

## 2.4 Newton - Raphson and secant method.

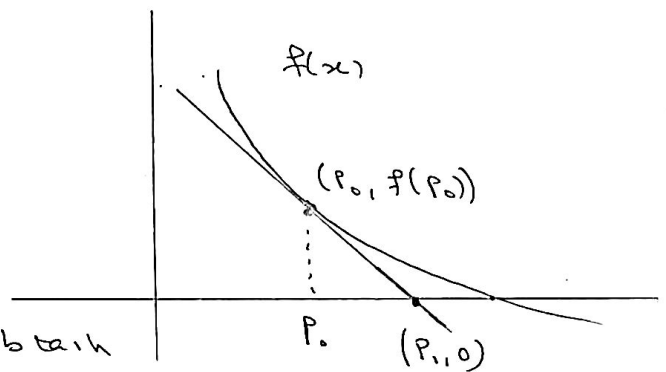
### 1) Newton - Raphson method:

(slope method for finding roots):

Graphical derivation of Newton Method:

$$f'(p_0) = \frac{0 - f(p_0)}{p_1 - p_0}, \text{ solving for } p_1, \text{ then.}$$

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)}$$



we repeat this process to obtain a sequence  $\{p_n\}$  that converges to  $\alpha$ .

$$(i.e.) \quad p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$$

Example:  $e^x - \cos x - 1 = 0$  on  $[0, 1]$

If  $p_0 = 1$ , using Newton Method:

$$p_{n+1} = p_n - \frac{(e^{p_n} - \cos p_n - 1)}{e^{p_n} + \sin p_n}$$

$$\Rightarrow p_1 = 0.669083898, \quad p_2 = 0.603760843$$

$$p_3 = 0.601349991, \quad p_4 = 0.601346767$$

Note  $|p_4 - p_3| < 10^{-5}$ , which is good.

Thm: Newton-Raphson theorem! (Convergence).

Assume  $f \in C^2[a, b]$  and  $\exists p \in [a, b]$  s.t.  $f(p) = 0$

If  $f'(p) \neq 0$ , then  $\exists \delta > 0$  s.t.  $\{P_n\}_{n=0}^{\infty}$  generated

by Newton-method  $P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$ ,  $n = 0, 1, 2, \dots$

will converge to  $p$  for any initial point  $P_0$

where  $P_0 \in [p - \delta, p + \delta]$ . ( $P_0$  close to  $p$ )

Proof: by hypothesis

$f(p) = 0 \Rightarrow$  Consider  $g(x) = x - \frac{f(x)}{f'(x)}$ ,  $g(p) = p$

$$g'(x) = 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2}$$

$$\Rightarrow g'(p) = \frac{f(p)f''(p)}{(f'(p))^2} = 0 < 1$$

beside:  $g(p) = p - \frac{f(p)}{f'(p)} = p$

we conclude: 1)  $g$  is cont. on  $[a, b]$  &  $g' \in C[a, b]$

2)  $p$  is a fixed point and  $g'(p) = 0 < 1$

therefore using fixed point thm, the sequence  $\{P_k\}_{k=0}^{\infty}$ ,  $k=0, 1, \dots$

will converge to the fixed point  $p$ , but  $p$  is

a root of  $f(x)$ , therefore  $\{P_k\}$  converges to  $p$ .  $\square$

The most Important Question is How fast the Iteration Converge.?

Def: If  $p$  is a root of  $f$ . The multiplicity of  $p$  is  $M$  iff  $f(p) = f'(p) = \dots = f^{(M-1)}(p) = 0$  &  $f^{(M)}(p) \neq 0$ .

Def: If  $p$  is a root of Multiplicity  $M$ , then  $\exists$  continuous function  $h(x)$  such that:

$$f(x) = (x - p)^M h(x), \text{ where } h(p) \neq 0.$$

Def: If  $M=1$ , we call  $p$  simple root.

