

- **Cost:** Total number of operations (+, −, ×, ÷)
- Let  $A, B$  be  $n \times n$  matrices,  $b$  an  $n \times 1$  vector,  $\alpha$  a scalar, and  $p$  a positive integer.

Expression	Cost
$A + B$	$n^2$
$A - B$	$n^2$
$\alpha A$	$n^2$
$AB$	$2n^3 - n^2$
$Ab$	$2n^2 - n$
$A^p$	$(p - 1)(2n^3 - n^2)$
$\det(A)$	$n! \sum_{k=0}^{n-1} \left(\frac{1}{k!}\right) - 1$

• Special sums:

$$(i) \sum_{k=1}^n k = \frac{n(n+1)}{2}$$

$$(ii) \sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}$$

$$(iii) \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$(iv) \sum_{k=1}^{n-1} k^2 = \frac{n(n-1)(2n-1)}{6}$$

$$(v) \sum_{k=1}^n c = c \cdot n$$

**Q.** If  $A, B$  are  $3 \times 3$  matrices, find the cost of evaluating  $2A + |B|B^3$

**Answer:** Cost =  $9 + 9 + 9 + 14 + 2(45) = 131$

\* Total cost of some linear methods \*

(1) Gaussian Elimination.

Total cost = Cost of transforming  $[A|b] \rightarrow [U|c]$  + Cost of Back Substitution

- transforming  $[A|b] \rightarrow [U|c]$ :

Step	$\pm$	$\times$	$\div$
1	$(n-1)n$	$(n-1)n$	$n-1$
2	$(n-2)(n-1)$	$(n-2)(n-1)$	$n-2$
.	.	.	.
.	.	.	.
$k$	$(n-k)(n-k+1)$	$(n-k)(n-k+1)$	$(n-k)$
.	.	.	.
.	.	.	.
$n-1$			

$$\text{Cost} = \sum_{k=1}^{n-1} (n-k)(n-k+1) + \sum_{k=1}^{n-1} (n-k)(n-k+1) + \sum_{k=1}^{n-1} (n-k) = \frac{4n^3 + 3n^2 - 7n}{6}$$

- Back Substitution: Cost =  $n^2$
- Total cost =  $\frac{4n^3 + 3n^2 - 7n}{6} + n^2 = \frac{4n^3 + 9n^2 - 7n}{6}$

**Example:** Solving a  $4 \times 4$  linear system  $Ax = b$ .

- transforming  $[A|b] \rightarrow [U|c]$ :

Step	$\pm$	$\times$	$\div$
1	3 (4)	3 (4)	3
2	2 (3)	2 (3)	2
3	1 (2)	1 (2)	1

$$\text{Cost} = 20 + 20 + 6 = 46$$

- Back Substitution: Cost =  $4^2 = 16$
- Total cost =  $46 + 16 = 62$

(2) LU Factorization.

Total cost = Cost of transforming  $A \rightarrow U$  + Cost of Forward substitution + Cost of Backward substitution

- transforming  $A \rightarrow U$ :

Step	$\pm$	$\times$	$\div$
1	$(n-1)^2$	$(n-1)^2$	$n-1$
2	$(n-2)^2$	$(n-2)^2$	$n-2$
$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$
$k$	$(n-k)^2$	$(n-k)^2$	$(n-k)$
$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$
$n-1$			

$$\text{Cost} = \sum_{k=1}^{n-1} (n-k)^2 + \sum_{k=1}^{n-1} (n-k)^2 + \sum_{k=1}^{n-1} (n-k) = \frac{4n^3 - 3n^2 - n}{6}$$

- Forward Substitution: Cost =  $n^2 - n$
- Back Substitution: Cost =  $n^2$
- Total cost =  $\frac{4n^3 - 3n^2 - n}{6} + n^2 - n + n^2 = \frac{4n^3 + 9n^2 - 7n}{6}$

**Example:** Solving a  $4 \times 4$  linear system  $Ax = b$ .

- transforming  $A \rightarrow U$ :

Step	$\pm$	$\times$	$\div$
1	3 (3)	3 (3)	3
2	2 (2)	2 (2)	2
3	1 (1)	1 (1)	1

$$\text{Cost} = 14 + 14 + 6 = 34$$

- Forward Substitution: Cost =  $4^2 - 4 = 12$
- Back Substitution: Cost =  $4^2 = 16$
- Total cost =  $34 + 12 + 16 = 62$

(3) Gauss-Jordan Reduction.

