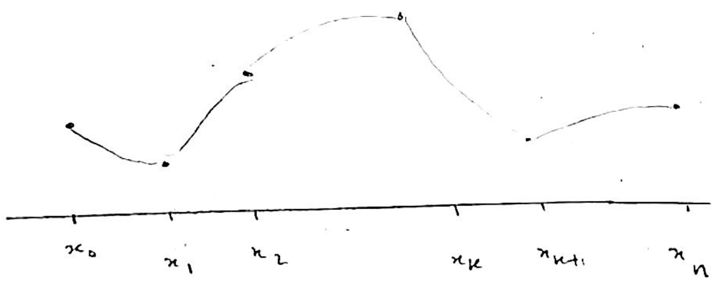
Chapter 5:

Interpolation by spline functions:

Let $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ be nodes.

a polynomial of degree(n) can have $(n-1)$ maxima and minima. and the graph can wiggle in order to pass through the points

Another way to Interpolate the function is to piece together the graphs of lower degree polynomial $S_k(x)$ and Interpolate the successive nodes (x_k, y_k) and (x_{k+1}, y_{k+1})



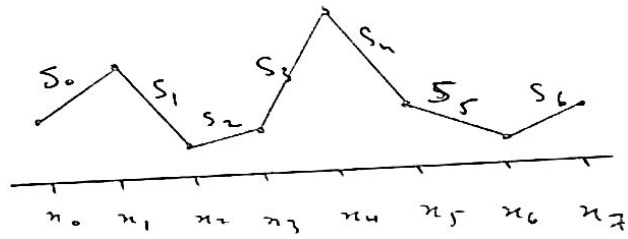
Def: A spline function is a function that consists of polynomial pieces Joined together with certain smoothness conditions.

Note: A spline Interpolation avoids the problem in which oscillation can occur between points when Interpolating using high degree polynomial.

First degree Spline:

First degree spline is a spline of degree 1 which consists of linear polynomials joined together to acheive continuity. (polygonal)

$x_0, \dots, x_7$ are called Knots.



In explicit form: the function must be defined piece by piece.

$$S(x) = \begin{cases} S_0(x) & : \quad x_0 \le x < x_1 \\ S_1(x) & : \quad x_1 \le x < x_2 \\ \vdots & \\ S_{n-1}(x) & : \quad x_{n-1} \le x < x_n \end{cases}$$

Note

$\lim_{x \to x_i^+} S(x) = \lim_{x \to x_i^-} S(x)$

cont.

where each $S_i(x)$ is a linear polynomial.

Therefore

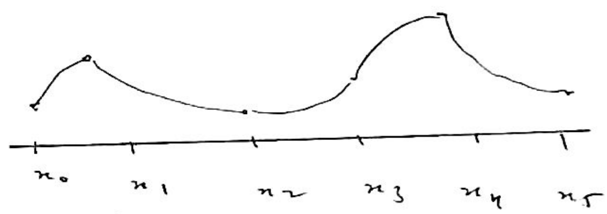
$$S_i(x) = a_i x + b_i \qquad (\text{equation of Line})$$

$$\Leftrightarrow \quad S_i(x) = y_i + d_i(x - x_i), \quad d_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$$

then
$$S(x) = \begin{cases} y_0 + d_0(x - x_0) & : \ x_0 \le x < x_1 \\ y_1 + d_1(x - x_1) & : \ x_1 \le x < x_2 \\ \vdots & \\ y_{n-1} + d_{n-1}(x - x_{n-1}) & : \ x_{n-1} \le x < x_n. \end{cases}$$

Second degree Spline: (piecewise quadratic function)

A function S is called a spline of degree 2 if



1) The domain of S is $[a, b]$

2) S & S' are continuous on $[a, b]$

3) There are points x_i (knots) such that:

$$a = x_0 < x_1 < \cdots < x_n = b$$

and S is a quadratic polynomial on each $[x_i, x_{i+1}]$.

