1.1 Review of Calculus

• Limits and Continuity

Assume $f(x)$ is defined on an open interval containing x_0.

Then
1. f has limit L at x_0 if $\lim_{x \to x_0} f(x) = L$
2. f is continuous at x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$
3. f is continuous on a set S if f is continuous at each point $x \in S$.

• $C^n(S)$: is the set of all functions f s.t f and its first n derivatives are continuous on S.

Exp: $f(x) = x^{4/3}$ is $C^1[-1,1]$

$f(x)$ and $f'(x) = \frac{4}{3} x^{\frac{1}{3}}$ are continuous on $[-1,1]$

but $f'' = \frac{4}{9} x^{-\frac{2}{3}}$ is not continuous at $x = 0$

• Convergent sequence

The sequence $\{x_n\}_{n=1}^{\infty}$ converges to a limit L if

$\lim_{n \to \infty} x_n = L$ (or we write $x_n \to L$ as $n \to \infty$)

• Error Sequence

$\{E_n\}_{n=1}^{\infty} = \{x_n - L\}_{n=1}^{\infty}$ is called an error sequence

---

$\forall$: for all $\exists$: there exists	s.t : such that $f^{(n)}(x)$: n^{th} derivative of f	$f \in C[a,b]$: f is continuous on $[a,b]$ $f \in C^1[a,b]$: f, f' are continuous on $[a,b]$ $f \in C^n[a,b]$: $f, f', \dots, f^{(n)}$ are cont. on $[a,b]$

Th • Assume $f(x)$ is defined on the set S.

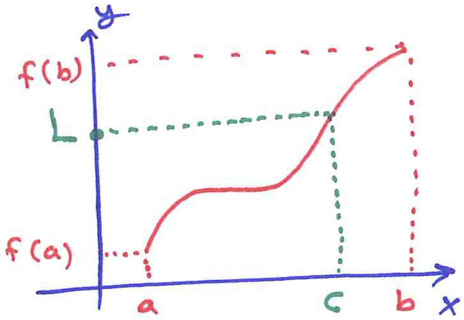
• Let $x_0 \in S$. Then the following are equivalent:

$$f \text{ is cont. at } x_0 \iff \text{if } \lim_{n\to\infty} x_n = x_0 \text{ then } \lim_{n\to\infty} f(x_n) = f(x_0)$$

Th (Intermediate Value Theorem)

• Assume $f \in C[a,b]$ and $f(a) < L < f(b)$.

• Then $\exists\, c \in (a,b)$ s.t $f(c) = L$



Exp • $f(x) = x^2$ is cont. on $[0,4]$

• Take $L = 9 \in \big(f(0), f(4)\big)$

• The solution of $f(c) = 9$ is

$c^2 = 9 \iff c = 3 \in (0,4)$

Th (Extreme Value Theorem)

• Assume that $f \in C[a,b]$, then f has abs. max

$f(b) = M = \max\limits_{a\le x\le b} \{f(x)\}$ and abs. min $m = \min\limits_{a \le x \le b}\{f(x)\} = f(a)$

• Differentiation

• f is diff at x_0 if $f'(x_0) = \lim\limits_{h\to 0} \dfrac{f(x_0+h) - f(x_0)}{h}$ exists

• f is differentiable on set S if f has derivative at each point in S.

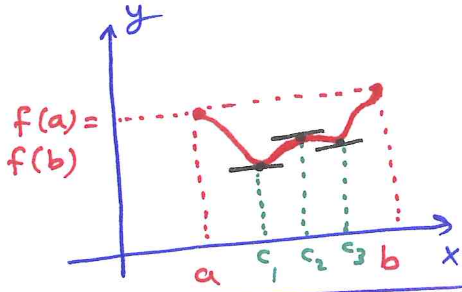
• If $f(x)$ is diff at x_0, then f is cont. at x_0.

