# Chapter 4: Interpolation and polynomial approximation

**4.1 + 4.2:** Given $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$

we are looking for the relation (function) between these points.

To know this function that passes through these points we approximate it.

**Def:** Interpolation: is an estimation of the Unknown function by polynomial of degree at most $n$ which passes through all the given points.

(i.e) $\quad P_n(x_i) \approx f(x_i)$

where $f$ is the unknown function and $P_n$ is the approximation polynomial. (we have $(n+1)$ points)

**Example:** $(1,2), (2,5), (3,10)$ a polynomial passes through 3 point. is quadratic.

$$\Rightarrow \quad f(x) = y = Ax^2 + Bx + C$$

$$\left.\begin{aligned} 2 &= A + B + C \\ 5 &= 4A + 2B + C \\ 10 &= 9A + 3B + C \end{aligned}\right\} \Rightarrow \begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned}$$

$$\Rightarrow \quad y = x^2 + 1$$

Therefore we can estimate any value of any point in $[1,3]$

Note: Sometimes we used Taylor polynomial to approximate function but sometimes its hard to compute higher derivatives and some times not available.

## 4.3. Lagrange Interpolation:

Given $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, we need to find the polynomial $P_n(x)$ which satisfies

$$P_n(x_i) = y_i \quad , \quad i = 0, 1, \ldots, n$$

- Given $(x_0, y_0), (x_1, y_1)$.

slope: $m = \dfrac{y_1 - y_0}{x_1 - x_0}$

then: $y - y_0 = m(x - x_0) = \left(\dfrac{y_1 - y_0}{x_1 - x_0}\right)(x - x_0)$

$$\Rightarrow \quad y = y_0 + y_1 \frac{(x - x_0)}{x_1 - x_0} - y_0 \frac{(x - x_0)}{x_1 - x_0}$$

$$= \frac{(x_1 - x)}{x_1 - x_0} y_0 + \frac{(x - x_0)}{(x_1 - x_0)} y_1$$

$$\Rightarrow \quad y = P_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} y_0 + \frac{(x - x_0)}{(x_1 - x_0)} y_1$$

where $P_1(x)$ has degree $\leq 1$ & has two linear factors.

we will denote:

$$\boxed{L_{1,0}(x) = \frac{x - x_1}{x_0 - x_1}} \quad \text{(s.t)} \quad L_{1,0}(x_0) = 1 \ , \ L_{1,0}(x_1) = 0$$

& $$\boxed{L_{1,1}(x) = \frac{x - x_0}{x_1 - x_0}} \quad \text{(s.t)} \quad L_{1,1}(x_1) = 1 \ , \ L_{1,1}(x_0) = 0$$

Note that $P_1(x)$ passes through $(x_0, y_0)$, $(x_1, y_1)$.

$$P_1(x_0) = y_0 \quad \& \quad P_1(x_1) = y_1$$

The terms $L_{1,0}(x)$ and $L_{1,1}(x)$ are called Lagrange coefficient polynomials.

& we can write: $$P_1(x) = \sum_{k=0}^{1} L_{1,k}(x)\, y_k$$

<u>Note:</u> If we use $P_1(x)$ to approximate $f(x)$ over the Interval $[x_0, x_1]$, we call $P_1(x)$ Interpolation

while If we use $P_1(x)$ to approximate $f(x)$ over $x < x_0$ or $x > x_1$, the $P_1(x)$ is called Extrapolation.

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Now Given $(x_0, y_0)$, $(x_1, y_1)$, $(x_2, y_2)$. (3 points)

$$P_2(x) := \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2 .$$

Similarly: If we have $(x_0,y_0), (x_1,y_1), (x_2,y_2), (x_3,y_3)$

$$P_3(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_0)} y_3$$

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In General: Given $(x_0,y_0), \ldots, (x_n,y_n)$.

$$P_n(x) = \frac{(x-x_1)\cdots(x-x_n)}{(x_0-x_1)\cdots(x_0-x_n)} y_0 + \cdots + \frac{(x-x_0)\cdots(x-x_{n-1})}{(x_n-x_0)\cdots(x_n-x_{n-1})} y_n$$

$$P_n(x) = \sum_{k=0}^{n} L_{n,k}(x)\, y_k \rightsquigarrow \text{lagrange polynomial.}$$

where: $$L_{n,k}(x) = \frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}.$$

