ch2 Solutions of Nonlinear equations $f(x) = 0$

2.2 Bracketing methods for locating a root: (solve $f(x)=0$):

I Bisection method : To solve $f(x) = 0$ for continuous function on $[a,b]$, where $f(a) \cdot f(b) < 0$

we take $c_0 = \frac{a+b}{2}$

1) If $f(c_0) = 0$, we find the root.
2) If $f(c_0) \cdot f(a) < 0$, then we assume $r \in [a, c_0]$
   & we call $[a, c_0] = [a_1, b_1]$
3) If $f(c_0) \cdot f(b) < 0$, then we assume $r \in [c_0, b]$
   & we call $[c_0, b] = [a_1, b_1]$

Then according to the case, we take $c_1 = \frac{a_1 + b_1}{2}$.

Again the three posiblites, then

$c_2 = \frac{a_2 + b_2}{2}$, ..., $c_n = \frac{a_n + b_n}{2}$.

Note : 1) Bisection method is used If $f(x)$ satisfies Bolzan's thm.

2) If this method satisfies Bolzano's then It is always converge.

<u>Example:</u> Find the root of $e^x - \cos x - 1 = 0$ on $[0,1]$

$f(0) = -1, \quad f(1) > 0$

	a_n	c_n	b_n	$f(c_n)$
$n=0$	0^-	0.5	1^+	-0.22886
$n=1$	0.5^-	0.75	1^+	$+0.38531$
$n=2$	0.5^-	0.625	0.75^+	$+0.057283$
$n=3$	0.5^-	0.5625	0.625^+	-0.090870
$n=4$	0.5625^-	0.59375	0.625	

Advantage: Always converges to the root

Disadvantage: Very slow.

<u>Note:</u> # of Iteration with accuracy $< 10^{-\square} = \epsilon \Rightarrow \frac{b-a}{2^{n+1}} < \epsilon$ & solve for n

<u>Thm:</u> Assume that $f \in C[a,b]$ and that $\exists\, r \in [a,b]$ (s.t) $f(r) = 0$. If $f(a) \cdot f(b) < 0$ and $\{c_n\}_{n=0}^{\infty}$ represents the sequence of mid points generated by Bisection method, then:

1) $$|r - c_n| \leq \frac{b-a}{2^{n+1}}, \quad n = 0, 1, \ldots$$

2) $\{c_n\}_{n=0}^{\infty}$ converges to the zero $x = r$

(i.e) $$\lim_{n \to \infty} c_n = r.$$

<u>Example:</u> above with accuracy $< 10^{-3}$

$\frac{1}{2^{n+1}} < 10^{-3} \Rightarrow 9.9 < n+1 \Rightarrow$ # of Iterat. $= \boxed{n+1 = 10}$ Since we started from $c_0 \to c_n$

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proof: Note first that r & $c_n \in [a_n, b_n]$

where $c_n = \dfrac{a_n + b_n}{2}$

this Implies that

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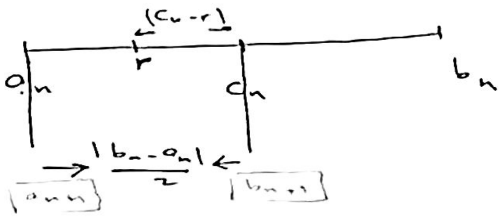
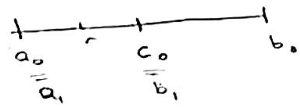
$$|r - c_n| \leq \frac{b_n - a_n}{2}, \forall n$$



observe the succesive interval widths :

Note: $b_1 - a_1 = \dfrac{b_0 - a_0}{2}$



$$b_2 - a_2 = \frac{b_1 - a_1}{2} = \frac{b_0 - a_0}{2^2}$$

$$\vdots$$

$$b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2} = \cdots = \frac{b_0 - a_0}{2^n}$$

$$\boxed{\Rightarrow |r - c_n| \leq \frac{b_0 - a_0}{2^{n+1}} \quad \forall n.}$$

Now, we want to show $\lim\limits_{n\to\infty} c_n = r$.

$$0 \leq |r - c_n| \leq \frac{b_0 - a_0}{2^{n+1}} = \frac{b - a}{2^{n+1}}$$

the as n Increasing $\lim\limits_{n\to\infty} \dfrac{b-a}{2^{n+1}} = 0 \Rightarrow$

$$\Rightarrow \lim_{n\to\infty} |r - c_n| = 0 \Rightarrow \lim_{n\to\infty} c_n = r.$$

