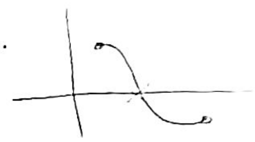
1.1 Review of Calculus.

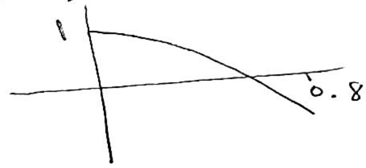
Bolzano Theorem: If f is cont. on $[a,b]$ & $f(a) \cdot f(b) < 0$, then $\exists$ at least one $c \in (a,b)$ such that $f(c) = 0$.



Example: $f(x) = \cos x - x$ on $[0,1]$

$f(0) > 0$, $f(1) < 0$

F is cont. on $[0,1]$, then $\exists\, c \in (0,1)$ such that $f(c) = 0$

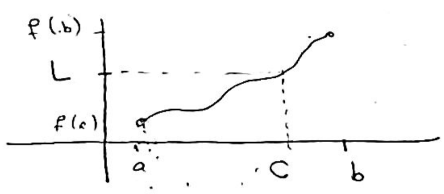


Note: If Bolzano's thm satisfies, then, we can conclude that there is a solution.

If Not, WE Don't know.. (Another way.)

Intermediate value thm: (IVT) or IVP

If f is Cont on $[a,b]$, & L is any number between $f(a)$ & $f(b)$, then $\exists\ c \in (a,b)$ such that $f(c) = L$



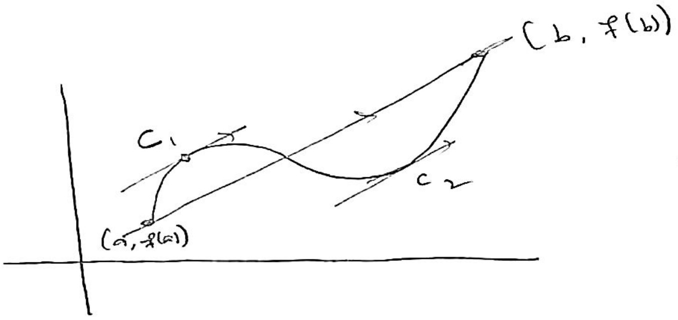
Mean Value theorem: (MVT)

If f is cont. on $[a,b]$ & differentiable on (a,b)

" Smooth function", then $\exists$ at least one $c \in (a,b)$

such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

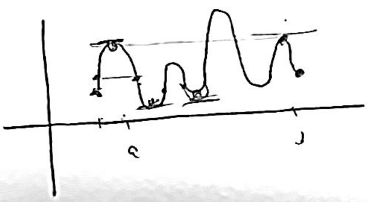
Geometrically: slope of the tangent line to the graph of $y=f(x)$ at $(c, f(c))$ = the slope of the secant line through the points $(a, f(a)), (b, f(b))$



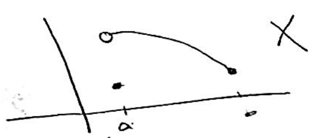
Rolle's Theorem:

If f is cont. on $[a,b]$ & diff on (a,b) & $f(a) = f(b)$. Then there exists at least one $c \in (a,b)$ such that

$f'(c) = 0$. (horizontal tangent)



Not cont



Def: <u>Taylor Series</u> :

1) Taylor series of $f(x)$ at $x = \overset{x_0}{a}$ is given by:

$$f(\underset{x_0}{a}) + f'(\underset{x_0}{a})(x - \underset{x_0}{a}) + \frac{f''(\underset{x_0}{a})(x - \overset{x_0}{a})^2}{2!} + \cdots + \frac{f^{(n)}(\overset{x_0}{a})(x - \underset{x_0}{a})^n}{n!} + \cdots$$

2) <u>Taylor Polynomial</u> of order n

(same as (1) but finite, then $\exists$ an error)

3) If $\overset{x_0}{a} = 0$, then the series is called

Maclaurin series.