OR $$L_{n,k}(x) = \frac{\prod\limits_{\substack{j=0 \\ j\neq k}}^{n} (x-x_j)}{\prod\limits_{\substack{j=0 \\ j\neq k}}^{n} (x_k-x_j)}$$

Note that: $L_{n,k}(x_i) = 1$ if $i = k$

$L_{n,k}(x_i) = 0$ if $i \neq k$

Moreover: $P_n(x_0) = y_0$

$$P_n(x_1) = 0y_0 + 1y_1 + 0y_2 + \cdots + 0y_n = y_1$$

$$\vdots$$

$$P_n(x_j) = L_{n,0}(x_j)\,y_0 + \cdots + L_{n,j}(x_j)\,y_j + \cdots + y_n L_{n,n}(x_j)$$

$$= 0.y_0 + \cdots + 1y_j + \cdots + 0y_n$$

$$= y_j$$

$$\Rightarrow \quad P_n(x_k) = y_k \quad , \quad \forall\ k = 0,1,\ldots,n.$$

Example: Given $f(x) = \cos x$ on $[0, 1.2]$

Find $P_1(x)$, $P_2(x)$, $P_3(x)$ and compare the answers of $P_1(0.35)$, $P_2(0.35)$, $P_3(0.35)$ with the exact answer.

To find $P_1(x)$ we need ~~two~~ points

Let $(x_0, y_0) = (0, \cos 0)$, $(x_1, y_1) = (1.2, \cos 1.2)$

$(x_0, y_0) = (0, 1)$ , $(x_1, y_1) = (1.2, 0.362358)$.

Then $P_1(x) = \sum_{k=0}^{1} L_{1,k}(x)\, y_k$

$$= \frac{(x - x_1)}{(x_0 - x_1)} y_0 + \frac{(x - x_0)}{(x_1 - x_0)} y_1$$

$$= \frac{(x - 1.2)}{0 - 1.2}(1) + \frac{(x - 0)}{(1.2 - 0)}(0.362358)$$

$\Rightarrow \quad P_1(x) = -0.833333\,(x-1.2) + 0.301965\,x$

& $\quad P_1(0.35) = 0.8140208$

& $\quad f(0.35) = 0.9393727$

error $\cong 0.1$

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_1)}\,y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\,y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\,y_2$$

Let $\quad (x_0, y_0) = (0, \cos 0) = (0, 1)$

$(x_1, y_1) = (0.6, \cos 0.6) = (0.6, 0.8253361)$

$(x_2, y_2) = (1.2, \cos 1.2) = (1.2, 0.362258)$

$$\Rightarrow \quad P_2(x) = \frac{(x-0.6)(x-1.2)}{(0-0.6)(0-1.2)}(1) + \frac{(x-0)(x-1.2)}{(0.6-0)(0.6-1.2)}(0.825336)$$

$$+ \frac{(x-0)(x-0.6)}{(1.2-0)(1.2-0.6)}(0.362258)$$

$\left[\dfrac{\text{length}}{\#\text{points}-1}\right]$ or $\boxed{h} = \dfrac{1.2}{2} = 0.6$

$\Rightarrow \quad P_2(x) = 1.388889\,(x-0.6)(x-1.2) - 2.292599\,(x-0)(x-1.2)$

$+ \quad 0.503275\,(x-0)(x-0.6)$

$\Rightarrow \quad P_2(0.35) = 0.93315053$

$f(0.35) = 0.939372713$

$\Rightarrow$ error $\cong 0.0062$

To find $P_3(x)$, we need 4 points, we choose them regular nodes.

$$h = \frac{\text{length}}{4 - 1} = \frac{1.2 - 0}{3} = 0.4$$

\# of pts

$\Rightarrow$ $(x_0, y_0) = (0, 1)$, $(0.4, \cos 0.4)$, $(0.8, \cos 0.8)$, $(1.2, \cos 1.2)$

$\Rightarrow$ $(x_0, y_0) = (0, 1)$, $(x_1, y_1) = (0.4, 0.921061)$

$(x_2, y_2) = (0.8, 0.696707)$, $(x_3, y_3) = (1.2, 0.362358)$

$P_3(x) = ---$

$$\left.\begin{aligned} P_3(0.35) &= 0.939607167 \\ f_3(0.35) &= 0.9393727713 \end{aligned}\right\} \Rightarrow \text{error} \cong -2.3 \times 10^{-4}$$

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Thm: Lagrange polynomial approximation:

Assume that $f \in C^{N+1}[a, b]$, $x_0, \ldots, x_n \in [a, b]$ are $n+1$ nodes. If $x \in [a, b]$, then:

$$f(x) = P_n(x) + E_n(x)$$

(i.e) $f(x) \approx P_n(x)$

where
$$E_n(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_n) f^{(n+1)}(c)}{(n+1)!}$$

$c \in [a, b]$.

