Ch 7 Numerical Integration:

7.1 Quadrature formulas:

Def: Given $a = x_0 < x_1 < \dots < x_n = b$

A formula of the form:

$$Q[f] = \sum_{k=0}^{n} w_k f(x_k) = w_0 f(x_0) + w_1 f(x_1) + \dots + w_n f(x_n)$$

with the property

$$\int_a^b f(x)\,dx = Q[f] + E[f]$$

is called a Numerical integration or Quadrature formula.

$E[f]$ is the truncation error for integration.

$\{x_k\}_{k=0}^{n}$ are called quadrature nodes.

$\{w_k\}_{k=0}^{n}$ are called the weights

Newton Cotes formulas: at equally spaced nodes:

1) Trapezoidal Rule:

$$\int_{x_0}^{x_1} f(x)\,dx \approx \frac{h}{2}(f_0 + f_1), \quad \text{with Error} = \frac{-h^3 f^{(2)}(c)}{12}$$

2) Simpson's $\frac{1}{3}$ Rule:

$x_0 \quad x_1 \quad x_2$

$$\int_{x_0}^{x_2} f(x)\,dx \approx \frac{h}{3}\left(f_0 + 4f_1 + f_2\right), \text{ with Error } = \frac{-h^5 f^{(4)}(c)}{90}$$

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3) Simpson's $\frac{3}{8}$ Rule:

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(f_0 + 3f_1 + 3f_2 + f_3\right), \text{with Error} = \frac{-3h^5 f^{(4)}(c)}{80}$$

Example: Estimate $\int_0^1 e^x\,dx = 1.7182818$ by:

1) $T(f) = \frac{1}{2}\left[e^0 + e^1\right] = 1.859141$

2) $S_{\frac{1}{3}}(f) = \frac{1}{2}/3\left[e^0 + 4e^{\frac{1}{2}} + e^1\right] = 1.718861$

3) $S_{\frac{3}{8}}(f) = \frac{3\left(\frac{1}{3}\right)}{8}\left[e^0 + 3e^{\frac{1}{3}} + 3e^{\frac{2}{3}} + e^1\right] = 1.718540$

Note: More partitions $\Longrightarrow$ More precise.

- To Derive the formulas we use Newton or Lagrange Interpolations.

Example: Derive Trapezoidal Rule and its truncation error?

$$P_1(x) = \left(\frac{x - x_1}{x_0 - x_1}\right) f_0 + \left(\frac{x - x_0}{x_1 - x_0}\right) f_1$$

$$\int_{x_0}^{x_1} P_1(x)\,dx = \frac{f_0}{-h}\left(\frac{(x - x_1)^2}{2}\right)\Bigg|_{x_0}^{x_1} + \frac{f_1}{h}\left(\frac{(x - x_0)^2}{2}\right)\Bigg|_{x_0}^{x_1}$$

$$= \frac{h^2 f_1}{2h} - \frac{h^2 f_0}{-2h} = \frac{h}{2}\left(f_0 + f_1\right)$$

- $$E_1(x) = \frac{(x - x_0)(x - x_1)\cancel{(x - x_2)}}{2!} f^{(2)}(c).$$

$$\int_{x_0}^{x_1} E_1(x)\,dx = \int_{x_0}^{x_1} \frac{(x - x_0)(x - x_1)\cancel{(x - x_2)}}{2} f^{(2)}(c)\,dx$$

Use change of variable:

$$x - x_0 = ht$$

$$x - x_1 = x - x_0 + x_0 - x_1$$

$$= ht - h = h(t-1).$$

$$\vdots$$

$$x - x_k = h(t - k)$$

$$dx = h\,dt.$$

$$\Rightarrow \int_{x_0}^{x_1} E_1(x)\,dx = \int_0^1 \frac{ht \cdot h(t-1)}{2} h\, f^{(2)}(c)\,dt = \frac{h^3}{2} f^{(2)}(c) \int_0^1 (t^2 - t)\,dt$$

$$= -\frac{h^3}{12} f^{(2)}(c)$$

Derive $S_{\frac{1}{3}}$ Rule:

$$\int_{x_0}^{x_2} P_2(x)\,dx = \int_{x_0}^{x_2} \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f_2$$