• $m = f'(x_0)$ is the slope of the tangent line to the graph $y = f(x)$ at the point $(x_0, f(x_0))$:

$$y - f(x_0) = m(x - x_0) \quad \text{tangent line}$$

Th (Rolle's Theorem)

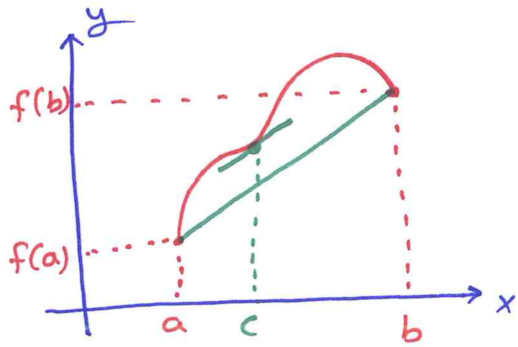
- Assume that $f \in C[a,b]$ and f is diff on (a,b).
- If $f(a) = f(b)$, then $\exists$ a number $c \in (a,b)$ s.t $f'(c) = 0$



Belzano here!

Th (Mean Value Theorem)

- Assume that $f \in C[a,b]$ and f is Diff on (a,b).
- Then, $\exists$ a number $c \in (a,b)$ s.t $f'(c) = \dfrac{f(b) - f(a)}{b - a}$



Th (First Fundemental Theorem)

- Assume $f \in C[a,b]$ and F is any antiderivative of f on $[a,b]$.
- Then, $\int_a^b f(x)\,dx = F(b) - F(a)$ where $F'(x) = f(x)$.

Th (Second Fundemental Theorem)

Assume $f \in C[a,b]$. Then $\dfrac{d}{dx} \int_a^x f(t)\,dt = f(x) \quad \forall x \in (a,b)$

Exp $\dfrac{d}{dx} \int_0^{x^2} \cos t \, dt = 2x \cos x^2$

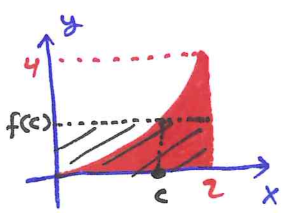
Th (Mean Value Theorem for Integrals)

Assume $f \in C[a,b]$. Then $\exists$ a number $c \in (a,b)$ s.t $\dfrac{1}{b-a} \int_a^b f(x)\,dx = f(c)$

$f(c)$ is the average value of f over the interval $[a,b]$

Exp $f(x) = x^2$ on $[0,2]$

- $av(f) = \frac{1}{2} \int_0^2 x^2\,dx$
$= \frac{1}{2} \cdot \frac{8}{3}$
$= \frac{4}{3} = f(c) = c^2 \Leftrightarrow c = \frac{2}{\sqrt{3}}$



- Area of rectangle is $(b-a)f(c) = 2\left(\frac{4}{3}\right) = \frac{8}{3}$

which is the red area

Taylor Series Expansion

Def Assume $f(x)$ is infinitely many differentiable at x_0.

Then the Taylor series of $f(x)$ at x_0 is

$$f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots$$

Exp Find the Maclaurin Series ($x_0 = 0$) of e^x, $\cos x$, $\sin x$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

- Series

Given an infinite series $a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=0}^{\infty} a_n$

- The n^{th} partial sum is $S_n = a_1 + a_2 + \cdots + a_n$
- The infinite series **converges** iff S_n converges $(\lim_{n\to\infty} S_n = L)$.

otherwise, it diverges.

Exp $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$, $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$ where $A = 1$, $B = -1$

$$S_n = \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) + \cdots + \left(\tfrac{1}{n} - \tfrac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

Hence, $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$ so the series converges to 1.