(b) $(2.5)-2)(2.5-3)(2.5-3.5)\ \frac{16}{27} \cdot \frac{1}{3!} \approx 0.02469$

Question 5 (5+5 points). Let $f(x) = \frac{1}{1-2x}$.

(a) Use Lagrange interpolation polynomial with nodes $x_0 = 2, x_1 = 3, x_2 = 3.5$ to find an estimation for $f(2.5)$

(b) Find an upper bound of the error when Estimed ~~…~~ $f(2.5)$

~~(b)~~ Find an upper bound of the error in the estimation in (a).. for $x \in [2, 3.5]$

(a)

$$f(x) \simeq P_2(x) = f_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$f(2.5) \simeq f(2) \frac{(2.5-3)(2.5-3.5)}{(2-3)(2-3.5)} + f(3) \frac{(2.5-2)(2.5-3.5)}{(3-2)(3-3.5)} + f(3.5) \frac{(2.5-2)(2.5-3)}{(3.5-2)(3.5-3)}$$

$$= \frac{-1}{3}\left[\frac{0.5}{+1.5}\right] + \frac{-1}{5}\left[\frac{-0.5}{-0.5}\right] + \frac{-1}{6}\left[\frac{-0.25}{0.75}\right]$$

$$= -\frac{1}{9} - \frac{1}{5} + \frac{1}{18} = \frac{-10-18+5}{90} = \frac{-23}{90}$$

(b) $$E_2(x) = \frac{(x-x_0)(x-x_1)(x-x_2)\, f^{(3)}(c)}{3!}$$

$$f(x) = \frac{1}{1-2x}$$

$$f'(x) = \frac{2}{(1-2x)^2}$$

$$f''(x) = \frac{8}{(1-2x)^3}$$

$$f'''(x) = \frac{48}{(1-2x)^4}$$

$$f^{(3)}(x) = \frac{48}{(1-2x)^4}, \quad x \in [2, 3.5]$$

① find max $f^{(3)}(x)$ over $[2, 3.5]$

$\frac{48}{(1-2x)^4}$ is decreasing, so it has a max at $x=2$, $\text{Max}\left[f^{(3)}(x)\right] = \frac{48}{(-3)^4} = \frac{16}{27}$

② We find $\text{Max}\,|(x-2)(x-3)(x-3.5)|$ over $[2, 3.5]$

$$g(x) = (x-2)(x-3)(x-3.5)$$

$$g'(x) = (x-3)(x-3.5) + (x-2)(x-3+x-3.5) = (x-3)(x-3.5) + (x-2)(2x-6.5)$$

$$= x^2 - 6.5x + 10.5 + 2x^2 - 10.5x + 13 = 3x^2 - 17x + 23.5$$

$$g'(x) = 0 \Rightarrow x_{1/2} = \frac{17 \pm \sqrt{(17)^2 - 12 \times 23.5}}{6} = \frac{17 \pm 2.646}{6} \Rightarrow x_1 = 3.274, \quad x_2 = 2.392$$

$g(2) = 0$, $g(3.5) = 0$, $|g(3.274)| = 0.0789$, $|g(2.392)| = 0.264$

so $\text{Max}\{|g(x)|\} = 0.264$

$$|E_2(x)| \leq \frac{0.264 \times \frac{16}{27}}{6} = 0.0260740741$$

Note: $E_n(x_i) = 0$ (i.e) The error at the nodes is equal zero

proof: We will prove it for $n=1$, the general case in the exercise: (i.e) $E_1(x) = \frac{(x-x_0)(x-x_1)}{2!} f^{(2)}(c)$

Define: $g(t) = f(t) - P_1(t) - E_1(x) \frac{(t-x_0)(t-x_1)}{(x-x_0)(x-x_1)}$ ←

Notice that $x, x_0, x_1$ are constants in $g(t)$.