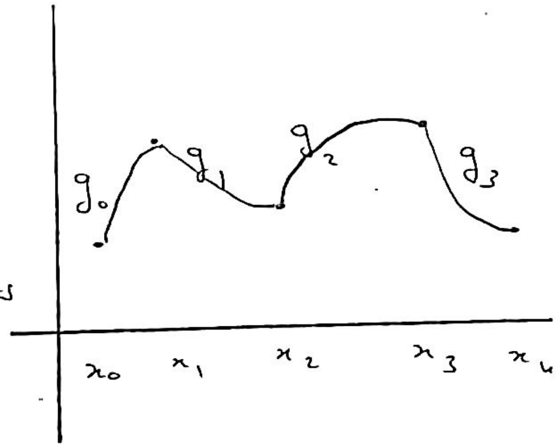
Example: $$S(x) = \begin{cases} (x+1)^2 - 1, & -2 \le x < 0 \\ 1 - (x-1)^2, & 0 \le x \le 2. \end{cases}$$

$S'(0) = 2$.

5.3 Cubic Spline:

Given $(x_0, y_0), \ldots, (x_n, y_n)$

The cubic spline is a function $g(x)$ such that it is a cubic polynomial between every two nodes and its of the form:



$$g_i(x) = a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i$$

on $[x_i, x_{i+1}]$ for $i = 0, 1, \ldots, n-1$

and that Satisfies: (we have $4n$ unknowns)

(1) $\left\{ \begin{array}{ll} g_i(x_i) = y_i, & i = 0, 1, \ldots, n-1 \\ g_{n-1}(x_n) = y_n & \Rightarrow \end{array} \right]$ ~~(n conditions)~~ $\underbrace{(n+1)}$ conditions.

(2) $g_i(x_{i+1}) = g_{i+1}(x_{i+1})$, $i = 0, \ldots, n-2$

(we have $\underbrace{(n-1)}$ conditions)

(3) $g_i'(x_{i+1}) = g_{i+1}'(x_{i+1})$, $i = 0, \ldots, n-2$

So we have $\underbrace{(n-1)}$ conditions

(4) $g_i''(x_{i+1}) = g_{i+1}''(x_{i+1})$, $i = 0, \dots, n-2$

So we have $(n-1)$ conditions.

Totally from (1), (2), (3) & (4) we have

$(n+1) + (n-1) + (n-1) + (n-1) = 4n - 2$ Conditions

we still need more two conditions, so

(5) one of the following should be satisfied:

(i) $g_0''(x_0) = g_{n-1}''(x_n) = 0$ (natural (free) boundary conditions)

(ii) $\left.\begin{aligned} g_0'(x_0) &= f'(x_0) \\ g_{n-1}'(x_n) &= f'(x_n) \end{aligned}\right\}$ (clamped boundary) conditions

Note: when (i) is satisfied, we call it Natural spline, & when (ii) is satisfied we call it clamped spline.

Example: Construct a natural Cubic spline that passes through the nodes $(1,2), (2,3), (3,5)$

$$g(x) = \begin{cases} g_0(x) & ; \quad x \in [1,2] \\ g_1(x) & ; \quad x \in [2,3] \end{cases}$$

$$= \begin{cases} a_0(x-1)^3 + b_0(x-1)^2 + c_0(x-1) + d_0, & [1,2] \\ a_1(x-2)^3 + b_1(x-2)^2 + c_1(x-2) + d_1, & [2,3] \end{cases}$$

There are 8 Unknowns, so we need 8 equations.

1. $g_i(x_i) = y_i \Rightarrow \begin{cases} g_0(x_0) = y_0 \Rightarrow \boxed{d_0 = 2} \quad \cdots (1) \\ g_1(x_1) = y_1 \Rightarrow \boxed{d_1 = 3} \quad \cdots (2) \end{cases}$

Moreover: $g_0(x_1) = g_1(x_1)$

$$a_0 + b_0 + c_0 + d_0 = d_1 = 3 \quad \cdots\cdots (3)$$

& $g_1(x_2) = f(x_2) = y_2$

$$a_1 + b_1 + c_1 + d_1 = 5 \quad \cdots\cdots (4)$$

$\boxed{2}$ $g_j'(x_{j+1}) = g_{j+1}'(x_{j+1}) \Leftrightarrow g_0'(2) = g_1'(2)$

& $g_j''(x_{j+1}) = g_{j+1}''(x_{j+1}) \Leftrightarrow g_0''(2) = g_1''(2)$

which Implies:

