the Error Term for Interpolation – Uniform Partition)

1. Given $(x_0, y_0), (x_1, y_1)$ "$n=1$"

with $E_1(x) = \dfrac{(x-x_0)(x-x_1)\, f''(c)}{2!}$



Then $|E_1(x)| \le \dfrac{h^2 M_2}{8}$ where $M_2 = \underset{x_0 \le x \le x_1}{\text{Max}} |f''(x)|$

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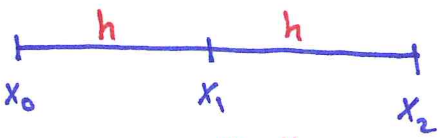
2. Given $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ "$n=2$"

with $E_2(x) = \dfrac{(x-x_0)(x-x_1)(x-x_2)\, f'''(c)}{3!}$



Then $|E_2(x)| \le \dfrac{h^3 M_3}{9\sqrt{3}}$ where $M_3 = \underset{x_0 \le x \le x_2}{\text{Max}} |f'''(x)|$ $\quad h = \dfrac{x_2 - x_0}{2}$

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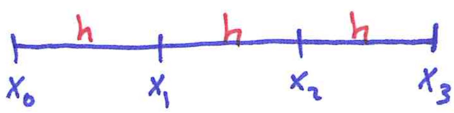
3. Given $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3)$ "$n=3$"

with $E_3(x) = \dfrac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)\, f^{(4)}(c)}{4!}$



Then $|E_3(x)| \le \dfrac{h^4 M_4}{24}$ where $M_4 = \underset{x_0 \le x \le x_3}{\text{Max}} |f^{(4)}(x)|$ $\quad h = \dfrac{x_3 - x_0}{3}$

Exp Given $f(x) = \ln(x+2)$ on $[1, 1.6]$

Find an upper bound for E_1, E_2, E_3 using uniform partition.

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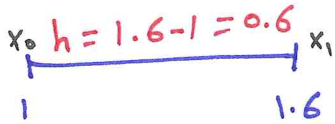
1. • $f'(x) = \dfrac{1}{x+2} \Rightarrow f''(x) = \dfrac{-1}{(x+2)^2} \Rightarrow |f''(x)| = \dfrac{1}{(x+2)^2}$

• The upper bound of E_1 is



$$|E_1(x)| \leq \frac{h^2 M_2}{8} \quad \text{where} \quad M_2 = \max_{1 \leq x \leq 1.6} |f''(x)|$$

• $|f''|$ is decreasing $\Rightarrow |f''(x)| \leq \dfrac{1}{(1+2)^2} = \dfrac{1}{9} = M_2$

• Hence, $|E_1(x)| \leq \dfrac{(0.6)^2\left(\frac{1}{9}\right)}{8} = 0.005$

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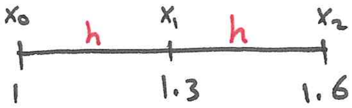
2. • The upper bound of E_2 is $|E_2(x)| \leq \dfrac{h^3 M_3}{9\sqrt{3}}$

• $M_3 = \max\limits_{1 \leq x \leq 1.6} |f'''(x)| \Rightarrow f'''(x) = \dfrac{2}{(x+2)^3}$



$h = \dfrac{1.6 - 1}{2} = 0.3$

• $|f'''(x)|$ is decreasing $\Rightarrow |f'''(x)| \leq \left|\dfrac{2}{(1+2)^3}\right| = \dfrac{2}{27} = M_3$

• Hence, $|E_2(x)| \leq \dfrac{(0.3)^3\left(\frac{2}{27}\right)}{9\sqrt{3}} = \dfrac{0.002}{9\sqrt{3}} = 0.000128$

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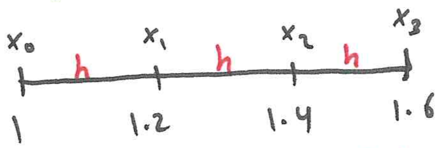
3. • The upper bound of E_3 is $|E_3(x)| \leq \dfrac{h^4 M_4}{24}$

• $M_4 = \max\limits_{1 \leq x \leq 1.6} |f^{(4)}(x)| \Rightarrow f^{(4)}(x) = \dfrac{-6}{(x+2)^4}$



• $|f^{(4)}(x)| = \dfrac{6}{(x+2)^4} \leq \dfrac{6}{(1+2)^4} = \dfrac{6}{81} = \dfrac{2}{27} = M_4$

$h = \dfrac{1.6 - 1}{3} = 0.2$

• Hence, $|E_3(x)| \leq \dfrac{(0.2)^4\left(\frac{2}{27}\right)}{24} = 0.00000494$