and $g(x) = f(x) - P_1(x) - E_1(x) \boxed{\frac{(x-x_0)(x-x_1)}{(x-x_0)(x-x_1)}} = 1$

$= f(x) - P_1(x) - E_1(x) = 0.$

No error at the nodes

Similarly: $g(x_0) = f(x_0) - P_1(x_0) - E_1(x_0) \cdot 0 = 0$

$g(x_1) = f(x_1) - P_1(x_1) - E_1(x_1) \cdot 0 = 0$

$x$
$x_0$
$x_1$

Consider the Interval $(x_0, x_1)$ & let $\underline{x} \in (x_0, x_1)$

$g$ cont & $g'$ exist on $(x_0, x_1)$

By Mean value thm: on $(x_0, x)$, $\exists\, c_1 \in (x_0, x)$

(s.t) $g'(c_1) = \frac{g(x) - g(x_0)}{x - x_0} = 0$

Similarly: $\exists\, c_2 \in (x, x_1)$ such that

$g'(c_2) = \frac{g(x_1) - g(x)}{x_1 - x} = 0.$

Now Consider the Interval $(c_1, c_2)$. , by MVT

on $g'$: $\exists\ c \in (c_1, c_2)$ such that.

$$g''(c) = \frac{g'(c_2) - g'(c_1)}{c_2 - c_1} = \frac{0-0}{c_2 - c_1} = 0$$

Now go back to $g(t)$ & Compute the derivative.

$$g'(t) = f'(t) - P_1'(t) - E_1(x)\,\frac{(t-x_0)+(t-x_1)}{(x-x_0)(x-x_1)}$$

$$g''(t) = f''(t) - 0 - E_1(x)\,\frac{2}{(x-x_0)(x-x_1)}$$

but: $\boxed{g''(c) = 0}$

$$\Rightarrow \quad g''(c) = f''(c) - E_1(x)\cdot\frac{2}{(x-x_0)(x-x_1)} = 0$$

$$\Rightarrow \quad E_1(x) = \frac{(x-x_0)(x-x_1)\,f^{(2)}(c)}{2!}$$

Example: for previous example: $f(x) = \cos x$ , $x \in [0, 1.2]$

$$E_2(x) = \frac{(x - x_0)(x - x_1)(x - x_2)}{(3)!} f^{(3)}(c)$$

$$= \frac{(x-0)(x-0.6)(x-1.2)}{6} f^{(3)}(c)$$

$$\Rightarrow E_2(0.35) = \frac{(0.35)(0.35-0.6)(0.35-1.2)}{6} f^{(3)}(c)$$

We can bound the error:

$$|E_2(x)| \le \left| \frac{x(x-0.6)(x-1.2)}{6} \right| \cdot \max_{x_0 < x < x_n} |f^{(3)}(x)|$$

$f^{(3)}(x) = \sin x$ which is Increasing function on $[0, 1.2]$

$$\Rightarrow \max |f^{(3)}(x)| \le \sin(1.2) = 0.932.$$

which Implies:

$$|E_2(0.35)| \le 0.01155.$$

in finding $P_2(0.35)$ we found it 0.0062 which is better due to $\max|f^{3}(c)|$

Note: The next Result adress the special Case when the nodes for lagrange polynomial are equally spaced.

(i.e) $x_k = x_0 + hk$ , for $k = 0, 1, --, n$ , $\boxed{h = \frac{b-a}{n}}$

$n$ = # points $-1$

& $P_n(x)$ used ~~the~~ inside $[x_0, x_n]$.

**Thm:** (Error Bounds for Lagrange Interpolation Equally spaced Nodes). (Uniform partition).

Assume that $f(x)$ is defined on $[a,b]$, which contains equally spaced nodes $x_k = x_0 + hk$, $h = \frac{b-a}{n}$

assume that $f(x)$ and the derivatives of $f(x)$ up to order $n+1$ ($f \in C^{(n+1)}_{[a,b]}$) are continuous and bounded on the special subintervals $[x_0, x_1]$, $[x_0, x_2]$, -- and $[x_0, x_3]$ resp. that is

$$|f^{(n+1)}(x)| \le M_{n+1} \quad , \text{ for } x_0 \le x \le x_n$$

**Then:** for $n = 1, 2, 3$, the error terms have the following useful bounds:

(1) $|E_1(x)| \le \frac{h^2 M_2}{8}$, $x \in [x_0, x_1]$

M means maximum the number under it is the number of the derivative

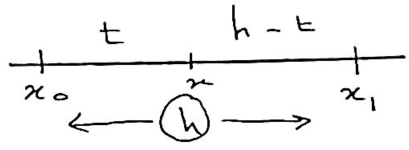
(2) $|E_2(x)| \le \frac{h^3 M_3}{9\sqrt{3}}$, $x \in [x_0, x_2]$