$g_0'(2) = g_1'(2) \Rightarrow c_0 + 2b_0 + 3a_0 = c_1$ ... (5)

$g_0''(2) = g_1''(2) \Rightarrow 2b_0 + 6a_0 = 2b_1$ ... (6)

$\boxed{3}$ Natural Conditions

$g_0''(1) = 0 \Rightarrow 2b_0 = 0$ ... (7)

$g_1''(3) = 0 \Rightarrow 2b_1 + 6a_1 = 0$ ... (8)

Solving the 8 equations in 8 Unknowns we have:

$$g(x) = \begin{cases} \frac{1}{4}(x-1)^3 + \frac{3}{4}(x-1) + 2 & , [1,2] \\ -\frac{1}{4}(x-2)^3 + \frac{3}{4}(x-2)^2 + \frac{3}{2}(x-2) + 3 & , [2,3] \end{cases}$$

Example: Construct a clamped spline for the previous example: with $g_0'(1) = 2$, $g_1'(3) = 1$

Sol: The first 6 conditions are the same.

$$g_0'(1) = 2 \implies C_0 = 2$$

$$g_1'(3) = 1 \implies c_1 + 2b_1 + 3a_1 = 1$$

Then solve the system.

Example: If the following function is a cubic spline over $[1,3]$

$$g(x) = \begin{cases} x^3 + x^2 + ax - a, & 1 \leq x < 2 \\ (x-1)^3 + b(x-1)^2 + 2, & 2 \leq x \leq 3 \end{cases}$$

Find a and b?

$g_0(x_1) = g_1(x_1) \implies 12 + 2a - a = b + 3 \implies$

$\boxed{12 + a = b + 3}$

& $g_0'(x_1) = g_1'(x_1) \implies \boxed{16 + a = 3 + 2b}$

solve (two equations), we have:

$a = -5$, $b = 4$.

Example: Construct a Natural cubic spline that passes through the nodes:

$(1,2), (2,5)$

$$g_0(x) = a_0(x-x_0)^3 + b_0(x-x_0)^2 + c_0(x-x_0) + d_0.$$
$$= a_0(x-1)^3 + b_0(x-1)^2 + c_0(x-1) + d_0$$

(s.t) $\begin{cases} g_j(x_j) = y_j & , \forall j = 0,1,\dots \\ g_{n-1}(x_n) = y_n \end{cases}$

$\Rightarrow$ $g_0(x_0) = y_0 \Rightarrow \boxed{d_0 = 2}$

$g_0(x_1) = y_1 \Rightarrow \boxed{d_0 + c_0 + b_0 + a_0 = 5.}$

Now, Natural spline:

$g_0''(x_0) = g_{n-1}''(x_n) = 0$

$\Rightarrow$ $g_0''(x_1) = 0$

$g_0'(x) = 3a_0(x-1)^2 + 2b_0(x-1) + c_0$

$g_0''(x) = 6a_0(x-1) + 2b_0$

$\Rightarrow$ $g_0''(x_0) = g_0''(1) = 2b_0 = 0 \Rightarrow \boxed{b_0 = 0}$

$g_0''(x_1) = g_0''(2) = \boxed{6a_0 + 2b_0 = 0}$

Birzeit University
Mathematics Department
Math 330

Second Exam
Student Name: ....Number:................ Section .... Instructor .....

Second Semester 2016-2017

Question 1 (5+5 points). (a) A natural cubic spline S is defined by

$$S(x) = \begin{cases} s_0(x) = 1 + A(x-1) - B(x-1)^3, & 1 \le x \le 2 \\ s_1(x) = 1 + C(x-2) - \frac{3}{4}(x-2)^2 + D(x-2)^3, & 2 \le x \le 3 \end{cases}$$

If S interpolates the data $(1,1)$, $(2,1)$ and $(3,0)$, find A, B, C, D.

