Chapter 9: 9.2+9.4

Numerical Solution of 1st order ODE's

1'st order ordinary differential equations.

$$\boxed{\begin{array}{l} y' = f(t, y) \\ y(t_0) = y_0 \end{array}} \leftarrow$$

Some Initial value problems can't be solved explicitly graph app. $\approx f(t)$

for example: $y' = t^3 + y^2$, with $y(0) = 0$

The first approach to solve such an I.V.P is called

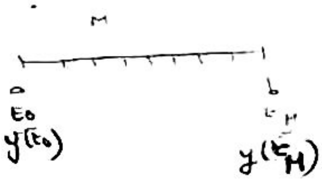
(9.2) Euler's Method : $O(h^1)$

this method has limited use, because of the larger error that is accumulated as the process proceeds

Let $[a, b]$ be the Interval, we want to find solution of $y' = f(t, y)$, with $y(a) = y_0$ I.V.P on $[a, b]$

we will approximate the solution using sets of points (t_k, y_k)

where $(y(t_k) \approx y_k)$



How we construct this set of points?

- we will use M subintervals of $[a, b]$, (s.t) $\boxed{h = \frac{b-a}{M}}$

and select the mesh points to be:

$$t_k = a + kh, \quad \text{for } k = 0, 1, \ldots, M.$$

نرسم مجموعة النقاط (t_k, y_k) بيطلع ال curve هو ال approximation للحل.

h: step size. Now we will solve approximately:

$$y' = f(t, y) \text{ over } \underset{[a,\ b]}{[t_0, t_M]} \text{ with } y(t_0) = y_0$$

Now assume that $y(t)$, $y'(t)$ and $y''(t)$ are continuous

using taylor expansion of $\boxed{y(t_1)}$ about $t = t_0$.

$$\Rightarrow \quad y(t_1) = y(t_0+h) = y(t_0) + h\, y'(t_0) + \frac{h^2}{2}\, y''(c_1)$$

If the step size h is chosen small enough, then we may neglect the second-order term (h^2) & get:

$$y(t_1) = y(t_0) + h\, y'(t_0)$$

$$y_1 = y_0 + h\, f(t_0, y_0) \qquad \text{Euler's Approximation.}$$

this process repeated & generates a sequence of points that approximates the solution Curve $y = y(t)$. $\rightarrow$ curve, (plot it by sequence of points).

General step for Euler's method:

$$t_{k+1} = a + (k+1)h$$
$$= a + kh + h$$

$$\boxed{t_{k+1} = t_k + h} \quad , \quad \boxed{y_{k+1} = y_k + h\, f(t_k, y_k)}$$

for $k = 0, 1, \dots, M-1$

Example: Use Euler's method to solve approximately the I.V.P.

$$y' = \boxed{R\, y}, \quad \text{over } [0, 1], \quad y(0) = y_0, \quad R \in \mathbb{R}.$$

$t_0 = a$

$$y_1 = y_0 + h\, f(t_0, y_0) = y_0 + h R y_0 = y_0 (1 + hR).$$

$$y_2 = y_1 + h\, f(t_1, y_1) = y_0(1+hR) + hR\,(y_0(1+hR))$$

$$= y_0\, (1+hR)(1+hR) = y_0\, (1+hR)^2$$

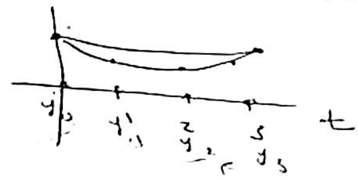
$\vdots$

$t_k = hk \quad , \quad y_k = y_{k-1}\,(1+hR) = y_0\,(1+hR)^k \quad , k = 0, 1, \dots, M.$

Example: Estimate the solution of $y' = \frac{t-y}{2}$, $y(0)=1$ on $[0,3]$

$\boxed{h=1}$

exact sol: $3e^{-t/2} - 2 + t$



$y_1 = y_0 + h\, f(t_0, y_0)$

$= 1 + h\left(\frac{-1}{2}\right) = 1 + (-0.5) = 0.5$

$t_1 = a + 1h = 0 + 1 = 1$

$y_2 = y_1 + h\, f(t_1, y_1) = 0.5 + 1\left(\frac{1 - 0.5}{2}\right) = 0.5 + 0.25 = 0.75$

$t_2 = a + 2h = 0 + 2 = 2$

$y_3 = y_2 + h\, f(t_2, y_2) = 0.75 + 1\left(\frac{2 - 0.75}{2}\right) = 0.75 + 0.625 = 1.375$

Total error: In General

$E(y,h) = \frac{h^2}{2} y''(c_1) + \frac{h^2}{2} y''(c_2) + \frac{h^2}{2} y''(c_3) + \cdots + \frac{h^2}{2} y''(c_M)$

$= \frac{h^2}{2}\left(\underbrace{y''(c_1) + \cdots + y''(c_M)}\right)$ Average.