(3) $|E_3(x)| \le \frac{h^4 M_4}{24}$, $x \in [x_0, x_3]$.

proof: (1) using change of variables:

$$x - x_0 = t \quad \& \quad x - x_1 = t - h$$

$\hookrightarrow x - (x_0 + h) = x - x_0 - h = t - h.$




Then: $E_1(x) = \dfrac{(x - x_0)(x - x_1) f^{(2)}(c)}{2!}$ $[x_0, x_1]$ by previous thm.

then $E_1(t) = E_1(x_0 + t) = \dfrac{t(t-h) f^{(2)}(c)}{2!}$ $[0,h]$, $0 \leq t \leq h$

Assume $|f^{(2)}(x)| \leq M_2$, $\forall \; x_0 \leq x \leq x_1$

we need to determine a bound for $t(t-h) = t^2 - th$

Let $\phi(t) = t^2 - ht$

$\phi'(t) = 2t - h$

$\Rightarrow$ $t = \frac{h}{2}$ is a critical point $(\phi'(t) = 0)$

The extreme value of $\phi(t)$ over $[0,h]$ occur either at the end points or at the critical point.

Note $\phi(0) = 0$, $\phi(h) = 0$ & $\phi\left(\frac{h}{2}\right) = -\frac{h^2}{4}$.

$\Rightarrow$ $|\phi(t)| = |t^2 - th| \leq \left|-\frac{h^2}{4}\right| = \frac{h^2}{4}$, $0 \leq t \leq h$.

$\Rightarrow$ $|E_1(x)| = \left|\dfrac{(t^2 - th)}{2!} f^{(2)}(c)\right| \leq \dfrac{h^2}{8} \cdot M_2.$

$\leq \frac{h^2}{4} \cdot \frac{M_2}{2!}$

Note: if we keep the $x$, we find $E_1^*\left(\frac{x_0 + x_1}{2}\right) = \frac{h^2}{4}$

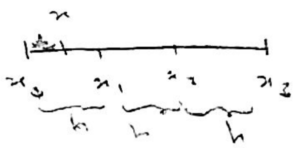
proof (3) : $|E_3(x)| \leq \dfrac{h^4 M_4}{24}$ , $x \in [x_0, x_3]$

$$E_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{4!} f^{(4)}(c)$$

Let $|f^{(4)}(x)| \leq M_4$ , $\forall\ x \in [x_0, x_3]$

then $|E_3(x)| \leq \dfrac{|(x-x_0)(x-x_1)(x-x_2)(x-x_3)|}{24} M_4$

we change the variables:



$x = x_0 + t \Rightarrow t = x - x_0$

$x_1 = x_0 + h \Rightarrow x - x_1 = x - x_0 - h = t - h$

$x_2 = x_0 + 2h \Rightarrow x - x_2 = x - x_0 - 2h = t - 2h$

$x_3 = x_0 + 3h \Rightarrow x - x_3 = x - (x_0 + 3h) = t - 3h$

$\Rightarrow |E_3(t)| = |E_3(x_0 + t)| \leq \dfrac{t(t-h)(t-2h)(t-3h)}{24} \cdot M_4$.

$[0, 3h]$

Let $\phi(t) = t(t-h)(t-2h)(t-3h) = t^4 - 6ht^3 + 11h^2t^2 - 6h^3t$.

$\phi'(t) = 4t^3 - 18ht^2 + 22h^2t - 6h^3$.

$\phi'(t) = 0 \Rightarrow t = 2.618033989\ h$

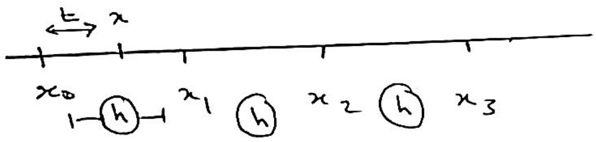
$t = 0.381966\ h$

$t = 1.5\ h$.

$$\phi(2.618033989\,h) = \phi(0.381966\,h) = |-h^4| = h^4$$

& $\phi(0) = \phi(3h) = 0$.

then the $\max|\phi(t)| = h^4$




$$\Rightarrow \quad |E_3(x)| \le \frac{h^4 M_4}{24}.$$

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Example: back to $\cos x$ , $[0, 1.2]$.