$$S_+(2) = S_-(2) \Rightarrow 1 + A - B = 1 \Rightarrow \boxed{A = B}$$

$$S'(x) = \begin{cases} A - 3B(x-1)^2, & 1 \le x \le 2 \\ C - \frac{3}{2}(x-2) + 3D(x-2)^2, & 2 \le x \le 3 \end{cases}$$

$$S'_+(2) = S'_-(2) \Rightarrow \boxed{A - 3B = C}$$

$$S''(x) = \begin{cases} -6B(x-1), & 1 \le x \le 2 \\ -\frac{3}{2} + 6D(x-2), & 2 \le x \le 3 \end{cases}$$

$$S''_+(2) = S''_-(2) \Rightarrow -6B = -\frac{3}{2} \Rightarrow \boxed{B = \frac{1}{4}} \Rightarrow \boxed{A = \frac{1}{4}}$$

$$\boxed{C = -1/2}$$

$$S''(1) = S''(3) = 0 \Rightarrow -3/2 + 6D = 0 \Rightarrow \boxed{D = \frac{1}{4}}$$

(b) Use Newton interpolation polynomial with nodes $x_0 - h, x_0, x_0 + 3h$ to derive the following formula

$$f''(x_0) \approx \frac{f_3 + 3f_{-1} - 4f_0}{6h^2}$$

$$f(t) \approx P_2(t) = a_0 + a_1(t - t_0) + a_2(t - t_0)(t - t_1)$$

Newton polynomial with nodes t_0, t_1, t_2.

$$\Rightarrow f''(t) \approx P_2''(t) = 2a_2$$

$$a_2 = f[t_0, t_1, t_2] = \frac{\frac{f(t_2) - f(t_1)}{t_2 - t_1} - \frac{f(t_1) - f(t_0)}{t_1 - t_0}}{t_2 - t_0}$$

take $t_0 = x_0$, $t_1 = x_0 - h$, $t_2 = x_0 + 3h$

$$\Rightarrow f''(x_0) \approx 2a_2 = 2 \cdot \frac{\frac{f(x_0+3h) - f(x_0-h)}{4h} - \frac{f(x_0-h) - f(x_0)}{-h}}{3h}$$

$$= 2 \cdot \frac{f_3 - f_{-1} + 4f_{-1} - 4f_0}{12h^2} = \frac{f_3 + 3f_{-1} - 4f_0}{6h^2}.$$

Q4) [15 points] Given the data:

x	0	1	2
$f(x)$	1	3	5

with $f'(0) = 2$ and $f'(2) = 3$

Set up the equations for the clamped cubic spline that interpolates these data. (Do not solve the equations)

1) $g(x) = \begin{cases} g_0(x) = a_0 x^3 + b_0 x^2 + c_0 x + d_0 & \text{on } [0,1] \\ g_1(x) = a_1(x-1)^3 + b_1(x-1)^2 + c_1(x-1) + d_1 & \text{on } [1,2] \end{cases}$

$g'(x) = \begin{cases} g_0'(x) = 3a_0x^2 + 2b_0x + c_0 & \text{on } (0,1) \\ g_1'(x) = 3a_1(x-1)^2 + 2b_1(x-1) + c_1 & \text{on } (1,2) \end{cases}$

2/8 $g''(x) = \begin{cases} g_0''(x) = 6a_0x + 2b_0 & \text{in } (0,1) \\ g_1''(x) = 6a_1(x-1) + 2b_1 & \text{in } (1,2) \end{cases}$

$g_0(0) = 1 = d_0$ --- (1)

$g_1(1) = 3 = d_1$ -- (2)

$g_1(2) = 5 = a_1 + b_1 + c_1 + 3$ -- (3)

$g_0'(1) = g_1'(1)$

$3a_0 + 2b_0 + c_0 = c_1$ --- (5)

$g_0(1) = g_1(1)$

$a_0 + b_0 + c_0 + 1 = d_1 = 3$ --- (4)

$g_0''(1) = g_1''(1)$

$\boxed{6a_0 + 2b_0 = 2b_1}$ -- (6)

7

$g_0'(0) = c_0 = f'(0) = 2$ -- (7)

$g_1'(2) = 3a_1 + 2b_1 + c_1 = f'(2) = 3$

$\left(\frac{4}{4}\right)(10/9)$

ch5 Curve fitting:

5.1 least Square Line:

Introduction: The set of points $(x_1, y_1), \dots, (x_n, y_n)$ are obtained from experimental observations and/or Numerical computations.