Use change of variables:

$$x - x_0 = h t \implies dx = h\,dt$$
$$x - x_1 = h(t-1)$$
$$x - x_2 = h(t-2)$$

$$\Rightarrow \int_0^2 \left[\frac{h(t-1)\,h(t-2)}{-h(-2h)} f_0 + \frac{ht(t-2)h}{h(-h)} f_1 + \frac{ht\,h(t-1)}{(2h)(h)} f_2 \right] h\,dt$$

$$= h \int_0^2 \left[\frac{f_0}{2}(t^2-3t+2) + f_1(2t-t^2) + \frac{f_2}{2}(t^2-t) \right] dt$$

$$= h \left[f_0 \left(\tfrac{1}{3}\right) + \tfrac{4}{3} f_1 + \frac{f_2}{3} \right] = \frac{h}{3}\left[f_0 + 4 f_1 + f_2 \right].$$

Example:

x	1	2	3	4
$f(x)$	20.1	22.5	25.6	28.9

$$\int_1^4 f(x)\,dx \approx S_{\frac{3}{8}}(f) = \frac{3(1)}{8}\left(f(1) + 3f(2) + 3f(3) + f(4) \right)$$

or by Trapezoidal:

$$\int_1^4 f(x)\,dx \approx \frac{3}{2}\left(f(1) + f(4) \right)$$

Def: The degree of precision (or accuracy) of a quadrature formula is the largest positive integer K such that the formula is exact for x^k, $k = 0, 1, 2, \ldots$

(i.e) $E_{trun}[Q_i[x]] = 0, \ \forall \ i \le K$ & $E[Q_{i+1}] \ne 0$

Example: Find D.O.P for $T(f)$.

	$f(x)$	Exact	formula	
$k=0$	1	$\int_0^1 1 dx = 1$	$\frac{1}{2}[1+1] = 1$	
$k=1$	x	$\int_0^1 x\, dx = \frac{1}{2}$	$\frac{1}{2}[0+1] = \frac{1}{2}$	
$k=2$	x^2	$\int_0^1 x^2 = \frac{1}{3}$	$\frac{1}{2}[0+1] = \frac{1}{2}$	$\times$

$\Rightarrow$ $\boxed{D.O.P = 1}$

Example: Find D.O.P for $S_{\frac{1}{3}}(f)$.

$f(x)$	Exact	formula
1	$\int_0^2 1 dx = 2$	$\frac{1}{3}[1+4+1] = 2$
x	$\int_0^2 x\, dx = 2$	$\frac{1}{3}[0+4+2] = 2$
x^2	$\int_0^2 x^2 dx = \frac{8}{3}$	$\frac{1}{3}[0+4+4] = \frac{8}{3}$
x^3	$\int_0^2 x^3 dx = 4$	$\frac{1}{3}[0+4+8] = 4$

x^4 $\qquad \int_0^2 x^4\,dx = \frac{32}{5}$ $\qquad \frac{1}{3}[0+4+16] = \frac{20}{3}$ X

$\Rightarrow$ D.O.P $(S_{\frac{1}{3}}(f)) = 3.$

$\Rightarrow$ Note:

D.P $(T(f)) = 1$

D.P $(S_{\frac{1}{3}}(f)) = 3$

D.O.P $(S_{\frac{3}{8}}(f)) = 3$

Thm: $\text{Error}_{\text{trun.}} = n f^{(K+1)}(c)$, where

K = D.O.P & n is a constant.

Example: Derive $E_{\text{trun}}(T(f))$

Since D.O.P $(T(f)) = 1 \Rightarrow E_{\text{trun}}(T(f)) = n f^{(2)}(c)$

Need to find n:

Since the error starts from 2nd degree poly. then

Let $f(x) = (x - x_0)^2 \Rightarrow f'(x) = 2(x - x_0)$, $f''(x) = 2$.

$$\text{Exact Error} = \int_{x_0}^{x_1} (x - x_0)^2\,dx - \frac{h}{2}[f_0 + f_1].$$

$$= \frac{h^3}{3} - \frac{h^3}{2} = -\frac{h^3}{6}.$$

Therefore: $-\frac{h^3}{6} = n \cdot 2 \Rightarrow n = -\frac{h^3}{12}.$

$$\Rightarrow E_{\text{trun.}} = -\frac{h^3}{12} f^{(2)}(c).$$

Example: Derive $S_{\frac{1}{3}}$ Rule's error:

Since D.O.P $(S_{\frac{1}{3}}(f)) = 3 \Rightarrow E_{trun.} = n f^{(4)}(c)$.