Note : If we want # of Iteration with accuracy $10^{-\square} = \epsilon$

we assume $\dfrac{b-a}{2^{n+1}} < \epsilon$ we solve for n $\quad n+1 \geq \dfrac{[\ln(b-a) - \ln \epsilon]}{\ln 2}$

# of Iteration $= n+1$ $(c_0 \longrightarrow c_n)$

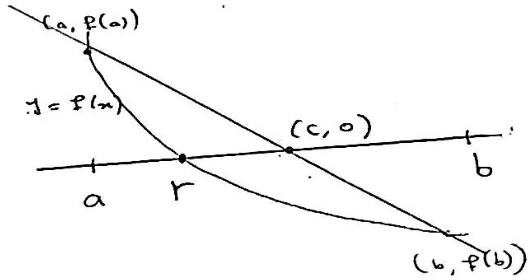
False position method :

This method was developed because the bisection method is very slow. [Faster than Bisection]

Assume $f(a) \cdot f(b) < 0$.

In bisection method we choose c to be mid point.

where In false position method we will ~~choose~~ find the point $(c, 0)$, where c is a point on the secant line L joining $(a, f(a))$, $(b, f(b))$.



To determine c, we compute the slope:

$$m = \frac{f(b) - f(a)}{b - a} \quad \cdots (1)$$

& $$m = \frac{0 - f(b)}{c - b} \quad \cdots (2)$$

$$(1) = (2) \implies c = b - \frac{f(b)(b-a)}{f(b) - f(a)}$$

Now, we continue, the three posibilities

1) If $f(a) \cdot f(c) < 0$, then the zero lies in $[a, c]$

2) If $f(c) \cdot f(b) < 0$, then the zero lies in $[c, b]$

3) If $f(c) = 0$, then c is a zero.

In General: $$C_n = b_n - \frac{f(b_n)(b_n - a_n)}{f(b_n) - f(a_n)}$$

Notes: 1) False position is faster than Bisection method

2) We can't know the error before Solving.

3) Always there exist one end point fixed and another one is changable.

Example: $e^{x} - \cos x - 1 = 0$, $[0,1]$

	a_n	c_n	b_n	$f(c_n)$
c_0	0^-	0.459	1^+	-0.31372
c_1	0.459^-	0.5726	1^+	--

$$c_0 = 1 - \frac{f(1)(1-0)}{f(1) - f(0)} = 0.459$$

$$c_1 = 1 - \frac{f(1)(1-0.459)}{f(1) - f(0.459)} = 0.5726.$$

to reach 0.5 in bisection we needed 4 Iterations

Example: Find an approximation to $\sqrt[3]{25}$ correct to within 10^{-2} using the bisection method? false position method?

$x = \sqrt[3]{25} \Rightarrow x^3 - 25 = 0$ on $[2^-, 3^+]$

$c_0 = 2.5$

With accuracy 10^-2
Let x = 25^(1/3)
f(x) = x^3 - 25 = 0 --> [2,3]
f(2) = -17, f(3) = 2

Example: Solve $x \sin x = 1$ on $[0, 2]$.

$$f(x) = x \sin x - 1$$

$$f(0) = -1 \quad , \quad f(2) = 0.81859485$$

$$c_0 = b_0 - \frac{f(b_0)(b_0 - a_0)}{f(b_0) - f(a_0)} = 2 - \frac{0.81859485\,(2-0)}{0.81859485 - (-1)}$$

$$= 1.09975017.$$

Now $f(c_0) = -0.02001912$.

We choose $[a_1, b_1] = [1.09975017, 2]$

$$c_1 = b_1 - \frac{f(b_1)(b_1 - a_1)}{f(b_1) - f(a_1)} = 1.12124074$$

$$f(c_1) = 0.00983461.$$

then $[a_2, b_2] = [1.09975017, 1.12124074]$

a_n	c_n	b_n	$f(c_n)$
0^-	1.09975017	2^+	-0.02001912
1.09975017^-	1.12124074	2^+	$^+0.00983461$
1.09975017^-	1.11416120	1.12124074	$^+0.00000563$
1.09975017	1.11415714	1.11416120	0.00000000

we fixed 5 digits