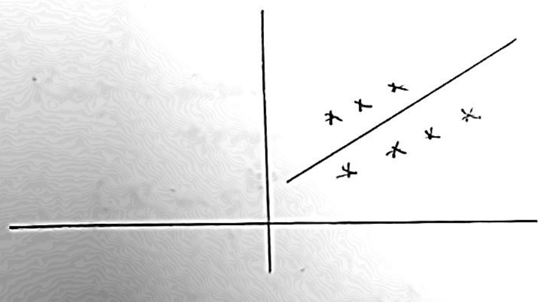
for example: $(1, 2.5)$, $(0, 0.9)$, $(-1, -0.7)$ are experimental observations for $f(x) = 2x+1$.

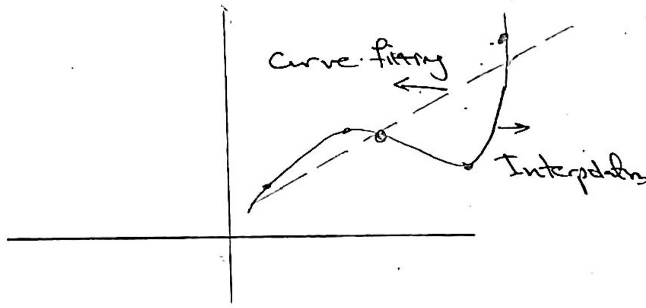
there fore the error: 0.5, 0.1, 0.2.

Note: In Interpolation, we construct a curve that passes through the points $(P_n(x_i) = f(x_i))$

but In Curve fitting, we want to find a smooth curve that approximates the data in some sense.

Thus the curve does not have to pass through the points





The task of Curve fitting is to find a smooth Curve that fits the data " In avarge " we know: $f(x_K) = y_K + e_K$

where e_K is the measurment error.

y_K Numerical Computation.

The major question is:

How do we find the best Linear approximation of the form $y = f(x) = Ax + B$ that goes near (not always through) the points?

$$e_K = f(x_K) - y_K \qquad , 1 \le K \le n.$$

There are several norms that can be used with the residuals to measure how far the curve $y = f(x)$ Lies from the data.

(1) Maximum error: $E_\infty(f) = \max\limits_{1 \le k \le n} \{ |f(x_k) - y_k| \}$

(2) Avarge error: $E_1(f) = \frac{1}{n} \sum\limits_{k=1}^{n} |f(x_k) - y_k|$.

(3) Root-mean square error (RMS): $E_2(f) = \left(\frac{1}{n} \sum\limits_{K=1}^{n} |f(x_k) - y_k|^2 \right)^{\frac{1}{2}}$

the best to minimize the error is #3, because it's easy to minimize computationally.

So we use it in Least square Line.

Least square Line :

Let $\{(x_k, y_k)\}_{k=1}^{n}$ be a set of n points where $\{x_k\}$ are distinct

The least square Line $y = f(x) = Ax + B$ is the line that minimize the root - mean - square error (rms) $E_2(f)$ (best fitting).

Note: $E_2(f)$ is minimum iff

$$n\left(E_2(f)\right)^2 = \sum_{k=1}^{n} (A x_k + B - y_k)^2 \text{ is minimum.}$$

To find the best fitting line $f(x) = Ax + B$

Let $E(A, B) = \sum_{k=1}^{n} (A x_k + B - y_k)^2$.

$$\frac{\partial E}{\partial A} = \sum_{k=1}^{n} 2\left|(A x_k + B - y_k)\right| \cdot x_k = 0 \quad \cdots (1)$$

$$\frac{\partial E}{\partial B} = \sum_{k=1}^{n} 2\left|(A x_k + B - y_k)\right| \cdot 1 = 0 \quad \cdots (2)$$

reordering the terms: we have

$$A \sum_{k=1}^{n} x_k^2 + B \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} y_k x_k \quad \cdots \quad (1^*)$$

$$A \sum x_k + n.B = \sum_{k=1}^{n} y_k \quad \cdots \quad (2^*)$$

Equations (1^*) & (2^*) are called Normal equations.

we solve the normal equations to get A & B.

Example: find the best line (least square line) fit of the data

$(-1,0)$, $(0,2)$, $(1,3)$, $(2,5)$, $(3,1)$.