$= \frac{h^2}{2}\left(M\, y''(c)\right)$

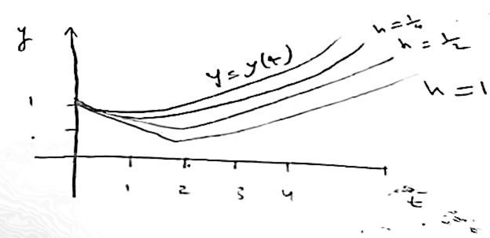
$= \frac{h^2}{2}\left(\frac{b-a}{h}\right) y''(c) = \frac{(b-a)h}{2} y''(c) =$

$= \frac{(b-a)}{2} y''(c) \cdot h \approx c\, h = \boxed{O(h^1)}$

Note: If, we take $\frac{h}{2}$ instead of h, the

$E\left(y, \frac{h}{2}\right) = c\left(\frac{h}{2}\right) = \frac{1}{2}(ch) = \frac{1}{2} E(y,h)$

لما نصغر h لما نكبر أصـ   نفس الأفكار السابقة .



9.4

Taylor method $O(h^2)$: Expand $y(t_1)$ about $t = t_0$.

→ Derive a formula of total error with order $\boxed{(O(h^2))}$ to solve $\boxed{f(t,y) = y', \ y(0) = y_0}$

$$y_1 = y_0 + h\, f(t_0, y_0) + \frac{h^2}{2} f'(t_0, y_0) + \frac{h^3}{3!} f''(t_0, C)$$
$$= y_0 + h\, y' + \frac{h^2}{2} y''$$

$$\Rightarrow \quad y_1 = y_0 + h\, y'(t_0) + \frac{h^2}{2} y''(t_0) + \frac{h^3}{3!} y'''(C)$$

Step error : $\dfrac{h^3}{3!} y'''(c)$

repeat the steps above, then for $k = 0, 1, \dots, M$, $\boxed{h = \dfrac{b-a}{M}}$

$$y_{k+1} \cong y_k + h\, f(t_k, y_k) + \frac{h^2}{2} \frac{d}{dt} f(t_k, y_k).$$

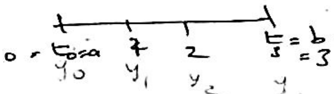
$k = 0, 1, \dots, M-1$, where M is the # of subintervals of $[a,b]$

Total error in step ~~(K+1)~~

$$E(y,h) = \left(\frac{h^3}{3!} \left(y'''(c_1) \right) + \dots + \frac{h^3}{3!} \left(y'''(c_M) \right) \right)$$

$$= \frac{h^3}{3!} \left(M\, y'''(c) \right) = \frac{(b-a)}{3!} y'''(c) \cdot h^2 \approx C h^2 = \boxed{O(h^2)}$$

Example: Solve: $y' = \dfrac{t-y}{2}$ on $[0,3]$, with $y_0 = 1$, $h = 1$.



$$y_1 = y_0 + h\, f(t_0, y_0) + \frac{h^2}{2} \frac{d}{dt} f(t_0, y_0)$$
$$= 1 + 1(-0.5) + \frac{1}{2}\left(\frac{1}{2} - \frac{(0-1)}{4} \right) = 0.875$$

$$y_2 = y_1 + h\, f(t_1, y_1) + \frac{h^2}{2} \frac{d}{dt} f(t_1, y_1)$$
$$= 0.875 + 1\left(\frac{1 - 0.875}{2} \right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}(1 - 0.875) \right)$$
$$= 1.171875$$

$$y' = \frac{t-y}{2}$$
$$y'' = \frac{1}{2} - \frac{y'(t)}{2}$$
$$= \frac{1}{2} - \frac{(t-y)}{4}$$

$y_3 = \dots$

Example: Use taylor method of $O(h^4)$ to estimate the solution of $y' = \frac{t-y}{2}$, $y(0)=1$ on $[0,3]$, $h=1$