• 1) $|E_1(x)| \le \dfrac{h^2}{8} M_2$ , $h = x_1 - x_0 = 1.2$.

$|f^{(2)}(c)| \le M_2$ : $f^{(2)}(x) = -\cos x$

$\cos 0 = 1$ , $\cos(1.2) = 0.362357754$

then $\cos x$ is decreasing $\Rightarrow$ $\max\limits_{0 \le x \le 1.2} |\cos x| = \cos 0 = 1$

$$\Rightarrow \quad |E_1(x)| \le \frac{(1.2)^2 \cdot 1}{8} = 0.180000$$

• 2) $|E_2(x)| \le \dfrac{h^3 M_3}{9\sqrt{3}}$ , $h = \dfrac{1.2 - 0}{2} = 0.6$

$|f^{(3)}(x)| = |\sin x| \le \sin(1.2) = 0.932039 = M_3$

$$\Rightarrow \quad |E_2(x)| \le \frac{(0.6)^3 \cdot 0.932039}{9\sqrt{3}} = 0.012915.$$

# 4.4 Newton Polynomials:

In lagrange Polynomial there is No constructive relationship between $P_{n-1}(x)$ & $P_n(x)$

Each polynomial has to Construct Individually and the work requires to compute the higher-degree polynomials involves many computations.

Therefore: we need to find a new approach and Construct Newton Polynomials that have the reursive pattern.

Given $(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$, $P_n(x_i) = y_i$

Let $P_1(x) = a_0 + a_1(x - x_0)$

$$P_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$
$$= P_1(x) + a_2(x - x_0)(x - x_1)$$

$$P_3(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$
$$= P_2(x) + a_3(x - x_0)(x - x_1)(x - x_2)$$

$\vdots$

$$P_n(x) = P_{n-1}(x) + a_n(x - x_0) \cdots (x - x_{n-1})$$

$P_n(x)$ is called Newton polynomial with $n$ centers $x_0, x_1, \cdots, x_{n-1}$ with degree $\leq n$.

We need Now to determine $a_i$'s.

$$y_0 = P_1(x_0) = a_0 + a_1(x_0 - x_0) = a_0$$

$$\Rightarrow \quad a_0 = y_0 = f(x_0).$$ Zero Divided difference

Beside: $y_1 = P_1(x_1) = f(x_0) + a_1(x_1 - x_0) = f(x_1)$

solve for $a_1$:

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_0, x_1]$$

which is called first divided difference.

More over: $y_2 = f(x_2) = P_2(x_2) = f(x_0) + f[x_0, x_1](x_2 - x_0)$

$$+ a_2(x_2 - x_0)(x_2 - x_1)$$

then

$$a_2 = \frac{f(x_2) - f(x_0) - f[x_0, x_1](x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

Rearange:

$$a_2 = \frac{\dfrac{f(x_2) - f(x_0)}{(x_2 - x_0)} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{(x_2 - x_1)}$$

for computational purpose we will write it as:

$$a_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{(x_2 - x_0)} = f[x_0, x_1, x_2]$$

$a_2$ is the second divided difference.

where $f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$

In General: $a_k = f[x_0, x_1, \dots, x_k]$ is the

$k$ th divided difference.

Def: The divided differences for a function $f(x)$ are defined:

$f[x_k] = f(x_k)$ · Zero th divided difference.

$$f[x_{k-1}, x_k] = \frac{f[x_k] - f[x_{k-1}]}{x_k - x_{k-1}}$$

$$f[x_{k-2}, x_{k-1}, x_k] = \frac{f[x_{k-1}, x_k] - f[x_{k-2}, x_{k-1}]}{x_k - x_{k-2}}$$

& In General: higher order D.D:

$$f[x_{k-j}, x_{k-j+1}, \dots, x_k] = \frac{f[x_{k-j+1}, \dots, x_k] - f[x_{k-j}, \dots, x_{k-1}]}{x_k - x_{k-j}}$$

