Chapter 3: Solution of Linear Systems.

To solve a system we have 5 direct methods.

1) Gaussian Elimination : $[A|b] \rightarrow [U|c]$ + Back substitution

2) $(A|b) \longrightarrow (I|c)$: Gauss - Jordan elimination

3) Inverse method : $Ax = b \rightarrow x = A^{-1}b$.

4) $x_i = \dfrac{|A_i|}{|A|}$: Cramer's method.

5) $A = LU$. factorization.

speedness: $4 < 3 < 2 < 1 < 5$

To solve linear & Non linear system numerically, there exist 3 ways:

1) Fixed point iteration

2) Gauss Sidel method.

3) Newton method.

3.3 Upper triangular Linear system:

Back substitution:

$$3x_1 + 2x_2 + 4x_3 = 9$$

$$4x_2 + 6x_3 = 10$$

$$10x_3 = 10$$

Using B.S, we found $x_3 = 1$, $x_2 = 1$, $x_1 = 1$

The Augmented matrix $\left[\begin{array}{ccc|c} 3 & 2 & 4 & 9 \\ 0 & 4 & 6 & 10 \\ 0 & 0 & 10 & 10 \end{array}\right]$.

Def: An $n \times n$ matrix $A = [a_{ij}]$ is called an upper triangular if $a_{ij} = 0$, $\forall\, i > j$, and is called Lower triangular if $a_{ij} = 0$, $\forall\, i < j$

Total Cost for solving 3×3 systems of upper triangular using B.S

$$\left[\begin{array}{ccc|c} - & - & - & - \\ 0 & - & - & - \\ 0 & 0 & - & - \end{array}\right]$$

Step	$+, -$	$\times, \div$
1	—	1
2	1	2
3	2	3

In General: If we have upper triangular system of L.E.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n-1}x_{n-1} + a_{1n}x_n = b_1$$

$$a_{22}x_2 + \cdots + a_{2n-1}x_{n-1} + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{nn}x_n = b_n$$

Then the cost will be as follow:

Step	$+, -$	$\times, \div$
1	—	1
2	1	2
3	2	3
$\vdots$	$\vdots$	
k	$k-1$	K
$\vdots$	$\vdots$	
n	$n-1$	n

Then the cost of solving $n \times n$ system, will be:

- Cost $(+,-) :=$

$$= \sum_{k=1}^{n} (k-1)$$

$$= \sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$$

$$= \frac{n(n+1)}{2} - n$$

- Cost $(\times, \div) = \sum_{k=1}^{n} k = \frac{n(n+1)}{n}$

same as F.S cost $= n^2$

$\Rightarrow$ Total Cost for B.S $= \frac{n(n+1)}{n} + \frac{n(n+1)}{n} - n = n^2$

3.4 Gaussian Elimination & pivoting:

$$Ax = b$$

$$[A \mid b] \longrightarrow [U \mid c] + \text{B.S.}$$

we can use Row Operations:

1) Multiply a row by nonzero scalar
2) Switch any two rows.
3) Replace any row by adding to it a nonzero multiple by another row

Using the following equation:

$$\text{row } r_{new} = \text{row } r_{old} - m_{rp} \text{ row } p.$$

where $m_{rp} = \dfrac{a_{rp}}{a_{pp}}$, $r > p$.

Example:

$$x_1 + 2x_2 + x_3 + 4x_4 = 13$$

$$2x_1 + 4x_3 + 3x_4 = 28$$

$$4x_1 + 2x_2 + 2x_3 + x_4 = 20$$

$$-3x_1 + x_2 + 3x_3 + 2x_4 = 6$$

First we write the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 2 & 0 & 4 & 3 & 28 \\ 4 & 2 & 2 & 1 & 20 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right]$$

, here the pivot row is the 1st row.

we have

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{2}{1} = 2$$

$$m_{31} = \frac{a_{31}}{a_{11}} = \frac{4}{1} = 4$$

$$m_{41} = \frac{a_{41}}{a_{11}} = \frac{-3}{1} = -3.$$

$\Rightarrow$ 3 divisions in first step

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & -6 & -2 & -15 & -32 \\ 0 & 7 & 6 & 14 & 45 \end{array}\right]$$

, To get this matrix, we multiplyied the first row by m_{rp}, then adding it to row p.