Example:  $f(x) = (x-1) \ln x$ .

one root is  $p=1$ , since  $f(p)=0$ .

$$f'(p) = \frac{(p-1)}{p} - \ln p = \frac{(1-1)}{1} - \ln 1 = 0$$

so we can't use Newton method here for convergence.

$$f''(p) = 2 \Rightarrow p=1 \text{ of Multiplicity } \underline{2}$$

Example:

$$f(x) = x^3 - 3x + 2$$

$p=1$  is a root.

$$\begin{aligned} \Rightarrow f(x) &= (x-1)(x^2+x-2) \\ &= (x-1)(x-1)(x+2) \end{aligned}$$

$\Rightarrow$  1 is double root  
& 2 is a simple root.

$$\begin{array}{r} x^2 + x - 2 \\ x-1 \overline{) x^3 - 3x + 2} \\ \underline{-x^3 + x^2} \phantom{+ 2} \\ x^2 - 3x + 2 \\ \underline{-x^2 + x} \phantom{+ 2} \\ -2x + 2 \\ \underline{-2x + 2} \\ 0 \end{array}$$

Note: Newton method for simple root ( $M=1$ ) is faster than for multiple root ( $M > 1$ ).

### Speed of Convergence : (order of Convergence):

Def: Assume that  $\{P_n\}_{n=0}^{\infty}$  converges to  $P$

and set  $E_n = P - P_n$  for  $n \geq 0$ .

If there exist two positive constants  $A, R$

such that:

$$\lim_{n \rightarrow \infty} \frac{|P - P_{n+1}|}{|P - P_n|^R} = \lim_{n \rightarrow \infty} \frac{|E_{n+1}|}{|E_n|^R} = A$$

Then the sequence is said to converge to  $P$  with

order of convergence  $R$ ,  $A$  is asymptotic error constant

If  $R=1$ , then the convergence of  $\{P_n\}_{n=0}^{\infty}$  is linear.

If  $R=2$ , then the convergence of  $\{P_n\}_{n=0}^{\infty}$  is Quadratic.

Example: show that  $P_n = \frac{1}{n^3}$  converges to 0 linearly?  
root

$$\lim_{n \rightarrow \infty} \frac{|e_{n+1}|}{|e_n|} = \lim_{n \rightarrow \infty} \frac{\left| 0 - \frac{1}{(n+1)^3} \right|}{\left| 0 - \frac{1}{n^3} \right|} = \lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)^3}}{\left(\frac{1}{n^3}\right)}$$

$$= \lim_{n \rightarrow \infty} \frac{n^3}{(n+1)^3} = \left( \lim_{n \rightarrow \infty} \frac{n}{n+1} \right)^3 = \left( \lim_{n \rightarrow \infty} \left( \frac{1}{1 + \frac{1}{n}} \right) \right)^3 = 1$$

$\Rightarrow \frac{1}{n^3} \rightarrow 0$  linearly.

Example: Find Order of Convergence for  $P_n = 10^{-n}$   
 $P_n = 10^{-n} \rightarrow 0$  linearly??

$$\lim_{n \rightarrow \infty} \frac{|0 - 10^{-(n+1)}|}{|0 - 10^{-n}|} = A$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{10^{-(n+1)}}{10^{-nR}} = \lim_{n \rightarrow \infty} \frac{10^{-nR}}{10^{n+1}} = \lim_{n \rightarrow \infty} \frac{10^{-n(R-1)}}{10}$$

A should be positive constant

therefore if  $R > 1$ , then  $\lim_{n \rightarrow \infty} \frac{10^{-n(R-1)}}{10} = \infty$

if  $R = 1$ , then  $\lim_{n \rightarrow \infty} \frac{10^{-n(R-1)}}{10} = \frac{1}{10}$

if  $R < 1$ , then  $\lim_{n \rightarrow \infty} \frac{10^{-n(R-1)}}{10} = 0$

Note: for every  $R$ ,  $\exists!$  A. Here  $R = 1$  &  $A = \frac{1}{10}$ .

Note: If  $R$  is large, then  $\{P_n\}$  converges rapidly to  $P$

Note: we can write it as:

$$|E_{n+1}| \approx A |E_n|^R$$

3) Find the order of convergence of the sequence

1.5, 1.373333333, 1.365262015, 1.365230014, 1.365230013

$\underbrace{1.5}_{P_1}, \underbrace{1.373333333}_{P_2}, \underbrace{1.365262015}_{P_3}, \underbrace{1.365230014}_{P_4}, \underbrace{1.365230013}_{P_5}$

$$|E_1| = |P_1 - P_0| = 0.126666667$$

$$|E_2| = |P_2 - P_1| = 0.008071318$$

$$|E_3| = |P_3 - P_2| = 0.000032001$$

$$|E_4| = |P_4 - P_3| = 0.000000001$$

(2) or  $\mu$   
converges for  $R=2$

$$\frac{|E_2|}{|E_1|^2} = 0.50306 \quad \text{and} \quad \frac{E_3}{E_2^2} = 0.49122 \rightarrow$$

$R=2$ ,  $A \approx 0.5$

4) When estimating the root of the function  $f(x) = (x-1)^2 \ln x$  and using Newton Method, find the order of convergence  $R$  and the asymptotic error constant  $A$ .