Let $f(x) = (x - x_0)^4 \Rightarrow f^{(4)}(c) = 24$.

$$\text{Exact error} = \int_{x_0}^{x_2} (x - x_0)^4 \, dx - \frac{h}{3}\left[f_0 + 4 f_1 + f_2\right].$$

$$= \frac{32h^5}{5} - \frac{h}{3}\left[0 + 4h^4 + 16h^4\right]$$

$$= \frac{32h^5}{5} - \frac{20}{3}h^5 = \frac{-4h^5}{15}.$$

Now: $-\frac{4h^5}{15} = n \cdot 24 \Rightarrow n = \frac{-h^5}{90}$.

$$\Rightarrow E_{truncation} = -\frac{h^5}{90} f^{(4)}(c).$$

Example Find D.O.P of $\int_0^{3h} f(x)\,dx \approx \frac{3h}{4}\left(3f(h) + f(3h)\right)$

$f(x)$	Exact	formula.
1	$\int_0^{3h} f(x)\,dx = 3h$	$\frac{3h}{4}(3 \cdot 1 + 1) = 3h$
x	$\int_0^{3h} x\,dx = \frac{9h^2}{2}$	$\frac{3h}{4}(3 \cdot h + 3h) = \frac{9h^2}{2}$
x^2	$\int_0^{3h} x^2\,dx = 9h^3$	$\frac{3h}{4}(3h^2 + 9h^2) = 9h^3$
x^3	$\int_0^{3h} x^3\,dx = \frac{81h^4}{4}$	$\frac{3h}{4}(3h^3 + 27h^3) = \frac{90h^4}{4}$

$\Rightarrow$ D.O.P $= 2$

Example: Find D.O.P for:

$$\int_a^b f(x)\,dx \approx Q[f] = (b-a)\, f\left(\frac{a+b}{2}\right).$$

$f(x)$	Exact	formula.
1	$\int_a^b 1\,dx = b-a$	$(b-a)$
x	$\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}$	$(b-a)\left(\frac{a+b}{2}\right) = \frac{b^2}{2} - \frac{a^2}{2}$
x^2	$\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}$	$\neq (b-a)\left(\frac{a+b}{2}\right)^2$

$\Rightarrow$ D.O.P $= \boxed{1}$

Note: we can Use $a=0$ & $b=1$ to make it easier.

7.2 Composite Trapezoidal and Simpson's Rule:

Thm: Composite Trapezoidal Rule:

Suppose that the Interval $[a,b]$ is subdivided into M sub Intervals $[x_k, x_{k+1}]$ of width $h = \frac{b-a}{M}$.

$(x_k = a + hk \quad , \quad k = 0, 1, \dots, M)$.

The Composite trapezoidal rule for M subintervals can be expressed in any of three equivalent ways:

$$CT(f) = \frac{h}{2} \sum_{k=1}^{M} \left(f(x_{k-1}) + f(x_k) \right)$$

or
$$CT(f) = \frac{h}{2} \left(f_0 + 2f_1 + 2f_2 + \dots + 2f_{M-1} + f_M \right).$$

or
$$CT(f) = \frac{h}{2} \left(f(a) + f(b) \right) + h \sum_{k=1}^{M-1} f(x_k)$$

Note: In General, h not neccesserily be constant (Equal) for all partitions.