So we multiplied 1st row (3 times) i-e (4 x 3)

then adding 1st row to (the other rows) (i-e) (4x3)

then the cost for first step is

(4 x3) + 3 for $\times, \div$ & (4x3) for $+, -$

Again, continue in Gaussian elimination, the final Augmented matrix is:

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & 0 & -5 & -7.5 & -35 \\ 0 & 0 & 0 & -9 & -18 \end{array}\right]$$

$\Rightarrow$ Solving using B.S we have:

$x_4 = 2$, $x_3 = 4$

$x_2 = -1$, $x_1 = 3$

Then the total Cost for Reducing 4×4 into upper system

$\Rightarrow$ the cost for Reducing is $20 + 26 = 46$

Step	$+, -$	$\times, \div$
1	4×3	$(4\times3) + \boxed{3}$
2	3×2	$(3\times2) + \boxed{2}$
3	2×1	$(2\times1) + 1$
Total	20	26

multipliers m_i

There for Total for Solving the L.S is

$$46 + (4)^2 = 62$$

$$(G.E + B.S)$$

In General: The Cost of Reducing $A_{n\times n}$ into upper triangular:

step	$+, -$	$\times, \div$
1	$(n-1)\times n$	$(n-1)\times n + (n-1)$
2	$(n-2)\times(n-1)$	$(n-2)\times(n-1) + (n-2)$
3	$(n-3)\times(n-2)$	$(n-3)\times(n-2) + (n-3)$
$\vdots$	$\vdots$	
K	$(n-k)(n-k+1)$	$(n-k)(n-k+1) + (n-k)$
$\vdots$	$\vdots$	
$n-1$	1×2	$(1\times2) + 1$

the Cost for $(+, -)$

$$= \sum_{K=1}^{n-1} (n-k)((n-k)+1)$$

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$$= \sum n^2 + n + k^2 - (2n+1)K$$

$$= \sum_{k=1}^{n-1} (n-k)(n-k) + (n-k)$$

$$= \sum_{K=1}^{n-1} (n-k)^2 + \sum_{k=1}^{n-1} (n-k)$$

$$= \frac{n^3 - n}{3}$$

Now we know that, we can do change of variables (i.e)

Let $p = (n-k)$, therefore: $\sum_{p=1}^{n} p^2 = \frac{n(n+1)(2n+1)}{6}$, $\sum_{p=1}^{n} p = \frac{n(n+1)}{2}$

⇒ Cost for $(+, -)$ becomes:

$$\sum_{p=1}^{n-1} p^2 + \sum_{p=1}^{n-1} p \quad , \text{ therefore:}$$

$$= \frac{(n-1)\,n\,(2(n-1)+1)}{6} + \frac{(n-1)((n-1)+1)}{2}$$

$$= \frac{(n-1)\,n\,(2n-1)}{6} + \frac{(n-1)\,n}{2} = \frac{n^3-n}{3}$$

Total cost for $(\times, \div)$

$$= \sum_{k=1}^{n-1} (n-k) + (n-k)(n-k+1)$$

$$= \underbrace{\sum_{k=1}^{n-1} (n-k)}_{\frac{n^2}{2}-\frac{n}{2}} + \underbrace{\overbrace{\sum_{k=1}^{n-1} (n-k)(n-k+1)}^{\text{sum}}}_{\frac{n^3-n}{3}} \quad , \quad \xrightarrow{\text{take } n-k=p} \sum_{1}^{n-1} p^2 + \sum_{1}^{n-1} p$$

$$= \frac{(n-1)\,n}{2} + \frac{(n-1)\,n\,(2n-1)}{6} + \frac{(n-1)\,n}{2} = \frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6}$$

∴ Total Cost for <u>Reducing</u> $n \times n$ system into upper triangular:

$$= \frac{(n-1)\,n}{2} + \frac{(n-1)\,n(2n-1)}{6} + \frac{(n-1)\,n}{2} + \frac{(n-1)\,n(2n-1)}{6} + \frac{(n-1)\,n}{2}$$

= Reducing

$$\frac{4n^3+3n^2-7n}{6} \approx \frac{2}{3} n^3 .$$

Finally total Cost for <u>Solving</u> System using Gaussian Elimination

$$= \frac{4n^3+3n^2-7n}{6} + n^2 = \frac{2}{3} n^3 + \frac{3}{2} n^2 - \frac{7}{6} n \approx \frac{2}{3} n^3$$

Pivoting to Avoid $a_{pp}^{(p)} = 0$

Pivoting means to find the element in the row that is non zero element to do some calculation (Gaussian - Gauss J.)