$p=1$  is the root.

$$\begin{aligned} f(x) &= (x-1)^2 \ln x = (x-1)^2 \left( (x-1) \left( \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \dots \right) \right) \\ &= (x-1)^3 \left( 1 - \frac{(x-1)}{2} + \frac{(x-1)^2}{3} + \dots \right) \\ &= (x-1)^3 h(x), \quad h(1) = 1 \neq 0 \end{aligned}$$

$\Rightarrow \mu=3 \Rightarrow R=1, A = \frac{2}{3}$  or  $\mu$

$R: f' = \frac{(x-1)^2}{x} + 2(x-1) \ln x \Rightarrow f'(1) = 0$

$f'' = 2 \frac{x^2-1}{x^2} + 2 \ln x + 2 - \frac{2}{x} \Rightarrow f''(1) = 0$

$f''' = 2 \frac{2x(x-1)}{x^3} + 2 \frac{1}{x} - \frac{2}{x^2} = \frac{4x^2-4x+2x-2}{x^3} = \frac{4x^2-2x-2}{x^3} \Rightarrow f'''(1) = \frac{4-2-2}{1} = 0$

$\left(\frac{2}{3}\right)^4$

# Thm 1 Convergence Rate of Newton Method:

Assume  
If we use Newton-R. method such that  $\{P_n\} \rightarrow P$

where  $f(P) = 0$ , then:

Newton method is better for simple root because  $R$  is bigger

1) If  $P$  is simple root ( $M=1$ ), then  $R=2$ .

$$\text{and } A = \frac{|f''(P)|}{2|f'(P)|}$$

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$$

2) If  $P$  is a multiple root ( $M > 1$ ), then  $R=1$

$$\text{and } A = \frac{M-1}{M}$$

Find  $A$  and  $R$  both theoretically and Numerically

Note: always start with theoretically

Example: Quadratic Convergence at a simple root.

$f(x) = x^3 - 3x + 2$ , has  $-2$  as a simple root.

start with  $P_0 = -2.4$ , Using Newton method:

2

Numerically:	$P_k$	$P_{k+1} - P_k$	$E_k = P - P_k$	$\frac{ E_{k+1} }{ E_k ^2}$
	-2.40000000	0.323809524	0.40000000	0.476190475
	-2.076190476	0.072594465	0.076190476	0.619469086
	-2.003596011	0.003587422	0.003596011	0.664202613
	-2.000008589	0.000008589	0.000008589	⋮
	-2.00000000	0.00000000	0.00000000	✓

Theoretically: Simple root  $\Rightarrow R=2$  &  $A = \frac{|f''(P)|}{2|f'(P)|}$

$$A = \frac{|6(-2)|}{2|3(-2)^2 - 3|} = \frac{12}{18} \approx 0.66667$$

Now for the same function we will check

Linear convergence to a double Root.  $P=1$

Numerically	$P_k$	$P_{k+1} - P_k$	$E_k = P - P_k$	$\frac{ E_{k+1} }{ E_k }$
0	1.20000000	-0.09696967	-0.20000000	0.515151515
1	1.103030303	-0.050673883	-0.103030303	0.508165253
2	1.052356420	-0.025955609	-0.052356420	0.496751115
3	1.02640081	-0.013143081	-0.026400811	0.509753688
4	1.013257730	-0.006614311	-0.013257730	0.501097775
5	1.006643419	-0.003318055	-0.006643419	0.500550093
	⋮	⋮	⋮	⋮

Theoretically  $R=1$  since we have double root ( $M > 1$ )

and  $A = \frac{M-1}{M} = \frac{2-1}{2} = \frac{1}{2}$ .