Example: Use Composite Trapezoidal of 3 Compositions to estimate $\int_1^2 \frac{e^{x^2}}{\sin x}\, dx$.

$$CT(f) = \frac{1/3}{2} \left(f(1) + 2f\left(1\tfrac{1}{3}\right) + 2f\left(1\tfrac{2}{3}\right) + f(2) \right)$$

$$= \frac{1}{6} \left(3.2303 + 2 \frac{e^{(1\frac{1}{3})^2}}{\sin(1\frac{1}{3})} + 2 \frac{e^{(1\frac{2}{3})^2}}{\sin(1\frac{2}{3})} + \frac{e^{2^2}}{\sin 2} \right) =$$

Proof: Apply the trapezoidal rule over each sub Interval $[x_{k-1}, x_k]$: then

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{M} \frac{h}{2}\left(f(x_{k-1}) + f(x_k)\right)$$

$$= \frac{h}{2}\left(f(x_0) + f(x_1) + f(x_1) + f(x_2) + \cdots + f(x_{M-1}) + f(x_M)\right)$$

$$= \frac{h}{2}\left(f(x_0) + 2f(x_1) + \cdots + 2f(x_{M-1}) + f(x_M)\right).$$

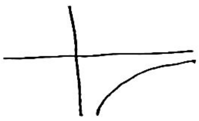
Error of Composite Trapezoidal:

$$= \frac{-h^3 f^{(2)}(c)}{12} \cdot M = -\frac{h^3 f^{(2)}(c)}{12}\left(\frac{b-a}{h}\right) = \frac{-(b-a)h^2 f^{(2)}(c)}{12}$$

Example: Find # of Compositions needed to estimate $\int_1^2 \ln(x+1)\,dx$ with accuracy 10^{-4}.

$$f(x) = \ln(x+1) \Rightarrow f'(x) = \frac{1}{x+1} \Rightarrow f''(x) = \frac{-1}{(1+x)^2}.$$

$$M_2 = \max_{1 \le x \le 2} |f^{(2)}(x)| = \frac{1}{4}$$



$$\Rightarrow E = -\frac{1}{12}\left(\frac{2-1}{M}\right)^2 (2-1)\left(\frac{1}{4}\right) < 10^{-4}.$$

$$\Rightarrow M^2 > 208.33$$

$$\Rightarrow M > 14.4 \Rightarrow \boxed{M = 15}$$

# of points $= 16$.

Note: Although the error in $CT(f)$ contains $o(h^2)$ but while in $T(f)$ contains $o(h^3)$, but $CT(f)$ is more accurate since h is smaller because of division.

Example:

t	0	1	2	3
$f(t)$	10	12	13	15

$$\int_0^3 f(t)\,dt = \frac{1}{2}\left(f(0) + 2f(1) + 2f(2) + f(3)\right) = 37.5$$

Example:

t	1	3	4	5
$f(t)$	10	15	20	21

$$\int_1^5 f(t)\,dt = \frac{2}{2}\left(f(1) + f(3)\right) + \frac{1}{2}\left(f(3) + f(4)\right) + \frac{1}{2}\left(f(4) + f(5)\right)$$

$$= \frac{2}{2}(10 + 15) + \frac{1}{2}\left(15 + 2(20) + 21\right) = 73.$$

Example: Estimate: $\int_{-1}^{3} x e^{-x}\,dx$ Using two compositions of $T(f)$

$$CT(f) = \frac{h}{2}\left(f(-1) + 2f(1) + f(3)\right)$$

Thm: Composite Simpson's $\frac{1}{3}$ Rule:

Suppose that $[a, b]$ is subdivided into $2M$ subIntervals $[x_k, x_{k+1}]$ of equal width $h = \frac{b-a}{2M}$, $x_k = a + hk$.

for $k = 0, 1, 2, \ldots, 2M$. Then the Composite Simpson's rule for $2M$ subIntervals Can be expressed as

$$CS(f) = \frac{h}{3} \sum_{k=1}^{M} f(x_{2k-2}) + 4 f(x_{2k-1}) + f(x_{2k}))$$

or $$CS(f) = \frac{h}{3}\left(f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 4f_{2M-1} + f_{2M}\right)$$

or $$CS(f) = \frac{h}{3}\left(f(a) + f(b)\right) + \frac{2h}{3} \sum_{k=1}^{M-1} f(x_{2k}) + \frac{4h}{3} \sum_{k=1}^{M} f(x_{2k-1})$$