Sol: Need to calculate y_1, y_2, y_3, but need to know the formula for $O(h^4)$

$t_0 \quad t_1 \quad t_2 \quad t_3$

$0 \quad 1 \quad 2 \quad 3$

First: $y(t_1) = y(t_0) + h\,y'(t_0) + \frac{h^2}{2!} y''(t_0) + \frac{h^3}{3!} y'''(t_0) + \frac{h^4}{4!} y^{(4)}(t_0) + \underbrace{\frac{h^5}{5!} y^{(5)}(c)}_{\text{step error}}$

As before, to have error of order $O(h^4)$ we need to add the errors in M sub Intervals.

$$E = \frac{h^5}{5!} y^{(5)}(c_1) + \cdots + \frac{h^5}{5!} y^{(5)}(c_M) =$$

$$= \frac{h^5}{5!}\left(M\, y^{(5)}(c)\right) = \frac{(b-a)h^4}{5!} y^{(5)}(c) = \frac{(b-a)y^{(5)}(c)}{5!} h^4 = O(h^4)$$

therefore: $y_1 = y_0 + h\,y'(t_0) + \frac{h^2}{2} y''(t_0) + \frac{h^3}{3!} y'''(t_0) + \frac{h^4}{4!} y''''(t_0)$

$\Rightarrow y_1 = 1 + 1(-0.5) + \frac{1}{2} y''(t_0) + \frac{h^3}{3!} y'''(t_0) + \frac{h^4}{4!} y''''(t_0)$

$y' = \frac{t-y}{2}$, $y(0)=1$, $y'' = \frac{1}{2} - \frac{t-y}{4}$, $y''' = -\frac{1}{2}\left(\frac{1}{2} - \frac{t-y}{4}\right)$

$y'''' = -\frac{1}{2}\left(-\frac{1}{2}\left(\frac{1}{2} - \frac{t-y}{4}\right)\right)$

then

$y_1 = 1 + 1(0.5) + \frac{1}{2}(0.75) + \frac{1}{6}(-0.375) + \frac{1}{24}(0.1875)$

$= 0.8203125$

$t_1 = 1$

$y_1 = 0.8203125$

$y_2 = 1.1045125$

$y_3 = 1.6701860$

$E(y,h) = O(h^4) = C h^4$

then $E(y, \frac{h}{2}) = C\left(\frac{h}{2}\right)^4 = C\frac{h^4}{16}$

$E(y, 10^{-2}h) = C(10^{-2}h)^4 =$

$= C(10^{-8})h^4$

Modified Euler Method: (Heun's Method): $O(h^2)$

$y' = t^3 + y^2$

This method is used to solve the I.V.P.:

$y'(t) = f(t, y(t))$ over $[a, b]$ with $y(t_0) = y_0$

To obtain the solution point (t_1, y_1), we can use the fundamental theorem of Calculus and Integrate $y'(t)$ over $[t_0, t_1]$

Initial point

$$\int_{t_0}^{t_1} f(t, y(t))\, dt = \int_{t_0}^{t_1} y'(t)\, dt = y(t_1) - y(t_0)$$

y_1

$t_0 \to t_1$

Solve this equation for $y(t_1)$, then

$t_1 = a + h$
$t_1 = t_0 + h$

$$y(t_1) = y(t_0) + \boxed{\int_{t_0}^{t_1} f(t, y(t))\, dt}$$

Now use Numerical method to find $\int_{t_0}^{t_1} f(t, y(t))\, dt$.

If we use trapezoidal rule with $h = t_1 - t_0$, then:

$$y(t_1) \approx y(t_0) + \frac{h}{2}\left(f(t_0, y(t_0)) + f(t_1, y(t_1))\right)$$

Implicit function.

Euler

error for Trapezoidal $= \dfrac{-h^3 y''(c)}{12}$

To get rid of the $y(t_1)$ Inside $f(t_1, y(t_1))$ we use

Euler's solution (i.e $y_1 \cong y_0 + h f(t_0, y_0)$)

error for Euler $\dfrac{(b-a)h}{2} y''(c)$

therefore

$$y(t_1) \approx y(t_0) + \frac{h}{2}\left(f(t_0, y_0) + f(t_1, y_0 + h f(t_0, y_0))\right)$$

This formula is called Heun's method

we can repeat the previous steps using Trapezoidal.

and Euler method is used as a prediction.