Assume
$$A = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ \vdots & & & \\ a_{n1}^{(1)} & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} \end{bmatrix}$$

If $a_{pp}^{(p)} = 0$, then row p can't be used to eliminate the elements in Column p below the main diagonal therefore we find row k, where $a_{kp}^{(p)} \neq 0$ and we change rows, this criteria is called Pivoting strategy.

- Trivial pivoting: If $a_{pp}^{(p)} \neq 0$, we don't change rows otherwise we have to change with row with $a_{kp}^{(p)} \neq 0$.

Pivoting is used to Reduce the Error.

Because the Computer uses fixed - precision arithmetic, it is possible that a small error will be introduced each time that an arithmetic operation is performed

<u>Example:</u> The values $x_1 = x_2 = 1.000$ are the solutions of

$$1.133\,x_1 + 5.281\,x_2 = 6.414$$
$$24.14\,x_1 - 1.210\,x_2 = 22.93$$

Use four digit arithmetic and Gaussian elimination with trivial pivoting to find an approximation for the solution.

$$\left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 24.14 & -1.210 & 22.93 \end{array}\right]$$

$m_{21} = \dfrac{24.14}{1.133} \approx 21.31$, where $m_{rp} = \dfrac{a_{rp}}{a_{pp}}$, $r > p$

& row 2 = row 2 − m_{21} · row 1

$\Rightarrow$ $a_{22}^{(2)} = -1.210 - 21.31\,(5.281) = -113.7$

$a_{23}^{(2)} = 22.93 - 21.31\,(6.414) = -113.8$

$$\Rightarrow \left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 0 & -113.7 & -113.8 \end{array}\right] \Rightarrow \text{B.S to find } x_1 \,\&\, x_2$$

$\Rightarrow$ $x_1 = 0.9956$ & $x_2 = 1.001$.

(The error start to apears from m_{21}).

We will solve the same example using changing the rows:

$$24.14\,x_1 - 1.210\,x_2 = 22.93$$
$$1.133\,x_1 + 5.281\,x_2 = 6.414$$

$$m_{21} = \frac{1.133}{24.14} \approx 0.04693$$

$$\Rightarrow a_{22}^{(2)} = 5.281 - (0.04693)(-1.210) \cong 5.338$$

$$a_{23}^{(2)} = 6.414 - (0.04693)(22.93) \cong 5.338$$

$$\Rightarrow \left[\begin{array}{cc|c} 24.14 & -1.210 & 22.93 \\ 0 & 5.338 & 5.338 \end{array}\right] \Rightarrow \text{B. S to solve } x_1, x_2$$

$$\Rightarrow x_2 = x_1 = 1$$

<u>Note</u>: the error in the first example is due to the magnitude of the multiplier $m_{21} \cong 21.31$

- Example (1) was without partial pivoting
- Example (2) was with <u>partial</u> pivoting.

Therefore we will use two ways to avoid propagation of error.

1) <u>Partial Pivoting:</u> we take the pivot element to be the largest (In magnitude) in the <u>Remaining</u> Column P (on the diagonal and below), then we locate row <u>K</u> that is :

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$$|a_{kp}| = \max \{ |a_{pp}|, |a_{(p+1)p}| \cdots , |a_{(n-1)p}|, |a_{np}| \}$$

then we change row p with row k , if $k > p$

In this case the multipliers $m_{rp} < 1$.

(see the last example).