Proof:
$$\int_0^{x_{2m}} f(x)\,dx = \int_0^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{2m-2}}^{x_{2m}} f(x)\,dx$$

$$= \frac{h_1}{3}(f_0 + 4f_1 + f_2) + \frac{h_2}{3}(f_2 + 4f_3 + f_4) + \cdots + \frac{h_m}{3}\left(f_{2m-2} + 4f_{2m-1} + f_{2m}\right)$$

If h_i are equal:

$$= \frac{h}{3}\left(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \cdots + 4f_{2m-1} + f_{2m}\right).$$

$$= \frac{h}{3} \sum_{k=1}^{m} \left(f_{2k-2} + 4f_{2k-1} + f_{2k}\right).$$

Error for $CS(f)$ (we apply $S_{\frac{1}{3}}$: M times)

$$= -\frac{h^5}{90} f^{(4)}(c) \cdot M = -\frac{h^5}{90} f^{(4)}(c) \cdot \left(\frac{b-a}{2h}\right)$$

$$= \frac{-h^4}{180} f^{(4)}(c) \cdot (b-a).$$

Example Find the number M and the step size h needed to estimate

$$\int_1^2 \ln(x+1)\,dx \quad \text{with} \quad E < 10^{-6}$$

$$f^{(4)}(x) = \frac{-6}{(1+x)^4} \Rightarrow M_4 = \underset{1 \le x \le 2}{\text{Max}} |f^{(4)}(x)| = 0.375$$

then:

$$\frac{\left(\frac{2-1}{2M}\right)^4 \cdot 0.375 \cdot (2-1)}{180} < 10^{-6}$$

$$(2M)^4 > 2083.333 \Rightarrow M^4 > 130.2083$$

$$\Rightarrow M > 3.378 \Rightarrow \boxed{M = 4}$$

& $h = \frac{2-1}{2(4)} = \frac{1}{8} = 0.125.$

7.5 Gauss - Legendre Integration.

Degree of precision for $G_n(f)$ $= 2n-1$, where $n = \#$ of points

Case 1: Assume we have 2-points:

$\Rightarrow$ D.O.P $(G_2(f)) = 2(2)-1 = 3.$

$$\int_{-1}^{1} f(x)\,dx \approx w_1 f(x_1) + w_2 f(x_2) = G_2(f)$$

We need 4 equations to find x_1, x_2, w_1, w_2;

1) $E(1) = 0 \Rightarrow \int_{-1}^{1} 1\,dx = 2 = w_1(1) + w_2(1)$

2) $E(x) = 0 \Rightarrow \int_{-1}^{1} x\,dx = 0 = w_1 x_1 + w_2 x_2$

3) $E(x^2) = 0 \Rightarrow \int_{-1}^{1} x^2\,dx = 2/3 = w_1 x_1^2 + w_2 x_2^2$

4) $E(x^3) = 0 \Rightarrow \int_{-1}^{1} x^3\,dx = 0 = w_1 x_1^3 + w_2 x_2^3$

Solve the four equations:

$\Rightarrow w_1 = w_2 = 1$ & $x_1 = \frac{1}{\sqrt{3}}$, $x_2 = -\frac{1}{\sqrt{3}}$

$\Rightarrow G_2(f) = f\left(\frac{1}{\sqrt{3}}\right) + f\left(\frac{-1}{\sqrt{3}}\right).$

Example: Estimate $\int_{-1}^{1} \frac{1}{x+2}\, dx$ using:

1) $T(f) = \frac{2}{2}\left(f(-1) + f(1) \right) = 1.3333$

2) $S_{\frac{1}{3}}(f) = \frac{1}{3}\left(f(-1) + 4f(0) + f(1) \right) = 1.11111$

3) $G_2(f) = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 1.09091$

Exact: 1.09861

Example: Find the Error for $G_2(f)$.