Upper Bound of the Error Term For interpolation - Non Uniform Partition

- Given $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ "$n=2$"

  with $E_2(x) = \dfrac{(x-x_0)(x-x_1)(x-x_2)\,f'''(c)}{3!}$

- Then $|E_2(x)| \le \dfrac{|\phi_2(x)|\,|f'''(c)|}{6}$ where

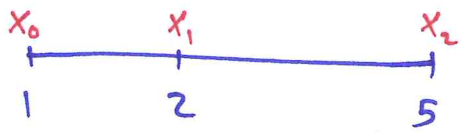
  $|\phi_2(x)|$ is an upper bound of $\phi_2(x) = (x-x_0)(x-x_1)(x-x_2)$ and

  $|f'''(c)|$ is an upper bound of $f'''(x)$.

Exp Given $f(x) = x^2 - \dfrac{2}{x}$ and $(1, f(1)), (2, f(2)), (5, f(5))$.

Find an upper bound for the error term of interpolation.

- $n = 2$



- The upper bound of error is

  $|E_2(x)| \le \dfrac{|\phi_2(x)|\,|f'''(c)|}{6}$

- $f'(x) = 2x + \dfrac{2}{x^2} \Rightarrow f''(x) = 2 - \dfrac{4}{x^3} \Rightarrow$

  $f'''(x) = \dfrac{12}{x^4} \Rightarrow |f'''(x)| = \dfrac{12}{x^4} \le 12 = M_3$

- $\phi_2(x) = (x-1)(x-2)(x-5) = (x^2 - 3x + 2)(x-5)$

  $\phi_2'(x) = (x^2 - 3x + 2)(1) + (x-5)(2x-3) = 3x^2 - 16x + 17 = 0$

  $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{16 \pm \sqrt{(16)^2 - 4(3)(7)}}{6} \Rightarrow x_1 = 3.8685$ or $x_2 = 1.4682$

  $|\phi(x_1)| = \boxed{6.06}^{\text{Max}}$ and $|\phi(x_2)| = 3.783$, $|\phi(1)| = |\phi(5)| = 0$ end points

- Hence, $|E_2(x)| \le \dfrac{(6.06)(12)}{6} = 12.12$

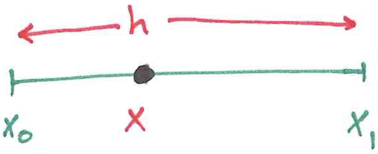
Proof of Th$^{\$}$

I $|E_1(x)| \le \dfrac{h^2 M_2}{8}$ where $M_2 = \underset{x_0 \le x \le x_1}{\text{Max}} |f''(x)|$

- The Error Term is $E_1(x) = \dfrac{(x-x_0)(x-x_1)\, f''(c)}{2} \Rightarrow$

$$|E_1(x)| = \frac{|\phi_1(x)|\,|f''(c)|}{2} \quad \text{where } |f''(c)| \le M_2$$

- Now we need to find an upper bound for $|\phi_1(x)|$.
  - $\phi_1(x) = (x-x_0)(x-x_1)$ using change of variables



  - Let $\boxed{x - x_0 = t}$
  - We have

$$x_1 = x_0 + h$$
$$-x_1 = -x_0 - h$$
$$x - x_1 = x - x_0 - h$$
$$\boxed{x - x_1 = t - h}$$

$$x_0 \le x \le x_1$$
$$0 \le x - x_0 \le x_1 - x_0$$
$$\boxed{0 \le t \le h}$$

- $\phi_1(x) = \phi_1(x_0 + t) = t(t-h) = t^2 - ht = \phi_1(t)$

$$\phi_1' = 2t - h = 0 \iff t = \frac{h}{2} \text{ critical point}$$

- $\left|\phi\left(\frac{h}{2}\right)\right| = \left|\frac{h^2}{4} - \frac{h^2}{2}\right| = \boxed{\frac{h^2}{4}}$ Max since $|\phi(0)| = |\phi(h)| = 0$ end points

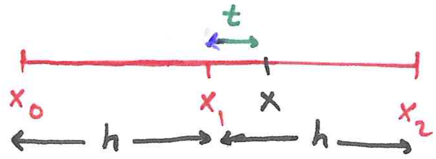
- Hence, $|E_1(x)| = \dfrac{|\phi_1(x)|\,|f''(c)|}{2} \le \dfrac{\frac{h^2}{4} M_2}{2} = \dfrac{h^2 M_2}{8}$

$\boxed{2}$ $|E_2(x)| \leq \dfrac{h^3 M_3}{9\sqrt{3}}$ where $M_3 = \underset{x_0 \leq x \leq x_2}{\text{Max}} |f'''(x)|$