Example: show that the Bisection method Converges

Linearly to 0? on  $[a, b]$

$$\lim_{n \rightarrow \infty} \frac{|P - P_{n+1}|}{|P - P_n|} \leq \frac{\frac{(b-a)}{2^{n+2}}}{\frac{(b-a)}{2^{n+1}}} = \frac{2^{n+1}}{2^{n+2}} = \frac{1}{2}$$



Newton and secant: Order of Convergence ( $P$  unknown)

(Q1) Consider the equation  $x = \cos x$

- (a) Use Newton's method with  $P_0 = 0.2$  to estimate the solution of this equation with error less than  $10^{-5}$
- (b) Find the order of convergence and the asymptotic error constant both numerically and theoretically.

Solution: (a)  $f(x) = x - \cos x$ ,  $f'(x) = 1 + \sin x$ ,  $f''(x) = \cos x$

Newton's iteration:  $P_{n+1} = P_n - \frac{P_n - \cos P_n}{1 + \sin P_n}$

$n$	$P_n$	$ P_n - P_{n-1} $
0	0.2	—
1	0.850777122	0.650777122
2	0.741530193	0.109246929
3	0.739086449	0.002443744
4	0.739085133	0.000001316

So  $P \approx P_4 = 0.739085133$

(b) Theoretically:  $f'(P) = f'(0.739085133) = 1.673612029 \neq 0$ ,

So  $P$  is a simple root.

Therefore,  $R = 2$  and  $A = \frac{|f''(P)|}{2|f'(P)|} = 0.220805395$

Numerically:

$ E_n  \approx  P_n - P_{n-1} $	$\frac{ E_{n+1} }{ E_n ^2}$
0.650777122	0.257955435
0.109246929	0.204756281
0.002443744	0.220365941
0.000001316	

*Newton*

Proof : part ① Recall

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$$

Let us define  $g(x) := x - \frac{f(x)}{f'(x)}$

Note that  $g(p) = p$ .

2 Take Taylor second expansion of  $g$  about  $P$ .

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(c_n)(x-p)^2}{2}$$

$c_n$  is the error

substitute  $P_n$  in  $x$ :

$$P_{n+1} = g(P_n) = p + g'(p)(P_n - p) + \frac{g''(c_n)(P_n - p)^2}{2} \dots (*)$$

$$\text{Now: } g'(x) = \frac{f(x) f''(x)}{(f'(x))^2} \Rightarrow g'(p) = 0.$$

$$g''(x) = \dots$$

$$* \quad g''(p) = \frac{f''(p)}{f'(p)}$$

substitute in (\*), we have:

$$g(P_n) = p + \frac{g''(c_n)(P_n - p)^2}{2}$$

$$\Rightarrow |P_{n+1} - p| = \frac{g''(c_n)(P_n - p)^2}{2}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{|E_{n+1}|}{|E_n|^2} = \lim_{n \rightarrow \infty} \left| \frac{g''(c_n)}{2} \right| = \left| \frac{f''(p)}{2 f'(p)} \right|.$$

$c_n$  goes to  $p$  as  $n$  goes to infinity

proof (2) We can prove it same way as (1), or

Since  $f$  has  $P$  as a multiple root ( $M > 1$ ), then:

$$f(x) = (x-p)^M h(x) \quad \text{where } h(p) \neq 0$$

Now, we know that:

So we use this method for  $M=1$  &  $M>1$

$$P_{k+1} =: g(P_k) = P_k - \frac{f(P_k)}{f'(P_k)}$$

(1)

(i.e) we define  $g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{(x-p)^M h(x)}{M(x-p)^{M-1} h(x) + (x-p)^M h'(x)}$

$$\Rightarrow g(x) = x - \frac{(x-p)h(x)}{Mh(x) + (x-p)h'(x)}, \quad \text{Note } \boxed{g(p) = p} \text{ fixed point}$$

$$g'(x) = 1 - \frac{(Mh(x) + (x-p)h'(x))(x-p)h'(x) + h(x) - (x-p)h(x)[Mh'(x) + h'(x) + (x-p)h''(x)]}{(Mh(x) + (x-p)h'(x))^2}$$

(3)

$$\Rightarrow g'(p) = 1 - \frac{Mh^2(p)}{M^2h^2(p)} = 1 - \frac{1}{M} = \frac{M-1}{M}, \quad \text{need to show } M \neq 1$$