<u>Example:</u> at $\overset{x_0}{a} = 0$, find maclaurin series for:

1) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$

2) $\sin x = 0 + x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

3) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

Taylor's theorem: Assume $f \in C^{n+1}[a,b]$ & $x_0 \in (a,b)$

then $\forall\, x \in (a,b)$, $\exists\, c \in (x_0, x)$ such that

$$f(x) = P_n(x) + R_n(x).$$

where
$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)\,(x-x_0)^k}{k!}$$

and
$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\,(x-x_0)^{n+1}$$

(i-e)
$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + \cdots$$
$$+ \frac{f^{(n)}(x_0)(x-x_0)^n}{n!} + \frac{f^{(n+1)}(c)(x-x_0)^{n+1}}{(n+1)!}$$

Note: If we need linear estimation of $f(x)$, then

$$f(x) \simeq f(x_0) + f'(x_0)(x-x_0) \qquad \text{Linear.}$$

with error
$$\frac{f''(x_0)(x-x_0)^2}{2!} + \cdots \quad \text{or} \quad \underbrace{\left[\frac{f''(c)}{2!}(x-x_0)^2\right]}_{\text{error}}$$

$$f(x) \simeq f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} \qquad \text{Quadratic}$$

with error
$$\frac{f'''(x_0)(x-x_0)^3}{3!} + \cdots \quad \text{or} \quad \underbrace{\left[\frac{f'''(c)}{3!}(x-x_0)^3\right]}_{\text{error}}$$

In General:

$$f(x) \simeq f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)(x-x_0)^n}{n!}$$

with error $\dfrac{f^{(n+1)}(x_0)(x-x_0)^{n+1}}{(n+1)!} + \cdots$ (Infinite terms)

or simply: $\dfrac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$

Therefore we conclude that:

Taylor's Inequality

$$\text{Error} = \frac{f^{(n+1)}(c)(x-x_0)^{n+1}}{(n+1)!}$$, where $c \in (x_0, x)$

[illegible] لقيمة ايجاد c

$$\Rightarrow |\text{Error}| \leq \max_x \frac{|f^{(n+1)}(x)|\,(x-a)^{n+1}}{(n+1)!}$$

Example: $f(x) = \cos x$, $x_0 = 0$, Determine second taylor polynomial & the error.

$$\cos x = \underbrace{1 - \frac{x^2}{2}}_{\text{second}} + \underbrace{\boxed{\frac{f'''(c)}{3!}x^3}}_{\text{error}} = 1 - \frac{x^2}{2} + \frac{1}{6}x^3 \sin c$$

for $c \in (0, x)$

If we assume $x = 0.01$, then

$$\cos 0.01 = \underbrace{0.99995}_{\text{appr}} + \underbrace{\frac{10^{-6}}{6}\sin C}_{\text{error}}$$

$|\sin x| \leq |x|$

$\Rightarrow |\sin c| \leq |c|$

$C \in (0, 0.01)$

$$|\cos 0.01 - 0.99995| = 0.1\bar{6} \times 10^{-6} |\sin c| \leq \cancel{0.1\bar{6}\times 10^{-6} \times C}$$

$0 < C < 0.01$

$$\Rightarrow |\cos 0.01 - 0.9995| = \frac{10^{-6}}{6} \sin C \leq 0.1\bar{6} \times 10^{-8}.$$