Th (Taylor's Theorem)

Assume $f \in C^{n+1}[a,b]$. Let $x_0 \in [a,b]$. Then $\exists$ a number $c \in (x_0, x)$ s.t the Taylor formula $\underbrace{f(x)}_{\text{True value}} = \underbrace{P_n(x)}_{\text{approximated value about } x_0} + R_n(x)$ holds where

$P_n(x)$ is polynomial of degree n used to approximate $f(x)$ with error (or remainder) $R_n(x)$ given by:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \quad \text{and} \quad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

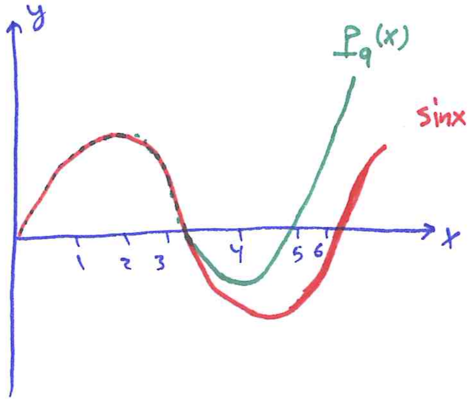
Exp $f(x) = \sin x$ with $x_0 = 0$. Then

$f(x) = \sin x = P_9(x) + R_9(x)$ where

$$P_9(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

$$R_9(x) = \frac{f^{(10)}(c)}{(n+1)!}(x-0)^{n+1} \leq \frac{|x|^{n+1}}{(n+1)!} \quad , c \in (0, x)$$

$M = 1$



$$\text{Error} = |R_n(x)| = \left|f^{(n+1)}(c)\right| \frac{|x - x_0|^{n+1}}{(n+1)!} \leq M \frac{|x - x_0|^{n+1}}{(n+1)!}$$

Linear Estimation:

$$f(x) = P_1(x) + R_1(x)$$

$$= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(c)}{2!}(x - x_0)^2 \quad , \quad c \in (x_0, x)$$

Exp • $f(x) = e^x$ with $x_0 = 0$

$$e^x = P_1(x) + R_1(x)$$

$$= f(0) + f'(0)(x-0) + \frac{f''(c)}{2!}(x-0)^2$$

$$= 1 + x + \frac{e^c x^2}{2!} \quad , \quad c \in (0, x)$$

Hence, $e^x \approx 1 + x$ with error $= \left| \frac{e^c x^2}{2!} \right| = \frac{e^c x^2}{2!}$

$e \approx 1 + 1$ with error $= \frac{e^c}{2!} < \frac{3}{2}$ $\qquad c \in (0,1)$

$0 < c < 1$

$e^0 < e^c < e^1$

$1 < e^c < 3 \Leftarrow 1 < e^c < e \Leftarrow$

• $e^x = P_2(x) + R_2(x)$

$$= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(c)}{3!}(x-0)^3$$

$$= 1 + x + \frac{x^2}{2!} + \frac{e^c x^3}{3!}$$

$0 < c < 0.1$

$1 < e^c < e^{0.1} < 2$

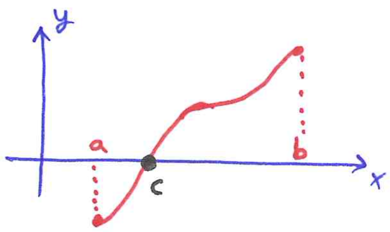
Hence, $e^x \approx 1 + x + \frac{x^2}{2!}$ with error $= \frac{e^c}{3!}|x^3|$

$e^{0.1} \approx 1 + 0.1 + \frac{(0.1)^2}{2!}$ with error $= \frac{e^c}{3!}(0.1)^3 < \frac{2}{3!}(0.001) < 10^{-3}$

Th (Belzano)

Assume $f(x) \in C[a,b]$ with $f(a)\,f(b) < 0$

Then $\exists$ a number $c \in (a,b)$ s.t $f(c) = 0$



means: c is root for $f(x)$
c is zero for $f(x)$
c is solution for $f(x) = 0$
f crosses x-axis at $x = c$
$x = c$ is the x-intercept

go to page 4