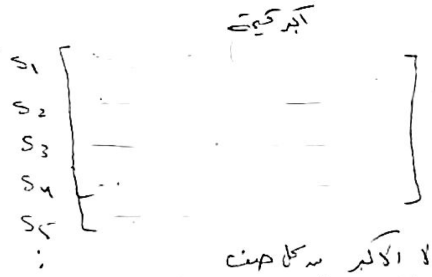
2) <u>Scaled Partial Pivoting:</u> (To avoid further error propagation)

A scaler factor must be computed first for each row (equation). Define : $S_i = \max_{1 \le j \le n} |a_{ij}|$, $(1 \le i \le n)$ الأكبر

then we use the row for which

the ration $\frac{|a_{i,1}|}{S_i}$ المقسوم is greatest.

as the pivotal row.



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<u>Example:</u>

$$0.0001x + y = 1 \qquad\qquad S_1 = 1$$
$$x + y = 2 \quad \Rightarrow \quad S_2 = 1$$

Now we take the ration : $\frac{0.0001}{1} = 0.0001 < \frac{1}{1}$

So the second equation is pivoting equation.

(2) Gauss - Jordan Elimination:

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \end{array}\right]$$

Example: Solve:

$$3x_1 + 2x_2 + 4x_3 = 9$$
$$x_1 - 2x_2 + 3x_3 = 2$$
$$3x_1 + 4x_2 - x_3 = 6$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 4 & 9 \\ 1 & -2 & 3 & 2 \\ 3 & 4 & -1 & 6 \end{array}\right] \xrightarrow{\div 3} \left[\begin{array}{ccc|c} 1 & 2/3 & 4/3 & 3 \\ 1 & -2 & 3 & 2 \\ 3 & 4 & -1 & 6 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 2/3 & 4/3 & 3 \\ 0 & -8/2 & 5/3 & -1 \\ 0 & 2 & -5 & -3 \end{array}\right] \longrightarrow \cdots$$

then the Cost for ~~Reducing~~ Solving 3×3 matrix using Gauss - Jordan Reduction is equal: 30

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steps	$+, -$	$\times, \div$
1	3×2	$3\times2+3$
2	2×2	$2\times2+2$
3	1×2	$1\times2+1$

In General: The Total Cost for ~~Reducing~~ solving $n \times n$ matrices is equal.

Steps	$+,-$	$\times, \div$
1	$n \times (n-1)$	$n \times (n-1) + n$
2	$(n-1) \times (n-1)$	$(n-1) \times (n-1) + (n-1)$
$\vdots$	$\vdots$	$\vdots$
k	$(n-k+1)(n-1)$	$(n-k+1)(n-1) + (n-k+1)$
$\vdots$		
n	$1 \times (n-1)$	$(n-1) \times 1 + 1$

$$\sum_{k=1}^{n} 2\big((n-k+1)(n-1)\big) + (n-k+1)$$

$$= \big(1+2(n-1)\big)\left(\sum_{k=1}^{n}(n-k+1)\right)$$

$$= (2n-1)\sum_{k=1}^{n}(n-k+1) = (2n-1)\left(n^2 - \frac{n(n+1)}{2} + n\right)$$

$$= \frac{2n^3+n^2-n}{2}.$$

Note that if we compare Gaussian elimination with Gauss-Jordan Reduction for large n

$$(\text{Gauss-Jord}) \quad \frac{2n^3+n^2-n}{2} > \frac{4n^3+3n^2-7n}{6} + n^2 \ (\text{Gaussian}).$$

less

$$= \frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n$$

$$\rightsquigarrow \quad n^3 > \frac{2}{3}n^3 \qquad A \to U$$

this Implies that Gaussian is faster than Gauss-Jordan Reduction. Cost for solving b:

$$\text{Gauss} \quad \frac{2n^3+n^2-n}{2} > \frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n \quad \text{Gaussian}$$

(3) Inverse method: $[A \mid I] \longrightarrow \cancel{[A^{-1} \mid I]}$ $[I \mid A^{-1}]$

$Ax = b \Rightarrow A^{-1} b = x$

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 1 & \end{array}\right]$$

for 3×3 matrix, the cost is equal

$$\sum(+,-) + \sum(\times,\div) + \text{cost}(A^{-1}b)$$

Steps	$+,-$	$\times,\div$
1	5×2	$5\times 2+5$
2	4×2	$4\times 2+4$
3	3×2	$3\times 2+3$

$$\text{Cost of } (A^{-1}b) = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$\Rightarrow$ Cost $(A^{-1}b) = 3$ (3 multiplication + 2 add)

$\Rightarrow$ Total Cost $= 60 + 15 = \boxed{75}$

In General: Cost for $n \times n$ system:

$\Rightarrow$ Total Cost $(+,-,\times,\div) = 3n^3 - \frac{5}{2}n^2 + \frac{1}{2}n$

& $\text{Cost}(A^{-1}b) = n$ (n multip. $+ (n-1)$ add)

$= n(2n-1)$.