X	Y	X Y	X^2
-1	0	0	1
0	2	0	0
1	3	3	1
2	5	10	4
3	1	3	9
5	11	16	15

Sum

$$\Rightarrow \quad \left.\begin{array}{l} 15A + 5B = 16 \\ 5A + 5B = 11 \end{array}\right\} \Rightarrow \quad \begin{array}{l} A = 0.5 \\ B = 1.7 \end{array}$$

then the best fitting Line is

$y = 0.5x + 1.7$ which gives minimum error $E_2(f)$.

Example: Find the normal equation of the best fitting parapola of the form $y = Ax^2 + Bx + C$

Sol: Let $E(A,B,C) = \sum_{k=1}^{n} (\overbrace{Ax_k^2 + Bx_k + C}^{f(x_k)} \overbrace{- y_k}^{-y_k})^2$

find $\frac{\partial E}{\partial A}$, $\frac{\partial E}{\partial B}$, $\frac{\partial E}{\partial C}$

$$\frac{\partial E}{\partial A} = 2 \sum_{k=1}^{n} (Ax_k^2 + Bx_k + C - y_k) \cdot x_k^2 = 0$$

$$\frac{\partial E}{\partial B} = 2 \sum_{k=1}^{n} (Ax_k^2 + Bx_k + C - y_k) \cdot x_k = 0$$

$$\frac{\partial E}{\partial C} = 2 \sum_{k=1}^{n} (Ax_k^2 + Bx_k + C - y_k) \cdot 1 = 0$$

Then the normal equations are:

$$A\left(\sum_{k=1}^{n} x_k^4\right) + B\left(\sum_{k=1}^{n} x_k^3\right) + C\sum_{k=1}^{n} x_k^2 = \sum_{k=1}^{n} x_k^2 y_k$$

$$A\left(\sum_{k=1}^{n} x_k^3\right) + B\left(\sum_{k=1}^{n} x_k^2\right) + C\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} x_k y_k$$

$$A\left(\sum_{k=1}^{n} x_k^2\right) + B\left(\sum_{k=1}^{n} x_k\right) + C\sum_{k=1}^{n} 1 = \sum_{k=1}^{n} y_k$$

Note: Normal equations can be any thing Not neccessery polynomial.

Example: $y = Ce^{Ax}$. find Best fitting . of the form $y = Ce^{Ax}$

Let $E(A,C) = \sum_{k=1}^{n} (Ce^{Ax_k} - y_k)^2$

$$\frac{\partial E}{\partial A} = 2 \sum_{k=1}^{n} (Ce^{Ax_k} - y_k) \cdot C x_k e^{Ax_k} = 0$$

$$\frac{\partial E}{\partial C} = 2 \sum_{k=1}^{n} (Ce^{Ax_k} - y_k) \cdot e^{Ax_k} = 0$$

Note that these two equations can't be solve to find A and C , therefore , we have to look for other method :

Therefore we will use Linearization method.

5.2 Linearization.

$y = Ce^{Ax}$, take (ln) for both sides.

$\Rightarrow \ln y = \ln C + Ax$.

we assume $\ln y = \bar{Y}$, $\ln C = B$, $x = \bar{X}$ $\Rightarrow$ $\bar{Y} = A\bar{X} + B$ Linear.

Data Linearization.

Example: Find the Best fitting curve of the form $f(x) = Ce^{Ax}$ for the following points.

$\cancel{(-1,0)}$, $(-1, 1.5)$, $(0,1)$, $(1,2)$, $(2,3)$, $(3,2)$

we drop it

Before filling the table, we should be careful of the the domain of $\bar{Y}$ and range $\bar{X}$

$\bar{Y} = A\bar{X} + B$.

$\bar{X}$	$\bar{Y}$	$\bar{X}^2$	$\bar{X}\bar{Y}$
-1	$\ln 1.5$	1	$-\ln 1.5$
0	$\ln 1$	0	0
1	$\ln 2$	1	$\ln 2$
2	$\ln 3$	4	$2\ln 3$
3	$\ln 2$	9	$3\ln 2$
5	2.89	15	4.56