Increasing

$$\frac{10^{-6}}{6} \times \max|\sin x| = \frac{10^{-6}}{6} \times 0.00999$$

$(\sin 0.01)$

Co

Example: $f(x) = e^{x}$ about $x_0 = 0$ (First polynomial)

$$e^{x} = \underbrace{1 + x}_{\text{first}} + \frac{x^2}{2} e^{c}$$

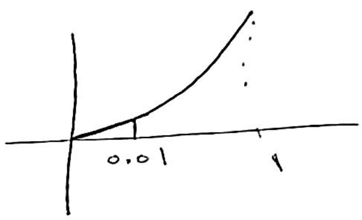
If we assume $x = 0.01$, then

$$e^{0.01} \simeq 1 + 0.01$$

with error: $\dfrac{(0.01)^2}{2} e^{c}$, $c \in (0, 0.01)$

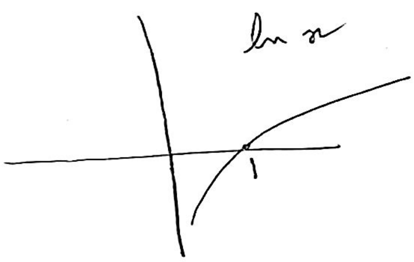
To bound the error:

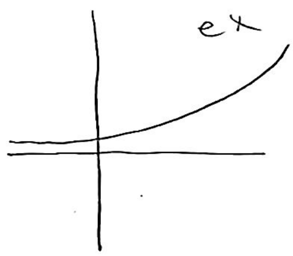
$$\frac{(0.01)^2}{2} e^{c} = 5 \times 10^{-5} e^{c}$$

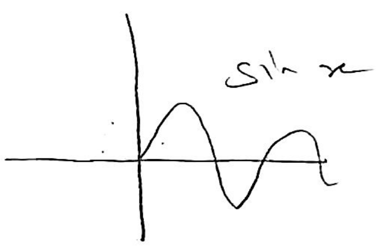


$$\leq 5 \times 10^{-5} e^{0.01} \simeq 5 \times 10^{-5} \times 1.01005 = 5.05025 \times 10^{-5}$$

Extreme Value theorem: If a real-valued function f is continuous in $[a, b]$ (closed & bounded) then f must attain a maximum and a minimum each at least once.







1.3 Error Analysis:

Def: If $\hat{p}$ is an approximation to p, then:

1) Absolute Error is $|p-\hat{p}| = E_p$

2) Relative Error is $R_p = \dfrac{|p-\hat{p}|}{|p|}$

Note: $R_p \longrightarrow E_p$ if $p \longrightarrow 1$.

Note: The relative error is more meaningful, because the relative error takes into consideration the size of the value.

Example: $p = 1.48$, $\hat{p} = 1.5$

$E_p = 0.02$, $R_p = 0.014$.

Example: $q = 1000000$, $\hat{q} = 999994$

$E_q = 6$, $R_q = 6 \times 10^{-6}$

Example: $m = 0.000000012$ (7), $\hat{m} = 0.000000009$ (8)

$E_m = 3 \times 10^{-9}$, $R_m = 0.25$

Def: The number $\hat{p}$ is said to approximate p to d significant digits if d is the Largest nonnegative integer for which:

$$\frac{|p-\hat{p}|}{|p|} \leq \frac{10^{1-d}}{2} \iff 2R_p \leq 10^{1-d}$$

Note: Some other books use the Def:

$$\frac{|p-\hat{p}|}{|p|} \leq \frac{10^{-d}}{2}.$$

Note(*) If we have 0.00123456, so we can approximate it for 3 significant digits as:

0.00123.

Example: If $p = 1.48$, $\hat{p} = 1.5$, then $R_p = 0.014$

$\Rightarrow$ $2R_p = 0.028.$

Lets start with $d = 0$

$d=0$ $\quad 0.028 < 10^{1}$ (✓)

$d=1$ $\quad 0.028 < 10^{0}$ (✓)

$d=2$ $\quad 0.028 < 10^{-1}$ (✓)

$d=3$ $\quad 0.028 < 10^{-2}$ (×)

$\Rightarrow \boxed{d=2}$

Example: $q = 1000000$, $\hat{q} = 999994 \Rightarrow R_q = 6 \times 10^{-6}$

$\Rightarrow 2R_q = 1.2 \times 10^{-5}$, start with $d = 0$, then.