$\Rightarrow$ Total Cost $= \cancel{5n^3 - \frac{7n^2}{2} + \frac{1}{2}n}$

check !! or

$$= 3n^3 - \frac{n^2}{2} - \frac{1}{2}n$$

Steps	$+,-$	$\times,\div$
1	$(2n-1)\times(n-1)$	$(2n-1)\times(n-1)+(2n-1)$
2	$(2n-2)\times(n-1)$	$(2n-2)(n-1)+(2n-2)$
$\vdots$	$\vdots$	$\vdots$
k	$(2n-k)(n-1)$	$(2n-k)(n-1)+(2n-k)$
$\vdots$	$\vdots$	$\vdots$
n	$n(n-1)$	$n(n-1)+n$
	$\frac{3}{2}n^3 - 2n^2 + \frac{1}{2}n$	$\frac{3}{2}n^3 - \frac{n^2}{2}$

(4) Cramer's Rule:

$$|x_i| = \frac{|A_i|}{|A|}, \quad |A| \neq 0$$

Recall: for 2×2, $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$

Note:

- Cramer's is faster than the Inverse in 2×2 matrix
- we used it if we are interested in one of the x_i's $\frac{|A_1|}{|A|}$, $\frac{|A_2|}{|A|}$
- Cost of solving 2×2 system ~~matrix~~ $= 3(2 \text{ multip} + 1 \text{ add}) + 2 \text{ division}$

$$= 11$$

- Cost for solving 3×3 system ~~matrix~~ $= 4$ determinants $+ 3$ division $\frac{|A_1|}{|A|}, \frac{|A_2|}{|A|}, \frac{|A_3|}{|A|}$

$$= 4\left(3\,(D_2) + \underline{3 \text{ multipl.}} + 2 \text{add}\right) + 3$$

$$= 4\left(3(3) + 3 + 2\right) + 3 = 59.$$

Note: Cramer's is the slowest (the most cost) comparing to the other methods.

Total Cost for solving $n\times n$ system: $(n+1)\,|n\times n| + n$

$$= (n+1)\left[n!\sum_{k=1}^{n-1}\frac{1}{k!} + (n! - 1)\right] + n$$

In General: by Induction: (The cost for 1 determinant)

the cost := $n! \sum_{k=1}^{n-1} \frac{1}{k!}$, $(\times, \div)$

& $(n! - 1)$, $(+, -)$.

proof: by Induction: when $n = 2$, $\det(A) = a_{11}a_{22} - a_{12}a_{21}$.

requires: 2 multip. + 1 addition

$\Rightarrow$ $2! \sum_{k=1}^{1} \frac{1}{k!} = 2$ $(\times, \div)$ & $2! - 1 = 1$

then it's true for $n = 2$. assume that it's true for m.

Let A be $(m+1) \times (m+1)$, then

$$\det(A) = \sum_{j=1}^{m+1} a_{ij} A_{ij} \quad , \quad \forall \; 1 \le i \le m+1$$

to compute A_{ij}, we need $\left(m! \sum_{k=1}^{m-1} \frac{1}{k!} \right) + (m! - 1)$

$\Rightarrow$ # of multiplication for $|A|$ is $\underbrace{(m+1)}_{\# \text{ of } A_{ij}} \left[m! \sum_{k=1}^{m-1} \frac{1}{k!} \right] + \underbrace{(m+1)}_{a_{ij} \cdot A_{ij}}$

$$= (m+1)! \left[\sum_{k=1}^{m-1} \frac{1}{k!} + \frac{1}{m!} \right] = (m+1)! \sum_{k=1}^{m} \frac{1}{k!}$$

& for addition: $\underbrace{(m+1)}_{\# \text{ of } A_{ij}} [m! - 1] + \underbrace{m}_{\text{جمع}} = (m+1)! - 1$.