Now, let's take the first order Taylor poly. of  $g(x)$  around  $(P)$

$$\Rightarrow g(x) = g(p) + g'(c)(x-p) \quad (2)$$

$$\Rightarrow P_{n+1} = g(P_n) = p + g'(c)(P_n - p)$$

$$\boxed{g'(c) \approx g'(p)}$$

$$\Rightarrow P - P_{n+1} \approx g'(p)(P - P_n)$$

$\lim_{n \rightarrow \infty} c_n = p$

$$\Rightarrow |E_{n+1}| \approx \frac{M-1}{M} |E_n|$$

Thm: Acceleration of Newton Method: ( $M > 1 \Rightarrow K' = 1$ )  
slow

If  $P$  is a root of multiplicity  $M > 1$ , then the iteration  $R=1$

$$P_{n+1} = P_n - M \frac{f(P_n)}{f'(P_n)} \quad \text{will converge quadratically to } P$$

Homework

proof:

We need to prove that the error term in Taylor

expansion of  $g(x)$  starts from  $(x-p)^2 g''(c)$ .

$$\text{Let } g(x) = x - \frac{M f(x)}{f'(x)} = x - \frac{M (x-p)^M h(x)}{M(x-p)^{M-1} h(x) + (x-p)^M h'(x)}$$

$$\Rightarrow g(x) = x - \frac{M(x-p)h(x)}{Mh(x) + (x-p)h'(x)}$$

$$g'(x) = 1 - \frac{M [Mh(x) + (x-p)h'(x)] [(x-p)h'(x) + h(x)]}{(Mh(x) + h'(x)(x-p))^2}$$

$$+ \frac{M(x-p)h(x) (Mh'(x) + (x-p)h''(x) + h'(x))}{(Mh(x) + h'(x)(x-p))^2}$$

$$g'(p) = 1 - \frac{M^2 h^2(p)}{M^2 h^2(p)} = 1 - 1 = 0.$$

$\Rightarrow$  Taylor expansion of  $g$  about  $p$ :

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(c)}{2!} (x-p)^2$$

$$\Rightarrow P_{n+1} = g(P_n) = g(p) + \frac{g''(c)}{2!} (P_n - p)^2$$

$$\Rightarrow |P - P_{n+1}| \approx \frac{g''(c)}{2!} |P - P_n|^2 \quad \text{Converges quadratically}$$

Example: Let  $f(x) = x^3 - 3x + 2 = (x-1)^2(x-2)$

1 has multiplicity 2

Using Acceleration Method, the Iteration is

$$P_{n+1} = P_n - 2 \frac{f(P_n)}{f'(P_n)}$$

$P_n$	$P_{n+1} - P_n$	$E_n = P - P_n$	$\frac{ E_{n+1} }{ E_n ^2}$
1.20000	-0.193939394	-0.20000	0.15151515150
1.0060606	-0.006054519	-0.006060606	0.165718578
1.000006087	-0.000006087	-0.000006087	
1.00--	0.0--	0.0--	

So, the Iteration Converges Quadratically.

Note: Using Newton method we need 6 Iteration.

but Using Acceleration method, we need only 4 Iteration

Example: Show that if  $g(p) = p$  &  $g'(p) = g''(p) = 0$ , then the sequence generated by  $P_{n+1} = g(P_n)$  converges <sup>at least</sup> cubically.

proof:  $g(x) = g(p) + g'(p)(x-p) + \frac{g''(p)}{2!}(x-p)^2 + \frac{g'''(c)}{3!}(x-p)^3$

$$P_{n+1} = g(P_n) = p + 0 + 0 + \frac{g'''(c)}{3!}(P_n - p)^3$$

$$\Rightarrow |P_{n+1} - p| = \frac{|g'''(c)|}{3!} |P_n - p|^3$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{|P_{n+1} - p|}{|P_n - p|^3} = \frac{|g'''(p)|}{6}$$

At least ↓

If $g'''(p) = 0 \Rightarrow R > 3$
If $g'''(p) \neq 0 \Rightarrow R = 3$

(47)

(Q) Let  $p$  be a fixed point of  $g(x)$ .