$1.2 \times 10^{-5} < 10^{1}$ (✓) ....

$d = 5$, $1.2 \times 10^{-5} < 10^{-4}$ (✓) $\Rightarrow$ $\boxed{d = 5}$

$d = 6$, $1.2 \times 10^{-5} < 10^{-5}$ (✗)

Example: $m = 0.000000012$, $\hat{m} = 0.000000009 \Rightarrow R_m = 0.25$

$\boxed{d = 1}$

Note: As d Increasing we will have less error and more accuracy.

Some other errors formed from Using Computer or Calculator

For example. $(\sqrt{3})^2 = 3$ Algebrically.

but $(\sqrt{3})^2 \approx 3$ Using Computer.

This kind of error is called Round off error.

Def: Consider any real number p that is expressed in the Normalized decimal form:

$$p = \pm\, 0.d_1 d_2 d_3 \cdots d_k d_{k+1} \cdots \times 10^n \quad , d_1 \neq 0$$

Suppose k is the maximum number of decimal digits carried in the floating point computations of computer, then p is represented in two ways.

1) The chopped floating point representation of p

$$fl_{chop}(p) = \pm\ 0.d_1 d_2 \ldots d_k \times 10^n \quad , d_1 \neq 0.$$

2) The rounded floating point representation

$$fl_{round}(p) = \pm\ 0.d_1 d_2 \ldots d_{k-1} r_k \times 10^n \ , \quad d_1 \neq 0$$

where r_k : is obtained by rounding the number $d_k d_{k+1} d_{k+2} \ldots$

to the nearst integer.

Example: Using 4 digits chopped $\frac{2}{3} = 0.6666$

Using 4 digits rounded $\frac{2}{3} = 0.6667$

(We approximate it to 4 significant digits).

Example: Approximate $\frac{22}{7}$ to 6 significant digits chopped.

$\frac{22}{7} = 3.142857142857142857.$

$fl_{chop}\left(\frac{22}{7}\right) = 3.14285 = 0.314285 \times 10^1$

Note: $\frac{22}{7}$ 6 significant digits Rounded.

$fl_{rond}\left(\frac{22}{7}\right) = 3.14286 . = 0.314286 \times 10^1$

Example: Use 3 significant digits rounded to approximate

$$\frac{\frac{2}{7}+\frac{8}{3}+\frac{9}{11}}{\underbrace{467 \times 9.4}_{4389.8}} = \frac{0.286+2.67+0.818}{4390}$$

$$= \frac{2.96+0.818}{0.439 \times 10^{4}} = \frac{3.78}{0.439 \times 10^{1}} = 8.61 \times 10^{-4} = 0.861 \times 10^{-3}$$

Loss of Significant:

When too many Significant digits Cancel.

Consider $p = 3.14159|26536$

$q = 3.14159|57341$

$p-q \approx 0 \Rightarrow p-q \approx 0.0000030805.$

In some Calculators the difference is given 0, which is Not true. this gives loss of Significant.

For Example: $p = 3.1567891 2346$

$q = 3.1567891 2355$

$\Rightarrow p-q = 0$ Using Calculator

but in fact its not true.

Example: Let $f(x) = x(\sqrt{x+1} - \sqrt{x})$

$$g(x) = \frac{x}{\sqrt{x+1} + \sqrt{x}}.$$

Use 6 significant digits rounded to estimate

$f(500)$, $g(500)$.

$$f(500) = 500(\sqrt{501} - \sqrt{500}) = 500(22.3830 - 22.3607)$$

$$= 500(0.0223) = 11.1500$$

$$g(500) = \frac{500}{\sqrt{501} + \sqrt{500}} = \frac{500}{22.3830 + 22.3607} = \frac{500}{44.7437} = 11.1748$$

$f(500) \not\approx g(500)$. Although $f(x)$ & $g(x)$ are

algebrically equivelant.