The General step for Heun's method is

$$\text{Euler} \leftarrow y_{k+1} = y_k + h\, f(t_k, y_k) \quad , \quad t_{k+1} = t_k + h$$

&

$$y_{k+1} = y_k + \frac{h}{2}\Big(f(t_k, y_k) + f\big(t_{k+1}, y_k + h\, f(t_k, y_k)\big)\Big)$$

Error:

1) Error for trapezoidal Rule (Integration) : $-y^{(2)}(c)\,\frac{h^3}{12}$

If this is the only error then we repeat the steps $\boxed{N}$ stop after M steps, then the error:

$$-\sum_{k=1}^{M} y^{(2)}(c_k)\,\frac{h^3}{12} \approx \frac{b-a}{12}\, y^{(2)}(c)\, h^2 = O(h^2)$$

$$\left.\begin{array}{ll} \text{Trapezoidal} & O(h^2) \\ \text{Euler} & O(h^1) \end{array}\right\rangle$$

we take min which is $O(h^2)$ since h is sufficiently small.

mple : Solve Using Heun's Method with $h=1$

$$y' = \frac{t-y}{2} = f(t,y) \quad , y(0)=1 \quad , [0,3]$$

$$y_1 = y_0 + \frac{h}{2}\left(f(t_0,y_0) + f(t_1, y_0 + h f(t_0,y_0))\right)$$

$$= 1 + \frac{1}{2}\left((-0.5) + f(1, 1 + f(0,1))\right)$$

$$= 1 + 0.5\,(-0.5 + f(1, 1-0.5))$$

$$= 1 + 0.5\,(-0.5 + f(1, 0.5))$$

$$= 1 + 0.5\left(-0.5 + \frac{1-0.5}{2}\right) = 0.875.$$

$$y_2 = y_1 + \frac{h}{2}\left(f(t_1,y_1) + f(t_2, y_1 + h f(t_1,y_1))\right)$$

$$= 0.875 + \frac{1}{2}\left(f(1,0.875) + f(2, 0.875 + f(1,0.875))\right)$$

$$= 0.875 + \frac{1}{2}\left(f(1,0.875) + f\left(2, 0.875 + \frac{1-0.875}{2}\right)\right)$$

$$= 0.875 + \frac{1}{2}\left(\frac{1-0.875}{2} + \frac{2-0.9375}{2}\right)$$

$$= 0.875 + \frac{1}{2}(0.0625 + 0.53125) = 1.171875.$$

$$y_3 = y_2 + \frac{h}{2}\left(f(t_2,y_2) + f(t_3, y_2 + h f(t_2,y_2))\right)$$

$$= 1.171875 + \frac{1}{2}\left(f(2,1.171875) + f(3,(1.171875 + f(2,1.17875)))\right)$$

$$= 1.171875 + \frac{1}{2}\left(\frac{2-1.171875}{2} + f\left(3, \left(1.171875 + \frac{2-1.17875}{2}\right)\right)\right)$$

$$= 1.171875 + \frac{1}{2}\left(0.4140625 + f(3, 1.58594)\right)$$

$$= 1.171875 + \frac{1}{2}\left(0.4140625 + \frac{3-1.58594}{2}\right) = 1.73242$$

9.5 Runge - Kutta Method's of order 4 (RK4)

Modified Simpson's method:

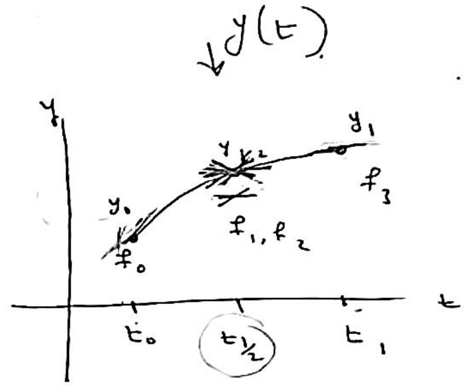
Consider the graph of the solution curve $y = y(t)$

over the first sub interval $[t_0, t_1]$

Let f_0 be the slope at the left

f_1, f_2 the estimation of the slope in the middle

f_3 the slop at the right



The next point (t_1, y_1) is obtained by integrating the slope function (using FT of calculus)

$$y(t_1) - y(t_0) = \int_{t_0}^{t_1} f(t, y(t))\, dt \qquad ---(**)$$

Assume we applied Simpson's rule with step size $\boxed{\frac{h}{2}}$

then the approximation of Integration in $(**)$: $\frac{h}{3}(f_0 + 4f_1 + f_2)$

h, $\frac{h}{2}$, $\frac{h}{2}$, t_0, $t_{\frac{1}{2}}$, t_1

to apply simpson we need to divid it

$$\int_{t_0}^{t_1} f(t, y(t))\, dt \simeq \frac{h}{6}\left(f(t_0, y(t_0)) + 4 f(t_{\frac{1}{2}}, y(t_{\frac{1}{2}})) + f(t_1, y(t_1)) \right)$$

when $t_{\frac{1}{2}}$ mid point of $[t_0, t_1]$.