(a) show that if  $g'(p) = g''(p) = \dots = g^{(k-1)}(p) = 0$  and  $g^{(k)}(p) \neq 0$ , then the fixed point iteration of  $g(x)$  will converge to  $p$  with  $R = k$  and  $A = \left| \frac{g^{(k)}(p)}{k!} \right|$

(b) Use part (a) to show that if  $p$  is a simple root of  $f(x)$ , then Newton's iteration will converge to  $p$  with  $R = 2$  and  $A = \left| \frac{f''(p)}{2f'(p)} \right|$

**Proof:** (a) Apply Taylor's expansion of  $g(x)$  about  $x = p$

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(p)}{2!}(x-p)^2 + \dots + \frac{g^{(k-1)}(p)}{(k-1)!}(x-p)^{k-1} + \frac{g^{(k)}(c)}{k!}(x-p)^k$$

but  $g(p) = p$  and  $g'(p) = g''(p) = \dots = g^{(k-1)}(p) = 0$

$$\rightarrow g(x) = p + \frac{g^{(k)}(c)}{k!}(x-p)^k$$

Substitute  $x = p_n \rightarrow g(p_n) = p + \frac{g^{(k)}(c)}{k!}(p_n - p)^k$ ,  $c$  between  $p_n$  and  $p$

but  $g(p_n) = p_{n+1} \rightarrow p_{n+1} - p = \frac{g^{(k)}(c)}{k!}(p_n - p)^k \rightarrow \frac{p_{n+1} - p}{(p_n - p)^k} = \frac{g^{(k)}(c)}{k!} \rightarrow \frac{|E_{n+1}|}{|E_n|^k} = \left| \frac{g^{(k)}(c)}{k!} \right|$

now take limit as  $n \rightarrow \infty$  and considering that  $c \approx p$  when  $n \rightarrow \infty$

$$\rightarrow \lim_{n \rightarrow \infty} \frac{|E_{n+1}|}{|E_n|^k} = \left| \frac{g^{(k)}(p)}{k!} \right| \rightarrow R = k \text{ and } A = \left| \frac{g^{(k)}(p)}{k!} \right|$$

(b) We know that Newton's iteration is a special case of FPI with  $g(x) = x - \frac{f(x)}{f'(x)}$

Therefore, based on part (a), we only need to prove that  $g'(p) = 0$  but  $g''(p) \neq 0$

Recall that  $p$  is a simple root of  $f(x)$  mean  $f(p) = 0$  and  $f'(p) \neq 0$

now,  $g'(x) = 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2} \rightarrow g'(p) = 0$

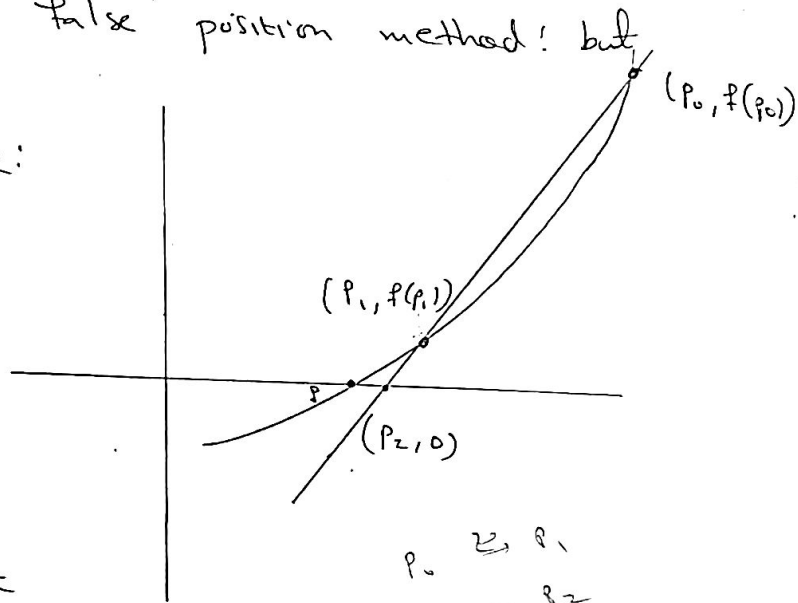
now,  $g''(x) = \frac{(f'(x))^2 [f(x)f'''(x) + f''(x)f'(x)]}{(f'(x))^4} = \frac{f(x)f'''(x) + f''(x)f'(x)}{(f'(x))^2} \rightarrow g''(p) = \frac{f''(p)}{f'(p)} \neq 0$

$$\rightarrow R = 2 \text{ and } A = \left| \frac{g''(p)}{2!} \right| = \left| \frac{f''(p)}{2f'(p)} \right|$$