Since D.O.P $= 3$, then Let $f(x) = x^4$

$$\text{Exact error} = \int_{-1}^{1} x^4\, dx - G_2(f)$$

$$= \frac{2}{5} - \left[\frac{1}{9} + \frac{1}{9} \right] = \frac{2}{5} - \frac{2}{9} = \frac{8}{45}.$$

then. $\frac{8}{45} = n\, f^{(4)}(c) = n \cdot 24 \Rightarrow \boxed{n = \frac{1}{135}}$

$$\Rightarrow E_{tru}(G_2(f)) = \frac{f^{(4)}(c)}{135}.$$

Note: If we assume $f(x) = (x+1)^4$, then

$\int (x+1)^4\, dx = \frac{32}{5}$, $f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right) = \frac{56}{9}$

$\Rightarrow$ Error $= \frac{32}{5} - \frac{56}{9} = \frac{8}{45}$

Case 2: Gauss - legendre of 3 points.

D.O.P $= 2 \cdot 3 - 1 = 5$.

Then $\int_{-1}^{1} f(x)\,dx \approx w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$

we need 6 equations. (As we did in $G_2(f)$)

$$\Rightarrow G_3(f) = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

with Truncation error : $E = \dfrac{f^{(6)}(c)}{15.750}$.

Case 3: Gauss - legendre of 1 point

D.O.P $= 2 \cdot 1 - 1 = 1$

$$\int_{-1}^{1} f(x)\,dx \approx w_1 f(x_1)$$

$\Rightarrow E(1) = 0 \Rightarrow \int_{-1}^{1} 1\,dx = 2 = w_1$

$\Rightarrow E(x) = 0 \Rightarrow \int_{-1}^{1} x\,dx = 0 = w_1 x_1$

$\Rightarrow G_1(f) = 2 f(0)$.

Thm: Gauss - legendre translation:

Suppose that the points $\{x_k\}$ & $\{w_k\}$ are given for N points Gauss- Legendre rule over $[-1,1]$

To apply the rule over the Interval $[a,b]$, use The change of Variable:

$$t = \frac{a+b}{2} + \frac{b-a}{2} x \qquad \begin{matrix} x=-1 \Rightarrow t=a \\ x=1 \Rightarrow t=b \end{matrix}$$

$$dt = \frac{b-a}{2} dx$$

$$\Rightarrow \int_a^b f(t)\,dt = \int_{-1}^{1} f\left(\frac{a+b}{2} + \frac{b-a}{2} x\right) \frac{b-a}{2} dx$$

$$\Rightarrow \int_a^b f(t)\,dt \approx \frac{b-a}{2} \sum_{k=1}^{N} w_k\, f\left(\frac{a+b}{2} + \frac{b-a}{2} x_k\right).$$

Example: $\int_1^5 \frac{dt}{t} = \ln(5) - \ln(1) = 1.609438$ (Exact)

Using $G_3(f)$: with $a=1$, $b=5$, $f(t) = \frac{1}{t}$

$$\Rightarrow G_3(f) = \frac{5-1}{2}\left(\frac{5}{9} f\left(\frac{5+1}{2} + \frac{5-1}{2}\left(-\sqrt{\tfrac{3}{5}}\right)\right) + \frac{8}{9} f\left(\frac{5+1}{2} + \frac{5-1}{2}(0)\right)\right.$$
$$\left. + \frac{5}{9} f\left(\frac{5+1}{2} + \frac{5-1}{2}\left(\sqrt{\tfrac{3}{5}}\right)\right)\right).$$

$$= 2\left(\frac{5}{9} f\left(3 - 2(0.6)^{\frac{1}{2}}\right) + \frac{8}{9} f(3) + \frac{5}{9} f\left(3 + 2(0.6)^{\frac{1}{2}}\right)\right)$$
$$= 1.602694.$$

Example: Estimate $\int_0^1 e^t \, dt$ using $G_2(f)$

$$t = \frac{a+b}{2} + \frac{b-a}{2} x = \frac{1}{2} + \frac{1}{2} x.$$

$$dt = \frac{1}{2} dx$$

$$\Rightarrow G_2(f) = \frac{1}{2} f\left(\frac{1}{2} + \frac{1}{2}\left(\frac{-1}{\sqrt{3}}\right)\right) + f\left(\frac{1}{2} + \frac{1}{2}\left(\frac{1}{\sqrt{3}}\right)\right)$$

$$= \frac{1}{2}\left[e^{\frac{1}{2} - \frac{1}{2\sqrt{3}}} + e^{\frac{1}{2} + \frac{1}{2\sqrt{3}}} \right]$$