Let $f(t_0, y(t_0)) = f_0$

$f(t_1, y(t_1)) = f_3$

& for the value in the middle we choose the average of f_1, f_2 :

$$f(t_{\frac{1}{2}}, y(t_{\frac{1}{2}})) \approx \frac{f_1 + f_2}{2}$$

then

$$\int_{t_0}^{t_1} f(t, y(t)) \approx \frac{h}{6}\left(f_0 + \frac{4(f_1 + f_2)}{2} + f_3\right)$$

$$= \frac{h}{6}(f_0 + 2f_1 + 2f_2 + f_3).$$

Hence, back to (**) & substitute this value, we have

$$y_1 \approx y_0 + \frac{h}{6}(f_0 + 2f_1 + 2f_2 + f_3)$$

where:

$$y_{\frac{1}{2}} = y_0 + \frac{h}{2} f(t_0, y_0)$$

$$f_0 = f(t_0, y_0)$$

$$f_1 = f(t_0 + \tfrac{h}{2}, y_0 + \tfrac{h}{2} f_0)$$

$$f_2 = f(t_0 + \tfrac{h}{2}, y_0 + \tfrac{h}{2} f_1)$$

$$f_3 = f(t_1, y_0 + h f_2)$$

$$\frac{-h^5}{90} f^{(4)}(c)$$ simpson's error

we take step size

Error: For simpson's method the error $= \frac{-h^5}{2880} f^{(4)}(c_1)$, $\boxed{\frac{h}{2}}$

If the only error at each step is given by simpson's error then after M steps $\Rightarrow$ Error $= \frac{-h^5}{2880} \cdot f^{(4)}(c_1) \cdot M =$

$$= \frac{-h^5}{2880} \cdot f^{(4)}(c) \boxed{\frac{(b-a)}{2h}} \approx \frac{+h^4}{5760}(b-a) f^{(4)}(c) \approx O(h^4)$$

Example: Solve: $y' = \frac{t-y}{2}$, $y(0)=1$, $[0,3]$, $h = \frac{1}{4}$.

Exact value: $\boxed{0.8974915}$

$$y_1 = y_0 + \frac{\frac{1}{4}}{6}\left(f_0 + 2f_1 + 2f_2 + f_3\right)$$

0 0.25 0.5 3
t_0 t_1
y_0 y_1

$$f_0 = f(t_0, y_0) = \frac{0-1}{2} = -0.5$$

$$f_1 = f\left(t_0 + \tfrac{h}{2}, y_0 + \tfrac{h}{2} f_0\right) = f\left(0 + \frac{0.25}{2}, 1 + \frac{0.25}{2}(-0.5)\right)$$

$$= f(0.125, 0.9375) = -0.40625$$

$$f_2 = f\left(0.125, 1 + \frac{0.25}{2}(-0.40625)\right) = f(0.125, 0.9492) = -0.4121094$$

$$f_3 = f(\underbrace{0.25}_{t_0+h}, 1 + 0.25(-0.4121094)) = f(0.25, 0.8969773) = -0.3234863$$

$$\Rightarrow y_1 = 1 + \frac{\frac{1}{4}}{6}\left(-0.5 + 2(-0.40625) + 2(-0.4121094) + (-0.3234863)\right)$$

$$\boxed{y_1 = 0.8974915}$$

$$y_2 = y_1 + \frac{h}{6}\left(f_0 + 2f_1 + 2f_2 + f_3\right)$$

$f_0 = f(t_1, y_1)$

$f_1 = f(t_1 + \frac{h}{2}, y_1 + \frac{h}{2} f_0)$

$f_2 = f(t_1 + \frac{h}{2}, y_1 + \frac{h}{2} f_1)$

$f_3 = f(t_2, y_1 + h f_2)$

$$y_3 = \quad + \frac{h}{6}\left(f_0 + 2f_1 + 2f_2 + f_3\right)$$

$f_0 = f(t_2, y_2)$

$f_1 = f(t_2 + \frac{h}{2}, y_2 + \frac{h}{2} f_0)$

$f_2 = f(t_2 + \frac{h}{2}, y_2 + \frac{h}{2} f_1)$

$f_3 = f(t_3, y_2 + h f_2)$