## 2) Secant method:

this method is similar to false position method! but faster. finding the slope:

$$\frac{f(p_1) - 0}{p_1 - p_2} = \frac{f(p_1) - f(p_0)}{p_1 - p_0}$$



Solving for  $p_2$ , we get

$$p_2 = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)}$$

$$\vdots$$

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$$

Note: In this method we need two Initial points  $p_0$  &  $p_1$

Moreover; this method is used only for simple roots with  $R = 1.618$  & for multiple root with  $R = 1$

Thm: If we use secant method so that  $p_n \rightarrow p$  (simple root)

then:

$$\lim_{n \rightarrow \infty} \frac{|E_{n+1}|}{|E_n|^{1.618}} = \left| \frac{f''(p)}{2f'(p)} \right|^{0.618} \quad \text{for simple}$$

(i.e)  $R \approx 1.618 \approx \frac{\sqrt{5}+1}{2}$

for simple root.

For Multiple root only Numerically.

(Q2) Consider the equation  $x = \cos x$ .

- (a) If  $P_0 = 0.5$  and  $P_1 = \frac{\pi}{4}$ , use the secant method to approximate the root of the equation with accuracy of  $10^{-4}$ .
- (b) Find the order of convergence and the asymptotic error constant both theoretically and numerically.

Solution: (a)  $f(x) = x - \cos x$ ,  $f'(x) = 1 + \sin x$ ,  $f''(x) = \cos x$

Secant's iteration:  $P_{n+2} = P_{n+1} - \frac{f(P_{n+1})(P_{n+1} - P_n)}{f(P_{n+1}) - f(P_n)}$

*secant*

$n$	$P_n$	$ P_n - P_{n-1} $
0	0.5	—
1	0.785398163	0.285398163
2	0.736384138	0.049014025
3	0.739058138	0.002674
4	0.739085149	0.000027011338

So  $P \approx P_4 = 0.739085149$

(b) Theoretically:  $f'(P) = f'(0.739085149) = 1.673612041 \neq 0$ ,

So  $P$  is a simple root.

Therefore,  $R = 1.618$  and  $A = \left| \frac{f''(P)}{2f'(P)} \right|^{0.618} = 0.393185938$

Numerically:

$ E_n  \approx  P_n - P_{n-1} $	$\frac{ E_{n+1} }{ E_n ^{1.618}}$
0.285398163	0.372735166
0.049014025	0.351741034
0.002674	0.393005788
0.000027011338	



**Table 2.8** Acceleration of Convergence at a Double Root

$k$	$p_k$	$p_{k+1} - p_k$	$E_k = p - p_k$	$\frac{ E_{k+1} }{ E_k ^2}$
0	1.200000000	-0.193939394	-0.200000000	0.151515150
1	1.006060606	-0.006054519	-0.006060606	0.165718578
2	1.000006087	-0.000006087	-0.000006087	
3	1.000000000	0.000000000	0.000000000	

**Table 2.9** Comparison of the Speed of Convergence

Method	Special considerations	Relation between successive error terms
Bisection		$E_{k+1} \approx \frac{1}{2} E_k $
Regula falsi		$E_{k+1} \approx A E_k $
Secant method	Multiple root	$E_{k+1} \approx A E_k $
Newton-Raphson	Multiple root	$E_{k+1} \approx A E_k $
Secant method	Simple root	$E_{k+1} \approx A E_k ^{1.618}$
Newton-Raphson	Simple root	$E_{k+1} \approx A E_k ^2$
Accelerated Newton-Raphson	Multiple root	$E_{k+1} \approx A E_k ^2$

**Example 2.17 (Acceleration of Convergence at a Double Root).** Start with  $p_0 = 1.2$  and use accelerated Newton-Raphson iteration to find the double root  $p = 1$  of  $f(x) = x^3 - 3x + 2$ .

Since  $M = 2$ , the acceleration formula (31) becomes

$$p_k = p_{k-1} - 2 \frac{f(p_{k-1})}{f'(p_{k-1})} = \frac{p_{k-1}^3 + 3p_{k-1} - 4}{3p_{k-1}^2 - 3},$$

and we obtain the values in Table 2.8. ■

Table 2.9 compares the speed of convergence of the various root-finding methods that we have studied so far. The value of the constant  $A$  is different for each method.