Total cost = Cost of transforming  $[A|b] \rightarrow [I|x]$

- transforming  $[A|b] \rightarrow [I|x]$ :

Step	$\pm$	$\times$	$\div$
1	$(n-1)n$	$(n-1)n$	$n$
2	$(n-1)(n-1)$	$(n-1)(n-1)$	$n-1$
.	.	.	.
.	.	.	.
$k$	$(n-1)(n-k+1)$	$(n-1)(n-k+1)$	$n-k+1$
.	.	.	.
.	.	.	.
$n$	.	.	.

- Total cost =  $\sum_{k=1}^n (n-1)(n-k+1) + \sum_{k=1}^n (n-1)(n-k+1) + \sum_{k=1}^n (n-k+1) = \frac{2n^3 + n^2 - n}{2}$

**Example:** Solving a  $4 \times 4$  linear system  $Ax = b$ .

- transforming  $[A|b] \rightarrow [I|x]$ :

Step	$\pm$	$\times$	$\div$
1	3 (4)	3 (4)	4
2	3 (3)	3 (3)	3
3	3 (2)	3 (2)	2
4	3 (1)	3 (1)	1

- Total cost =  $30 + 30 + 10 = 70$

(4) The Inverse Method.

Total cost = Cost of transforming $[A I] \rightarrow [I A^{-1}]$ + Cost of $A^{-1}b$
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- transforming  $[A|I] \rightarrow [I|A^{-1}]$ :

Step	$\pm$	$\times$	$\div$
1	$(n-1)(2n-1)$	$(n-1)(2n-1)$	$2n-1$
2	$(n-1)(2n-2)$	$((n-1)(2n-2))$	$2n-2$
.	.	.	.
.	.	.	.
$k$	$(n-1)(2n-k)$	$(n-1)(2n-k)$	$2n-k$
.	.	.	.
.	.	.	.
$n$			

$$\text{Cost} = \sum_{k=1}^n (n-1)(2n-k) + \sum_{k=1}^n (n-1)(2n-k) + \sum_{k=1}^n (2n-k) = \frac{6n^3 - 5n^2 + n}{2}$$

- $A^{-1}b$ : Cost =  $2n^2 - n$

- Total cost =  $\frac{6n^3 - 5n^2 + n}{2} + 2n^2 - n = \frac{6n^3 - n^2 - n}{2}$

**Example:** Solving a  $4 \times 4$  linear system  $Ax = b$ .

- transforming  $[A|I] \rightarrow [I|A^{-1}]$ :

Step	$\pm$	$\times$	$\div$
1	3 (7)	3 (7)	7
2	3 (6)	3 (6)	6
3	3 (5)	3 (5)	5
4	3 (4)	3 (4)	4

$$\text{Cost} = 66 + 66 + 22 = 154$$

- $A^{-1}b$ : Cost =  $2(4)^2 - 4 = 28$

- Total cost =  $154 + 28 = 182$

(5) Cramer's Rule.

$$\boxed{\text{Total cost} = \text{Cost of det} + \text{Cost of } \div}$$

- **det:** Cost =  $(n + 1) \times \left[ n! \sum_{k=0}^{n-1} \left(\frac{1}{k!}\right) - 1 \right] = (n + 1)! \sum_{k=0}^{n-1} \left(\frac{1}{k!}\right) - (n + 1)$

- **$\div$ :** Cost =  $n$

- Total cost =  $(n + 1)! \sum_{k=0}^{n-1} \left(\frac{1}{k!}\right) - (n + 1) + n = (n + 1)! \sum_{k=0}^{n-1} \left(\frac{1}{k!}\right) - 1$

**Example:** Solving a  $4 \times 4$  linear system  $Ax = b$ .

- Total cost =  $5$  ( cost of  $4 \times 4$  determinant ) +  $4 = 5(63) + 4 = 319$