The following table shows the constructive of the divided differences:

| $x_k$ | $f[x_k]$ | $f[x_{k-1}, x_k]$ | $f[x_{k-2}, x_{k-1}, x_k]$ | $f[\ ,\ ,\ ,\ ]$ | $f[\ ,\ ,\ ,\ ,\ ]$ |
|---|---|---|---|---|---|
| $x_0$ | $f[x_0] = a_0$ | | | | |
| $x_1$ | $f[x_1]$ | $f[x_0, x_1] = a_1$ | | | |
| $x_2$ | $f[x_2]$ | $f[x_1, x_2]$ | $f[x_0, x_1, x_2] = a_2$ | | |
| $x_3$ | $f[x_3]$ | $f[x_2, x_3]$ | $f[x_1, x_2, x_3]$ | $f[x_0, x_1, x_2, x_3] = a_3$ | |
| $x_4$ | $f[x_4]$ | $f[x_3, x_4]$ | $f[x_2, x_3, x_4]$ | $f[x_1, x_2, x_3, x_4]$ | $f[x_0, x_1, x_2, x_3, x_4] = a_4$ |

<u>Thm:</u> (Newton polynomial): Suppose that $x_0, x_1, \dots, x_n$ are $(n+1)$ distinct numbers in $[a,b]$, There exists a <u>unique</u> polynomial $P_n(x)$ of degree at most $n$ with the property: $f(x_j) = P_n(x_j)$, $j = 0, 1, \dots, n$

The Newton Polynomial is:

$$P_n(x) = a_0 + a_1(x - x_0) + \dots + a_n(x - x_0)\dots(x - x_{n-1}).$$

where $a_k = f[x_0, \dots, x_k]$, $k = 0, 1, \dots, n.$

Corollary: $f(x) = P_n(x) + E_n(x)$

where $E_n(x)$ is the same error as in Lagrange Interpolation.

Example: Let $f(x) = x^3 - 4x$

Construct divided difference for $x_0 = 1, x_1 = 2, \dots, x_5 = 6$.

and find $P_3(x)$

The points are $(1,-3), (2,0), (3,15), (4,48), (5,105), (6,192)$

| $x_k$ | $f[x_k]$ | first D.D | 2nd D.D | 3rd D.D. | 4th D.D |
|---|---|---|---|---|---|
| 1 | -3 | | | | |
| 2 | 0 | 3 | | | |
| 3 | 15 | 15 | 6 | | |
| 4 | 48 | 33 | 9 | 1 | |
| 5 | 105 | 57 | 12 | 1 | 0 |
| 6 | 192 | 87 | 15 | 1 | 0 |

then: $P_3(x) = -3 + 3(x-1) + 6(x-1)(x-2) + 1(x-1)(x-2)(x-3)$.

for example: $f[3,4,5] = \dfrac{f[4,5] - f[3,4]}{5-3} =$

$$= \frac{\frac{f[5]-f[4]}{5-4} - \frac{f[4]-f[3]}{4-3}}{5-3} = \frac{\frac{105-48}{1} - \frac{48-15}{1}}{2} = \boxed{12}$$

(100

Ex. Estimate $f(5.5)$ using Newton Interpolation polynomial $P_1$, $P_2$, $P_3$ for the following table.

| $x_k$ | $f[x_k]$ | 1st D.D | 2nd D.D | 3rd D.D | 4th D.D |
|---|---|---|---|---|---|
| 1 | 3 | — | — | — | — |
| 3 | 4.5 | 0.75 | — | — | — |
| 4.25 | 6 | | 0.138462 | — | — |
| 5.75 | 7.25 | | | 0.05722 | — |
| 6 | 8 | | | | 0.10287 |

$$P_1(x) = a_0 + a_1(x - x_0) = 3 + 0.75(x-1)$$

$$P_1(5.5) = 6.375 \approx P_1(5.5)$$

$$P_2(x) = P_1(x) + a_2(x-1)(x-3)$$

$$P_2(5.5) \approx 7.93267$$

$$P_3(x) = P_2(x) + a_3(x-1)(x-3)(x-4.25)$$

$$P_3(5.5) \approx 8.7373.$$

Notes: ① If $f(x) = x^5 - 4x^3 + 3x^2 + 7$

then $a_6 = 0$ & $a_i = 0$, $i > 5$.

② $a_5 = 1$, but $a_3 \neq -4$ & $a_2 \neq 3$, $a_4 \neq 0$

and so on.

③ $P_5(x) = f(x)$ for any nodes.

This Note is valid only for polynomials