$\Rightarrow$ $15A + 5B = 4.56$

$5A + 5B = 2.89$

$\Rightarrow$ $A = 0.167$, $B = 0.411$

$\Rightarrow$ $\bar{Y} = 0.167\bar{X} + 0.411$

$A = 0.167$

$C = e^{0.411}$

$\Rightarrow y = e^{0.411} \cdot e^{0.167x}$

$\approx 1.5e^{0.167x}$ (124

Example: Find the linearization form of

$$y = \frac{1}{Ax^2 + B}.$$

$$\Rightarrow \quad \frac{1}{y} = Ax^2 + B.$$

$$\bar{Y} = \frac{1}{y}, \quad \bar{X} = x^2 \quad \Rightarrow \quad \bar{Y} = A\bar{X} + B.$$

Example: $$y = \frac{x}{Ax+B} \quad \Rightarrow \quad \frac{1}{y} = A + B\left(\frac{1}{x}\right)$$

$$\bar{Y} = \frac{1}{y}, \quad \bar{X} = \frac{1}{x} \quad : \quad y, x \neq 0$$

If $A = 4$, $B = 5 \Rightarrow \bar{Y} = 5 + 4\bar{X}$

$$\Rightarrow \quad y = \frac{x}{5x+4}.$$

Example: $$y = \frac{1}{\sqrt{Ax^2+B}} \quad \Rightarrow \quad \bar{Y} = \frac{1}{y^2}, \quad \bar{X} = x^2$$

$$\bar{Y} = A\bar{X} + B.$$

Example: $$y = \frac{1}{(1 + Ce^{Dx})} \quad \Rightarrow \quad \bar{Y} = \ln\left(\frac{1}{y} - 1\right), \quad \bar{X} = x$$

$$\bar{Y} = D\bar{X} + \ln C$$

Example: $$y = \frac{x}{1 + Ce^{Dx}} \quad \Rightarrow \quad \bar{Y} = \ln\left(\frac{x}{y} - 1\right) = D\bar{X} + \ln C.$$

Example: [1] Write the normal equations of the best fit of the form $y = A\cos x + B\sin x$.

sol: $$E(A,B) = \sum_{k=1}^{n} (A\cos x_k + B\sin x_k - y_k)^2$$

$$\frac{\partial E}{\partial A} = 2\sum (A\cos x_k + B\sin x_k - y_k)\cdot\cos x_k = 0$$

$$\frac{\partial E}{\partial B} = 2\sum (A\cos x_k + B\sin x_k - y_k)\cdot\sin x_k = 0$$

$$\Rightarrow \quad A\sum \cos^2 x_k + B\sum \sin x_k \cos x_k = \sum \cos x_k\, y_k$$

$$A\sum \cos x_k \sin x_k + B\sum \sin^2 x_k = \sum \sin x_k\, y_k$$

[2] Use linearization to find the best fit of the form

$f(x) = \overset{D}{\cancel{A}}\cos x + \overset{C}{\cancel{B}}\sin x$, for $(1, f(1)), (2, f(2)), (3, f(3))$ when $f(x) = 3\cos x + 2\sin x$

Let $y = D\cos x_k + C\sin x_k$

$$\frac{y}{\cos x_k} = D + C\tan x_k \quad \text{or} \quad \frac{y}{\sin x_k} = D\cot x_k + C$$

$\Rightarrow$ Let $\bar{Y} = \dfrac{y_k}{\cos x_k}$, $\bar{X} = \tan x_k$, $B = D$, $C = A$

$$\Rightarrow \quad \bar{Y} = A\bar{X} + B$$

Regular equations

$$\left.\begin{aligned} A\sum x_k^2 + B\sum x_k &= \sum x_k y_k \\ A\sum x_k + 3B &= \sum y_k \end{aligned}\right\} \Rightarrow \text{table.}$$

$\left(\frac{1}{24}\right)$ (128

Q#2 (12points)Consider the data (1.1,0.4238),(1.2, 1.003), (1.3,1.662)

a- Using the above data, Find the best fit of the form $f(x) = Ax^3 + Bcosx$ Using Linearization.

b- Find the Normal equations when estimating the above data using a function of the form

$$f(x) = Ax^3 + Bcosx$$

$y = Ax^3 + B\cos x$

$\frac{y}{\cos x} = A\frac{x^3}{\cos x} + B$ $\qquad X = \frac{x^3}{\cos x}$

(2) $Y = AX + B$ $\qquad Y = \frac{y}{\cos x}$

Normal Equations

(1) $A\sum_{k=1}^{3} X_k^2 + B\sum_{k=1}^{3} X_k = \sum_{k=1}^{3} X_k Y_k$

$A\sum_{k=1}^{3} X_k + 3B = \sum_{k=1}^{3} Y_k$

(8)