$$f(x) = x(\sqrt{x+1} - \sqrt{x}) \cdot \left(\frac{\sqrt{x+1} + \sqrt{x}}{\sqrt{x+1} + \sqrt{x}}\right)$$

$$= \frac{x\left((\sqrt{x+1})^2 - (\sqrt{x})^2\right)}{\sqrt{x+1} + \sqrt{x}} = \frac{x(x+1-x)}{\sqrt{x+1} + \sqrt{x}} = g(x)$$

The Exact value: $f(500) = g(500) = 11.174755300747198$

So $g(x)$ has Less error.

Order of Approximation.

$$e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots$$

$$e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + O(h^4)$$

where $O(h^4)$ means "Constant $\times h^4$", we call it truncation error. or we say, the order of approximation is equal 4.

Similarly: $\sin h = h + O(h^3)$

$$\cos h = 1 - \frac{h^2}{2} + \frac{h^4}{4!} + O(h^6)$$

Def: Assume that $f(h)$ is approximated by the function $p(h)$ & that $\exists\, M > 0$ and $n \in \mathbb{Z}^+$ such that

$$\frac{|f(h) - p(h)|}{|h^n|} \leq M, \quad h \text{ sufficiently small}$$

then we say that $p(h)$ approximates $f(h)$ with order of approximation $O(h^n)$ and we write it as

$$f(h) = p(h) + O(h^n).$$

Example: show that $p(h) = 1+h$ estimate $f(h) = e^h$ with order $O(h^2)$.

$$\frac{|e^h - (1+h)|}{|h^2|} \overset{?}{\leq} M \quad , \quad e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots$$

$$\frac{\left|\frac{h^2}{2!} + \frac{h^3}{3!} + \cdots\right|}{|h^2|} = \frac{1}{2} + \frac{h}{3!} + \frac{h^2}{4!} + \cdots$$

$$< \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$$< \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}}\right) = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

Thm: If $f(h) = p(h) + O(h^n)$, $g(h) = q(h) + O(h^m)$

$r = \min\{m, n\}$, then:

- $f(h) \pm g(h) = p(h) + q(h) + O(h^r)$
- $f(h) \cdot g(h) = p(h) \cdot q(h) + O(h^r)$
- $\dfrac{f(h)}{g(h)} = \dfrac{p(h)}{q(h)} + O(h^r)$

provided $g(h)$ & $q(h) \neq 0$.

Example:
$$e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + o(h^4)$$

$$\cos h = 1 - \frac{h^2}{2!} + \frac{h^4}{4!} + o(h^6).$$

$$e^h + \cos h = 2 + h + \frac{h^3}{3!} + \frac{h^4}{4!} + o(h^4) + o(h^6)$$

$$= 2 + h + \frac{h^3}{3!} + o(h^4)$$

$$e^h \cdot \cos h = \left(\left(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!}\right) + o(h^4)\right)\left(\left(1 - \frac{h^2}{2!} + \frac{h^4}{4!}\right) + o(h^6)\right)$$

$$= 1 + h - \frac{h^3}{3} - \frac{5h^4}{24} - \frac{h^5}{24} + \frac{h^6}{48} + \frac{h^7}{144} + o(h^6)$$

$$+ o(h^4) + o(h^{10})$$

$$= 1 + h - \frac{h^3}{3} + o(h^4)$$

propagation error:

If $p = \hat{p} + \epsilon_p$, $q = \hat{q} + \epsilon_q$, then.

$$p \pm q = \hat{p} \pm q + (\epsilon_p + \epsilon_q)$$

$$p \cdot q = \hat{p} \cdot \hat{q} + \hat{p} \cdot \epsilon_p + \hat{q}\, \epsilon_q + \epsilon_p \epsilon_q$$

$p = 70 \pm 2$, $q = 60 \pm 2$

then $p - q = 10 \pm 4$