$\square$

3.5 (5) LU factorization

" Only specific matrices can be solved using this method "

$$\left[\begin{array}{ccc|c} 4 & 3 & -1 & 21 \\ 2 & 0.5 & 2 & 4 \\ 1 & 0.25 & 3 & 0 \end{array}\right] \rightsquigarrow \text{we write } A = LU$$

$$U = \begin{bmatrix} 4 & 3 & -1 \\ 0 & -1 & 2.5 \\ 0 & 0 & 2 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.25 & 0.5 & 1 \end{bmatrix}$$

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We solve: $Ax = b \Rightarrow LUx = b$.

Let $Ux = y$, then solve $Ly = b$ (Forward Substitution)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.25 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 21 \\ 4 \\ 0 \end{bmatrix} \Longrightarrow \begin{array}{l} y_1 = 21 \\ y_2 = -6.5 \\ y_3 = -2 \end{array}$$

Now solve $Ux = y$. (Backward Substitution).

$$\begin{bmatrix} 4 & 3 & -1 \\ 0 & -1 & 2.5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 21 \\ -6.5 \\ -2 \end{bmatrix} \Longrightarrow \begin{array}{l} x_1 = 2 \\ x_2 = 4 \\ x_3 = -1 \end{array}$$

<u>Note:</u> This method is better than Gaussian elimination, since if b is changed, the change occur in Step $\boxed{2}$ not from the beginning.

LU > Gaussian > Gauss-Jordan > Inverse > Cramer's.

Cost for factoring $A_{n\times n}$ into LU factorization.

First: the Cost for Reducing 3x3 system to LU factorization

$\Rightarrow$ Cost $(A \rightarrow U) = 13$

Steps	$+, -$	$\times, \div$
1	2×2	$2\times 2 + 2$
2	1×1	$1\times 1 + 1$

لأجل b

then Cost for Solving the system

$= U + L + B.S + F.S.$

$= 13 + 0 + 9 + 6 = 28.$

$n^2 - n$ [illegible] L $\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$ [illegible]

Note: If we use more than (1) b, then using LU factorization the cost for $b_1 = 28$, but for $b_2, b_3, \dots = 15$ Since the change starts from step $\boxed{2}$, that's why it's better to use LU factorization

In General: Cost for factoring $A_{n\times n}$ into LU

$\Rightarrow$ Cost: $\dfrac{2(n-1)n(2n-1)}{6} + \dfrac{(n-1)n}{2}$

$= \dfrac{4n^3 - 3n^2 - n}{6}$

steps	$+, -$	$\times, \div$
1	$(n-1)\times(n-1)$	$(n-1)\times(n-1)+(n-1)$
2	$(n-2)\times(n-2)$	$(n-2)\times(n-2)+(n-2)$
$\vdots$	$\vdots$	$\vdots$
k	$(n-k)(n-k)$	$(n-k)(n-k)+(n-k)$
$\vdots$	$\vdots$	$\vdots$
n-1	1×1	$1 \times 1 + 1$

$\Rightarrow$ Total Cost for Solving system

$= \dfrac{4n^3 - 3n^2 - n}{6} + 0 + (n^2 - n) + n^2$

$= \dfrac{4n^3 + 9n^2 - 7n}{6}$

نفس الـ cost Gaussian لـ elimination

- The cost for solving using Gaussian elimination

$$= \frac{4n^3 + 3n^2 - 7n}{6} + \overset{B.W.S}{n^2} = \frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n.$$

- The cost for solving other b using LU

$$= n^2 - n + n^2 = 2n^2 - n.$$

$$= BS + F.S.$$

- <u>Cost A^2</u> = ?? $\quad A_{3\times3}$

$$\begin{vmatrix} - & - & - \\ - & - & - \\ - & - & - \end{vmatrix} \times \begin{vmatrix} - & - & - \\ - & - & - \\ - & - & - \end{vmatrix}$$

$$3(3 + 3 + 3) + 3(2 + 2 + 2)$$