(3)

x_k	y_k	X_k	Y_k	X_k^2	$X_k Y_k$
1.1	0.4238	2.934	0.9343	8.608	2.741
1.2	1.003	4.769	2.768	22.74	13.20
1.3	1.662	8.213	6.213	67.45	51.03
Total	-	15.92	9.915	98.80	66.97

(1) $98.80A + 15.92B = 66.97$

$15.92A + 3B = 9.915$

$B = 1.001$
$A = -2.006$

$A = \frac{\begin{vmatrix} 66.97 & 15.92 \\ 9.915 & 3 \end{vmatrix}}{\begin{vmatrix} 98.8 & 15.92 \\ 15.92 & 3 \end{vmatrix}} = \frac{43.06}{42.95} = 1.003$

(1) $B = \frac{\begin{vmatrix} 98.8 & 66.97 \\ 15.92 & 9.915 \end{vmatrix}}{42.95} = \frac{-86.56}{42.95} = -2.015$

$y = 1.003x^3 - 2.015\cos x$

$f(1.15) \approx 0.7023$

~~Exact~~ 0.7039.00

(2/4)(126)

Q3) [20 points] Consider the points $(1,-2)$, $(2.5,-1.6)$, $(4,0)$.

(a) Find the normal equations of the least-square curve of the form $y = \dfrac{A\cos(\pi x) + Bx + C}{x}$.

1 $$E(A,B,C) = \sum_{k=1}^{n} \left(\frac{A\cos\pi x_k + Bx_k + C}{x_k} - y_k \right)^2$$

$$\frac{\partial E}{\partial A} = 0 = 2 \sum_{k=1}^{n} \left(\frac{A\cos\pi x_k + Bx_k + C}{x_k} - y_k \right) \cdot \frac{\cos\pi x_k}{x_k}$$

3

$$A \sum_{k=1}^{n} \frac{\cos^2\pi x_k}{x_k^2} + B \sum_{k=1}^{n} \frac{\cos\pi x_k}{x_k} + C \sum_{k=1}^{n} \frac{\cos\pi x_k}{x_k^2} = \sum_{k=1}^{n} \frac{y_k\cos\pi x_k}{x_k}$$

$$\frac{\partial E}{\partial B} = 0 = 2 \sum_{k=1}^{n} \left(\frac{A\cos\pi x_k + Bx_k + C}{x_k} - y_k \right) \cdot 1 = 0$$

3

$$A \sum_{k=1}^{n} \frac{\cos\pi x_k}{x_k} + nB + C \sum_{k=1}^{n} \frac{1}{x_k} = \sum_{k=1}^{n} y_k$$

$$\frac{\partial E}{\partial C} = 0 = 2 \sum \left(\frac{A\cos\pi x_k + Bx_k + C}{x_k} - y_k \right) \cdot \frac{1}{x_k} = 0$$

3

$$A \sum_{k=1}^{n} \frac{\cos\pi x_k}{x_k^2} + B \sum_{k=1}^{n} \frac{1}{x_k} + C \sum_{k=1}^{n} \frac{1}{x_k^2} = \sum_{k=1}^{n} \frac{y_k}{x_k}$$

(3/...) (126)

(b) Use linearization to find the fitting curve of the form $y = \dfrac{A\cos(\pi x) + B}{x}$.

$$xy = A\cos\pi x + B$$

$$Y = AX + B$$

3

x	y	$X = \cos\pi x_k$	$Y = xy$	XY	X^2
1	-2	-1	-2	2	1
2.5	-1.6	0	-4	0	0
4	0	1	0	0	1
		0	-6	2	2

5

$$A\sum X_k^2 + B\sum X_k = \sum X_k Y_k$$

$$A\sum X_k + 3B = \sum Y_k$$

$$2A + 0 = 2 \Rightarrow A = 1$$

$$3B = -6 \Rightarrow B = -2$$

$$y = \frac{\cos\pi x - 2}{x}$$

(34)(126)