- The Error Term is $E_2(x) = \dfrac{(x-x_0)(x-x_1)(x-x_2)\, f'''(c)}{3!}$ $\Rightarrow$

$|E_2(x)| = \dfrac{|\phi_2(x)|\,|f'''(c)|}{6}$ where $|f'''(c)| \leq M_3$

- Now we need to find an upper bound for $|\phi_2(x)|$.
  - $\phi_2(x) = (x-x_0)(x-x_1)(x-x_2)$
  - Using the change of variable:



$$x - x_1 = t$$
$$x - x_0 = t + h$$
$$x - x_2 = t - h$$

$$x_0 \leq x \leq x_2$$
$$x_0 - x_1 \leq x - x_1 \leq x_2 - x_1$$
$$-h \leq t \leq h$$

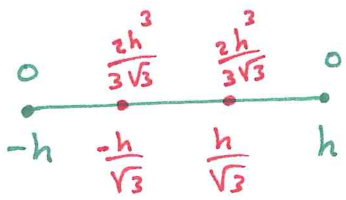
- $\phi_2(x) = \phi_2(x_1 + t) = (t+h)(t)(t-h)$

$$= t(t^2 - h^2)$$
$$= t^3 - t h^2$$
$$= \phi(t)$$

$\phi_2' = 3t^2 - h^2 = 0 \iff t = \pm \dfrac{h}{\sqrt{3}}$ critical points

- $\left|\phi_2\left(\frac{h}{\sqrt{3}}\right)\right| = \left|\dfrac{h^3}{3\sqrt{3}} - \dfrac{h^3}{\sqrt{3}}\right| = \dfrac{2h^3}{3\sqrt{3}}$

$\left|\phi_2\left(\frac{-h}{\sqrt{3}}\right)\right| = \left|\dfrac{-h^3}{3\sqrt{3}} + \dfrac{h^3}{\sqrt{3}}\right| = \dfrac{2h^3}{3\sqrt{3}}$



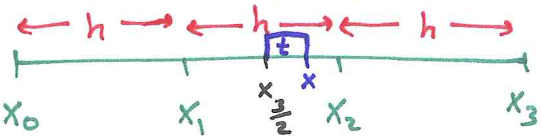
- Hence, $|E_2(x)| = \dfrac{|\phi_2(x)|\,|f'''(c)|}{6} \leq \dfrac{\frac{2h^3}{3\sqrt{3}} M_3}{6} = \dfrac{h^3 M_3}{9\sqrt{3}}$

$\boxed{3}$ $|E_3(x)| \le \dfrac{h^4 M_4}{24}$ where $M_4 = \underset{x_0 \le x \le x_3}{\text{Max}} |f^{(4)}(x)|$

- The Error Term is $E_3(x) = \dfrac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{4!} f^{(4)}(c) \Rightarrow$

$$|E_3(x)| = \frac{|\phi_3(x)|\,|f^{(4)}(c)|}{24} \quad \text{where} \quad |f^{(4)}(c)| \le M_4$$

- Now we need to find an upper bound for $|\phi_3(x)|$.
  - $\phi_3(x) = (x-x_0)(x-x_1)(x-x_2)(x-x_3)$



  - Using the change of variable:

$$x - x_0 = t + \tfrac{3}{2}h$$
$$x - x_1 = t + \tfrac{h}{2}$$
$$x - x_2 = t - \tfrac{h}{2}$$
$$x - x_3 = t - \tfrac{3}{2}h$$

$x_{\frac{3}{2}} = x_0 + \frac{3}{2}h$

$x_0 \le x \le x_3$

$x_0 - x_{\frac{3}{2}} \le x - x_{\frac{3}{2}} \le x_3 - x_{\frac{3}{2}}$

$-\frac{3}{2}h \le t \le \frac{3}{2}h$

- $\phi_3(x) = (t + \frac{3}{2}h)(t + \frac{h}{2})(t - \frac{h}{2})(t - \frac{3}{2}h)$

$$= \left(t^2 - \tfrac{9}{4}h\right)\left(t^2 - \tfrac{h^2}{4}\right) = t^4 - \tfrac{5}{2}h^2t^2 + \frac{9h^4}{16} = \phi_3(t)$$

- $\phi_3' = 4t^3 - 5h^2 t = 0 \Leftrightarrow t(4t^2 - 5h^2) = 0 \Leftrightarrow t = 0$ or $t = \pm\frac{\sqrt{5}}{2}h$
- $|\phi_3(0)| = \frac{9h^4}{16}$, $|\phi_3(\pm\frac{\sqrt{5}}{2}h)| = h^4$ Max

- Hence, $|E_3(x)| = \dfrac{|\phi_3(x)|\,|f^{(4)}(c)|}{24}$

$$\le \frac{h^4 M_4}{24}$$