Chapter Two

Solving non linear equations $f(x) = 0$

* In this chapter we will learn methods used to solve non linear equations numerically.

Exp ① solve $\sqrt{x}\, e^{\cos x} = 5 \Rightarrow \underbrace{\sqrt{x}\, e^{\cos x} - 5}_{f(x)} = 0$

② solve $e^{x} = x \Rightarrow \underbrace{e^{x} - x}_{f(x)} = 0$

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* Numerical Methods to solve $f(x) = 0$:

[1] Fixed Point Iteration (FPI)

[2] Bisection Method

[3] False Position Method (or Regula Falsi Method)

[4] Newton - Raphson Method

[5] Secant Method

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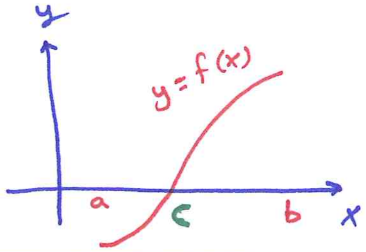
[2] and [3] are also called Bracketing Methods for locating a roots

Ch2: Solution of Nonlinear Equations $f(x)=0$

- Let $f(x)=0$
- Assume $\exists\ c \in (a,b)$ s.t $f(c)=0$
- How to estimate c ?



2.1 Iteration for Solving $x=g(x)$

Def Iteration is a repeated process until an answer is achieved.

- Iteration is used to find roots of equations, solution of linear and nonlinear systems of equations, solutions of differential equations, ---
- To build an iteration, $\{P_k\}$ we need a rule or function $g(x)$ and a starting point P_0.

Def (Fixed Point)

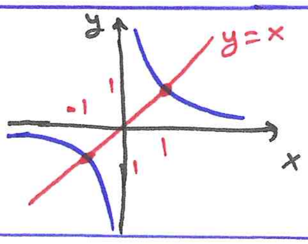
- A fixed point of a function $g(x)$ is a real number P s.t

$$P = g(P)$$

- That is, the fixed points of $y=g(x)$ are the points of intersection of $y=g(x)$ and $y=x$

Exp ① $g(x)=\frac{1}{x}$ $\Rightarrow$ fixed points are $1, -1$

since $P=g(P) \Leftrightarrow P=\frac{1}{P} \Leftrightarrow P^2=1$
$\Leftrightarrow P=\pm 1$



② $g(x)=x \Rightarrow$ all points are fixed points

③ $g(x)=x+1 \Rightarrow$ No fixed points

④ $g(x)=x^3 \Leftrightarrow x=x^3 \Leftrightarrow x(x^2-1)=0 \Leftrightarrow x=0,1,-1$

⑤ $g(x)=\cos x \Leftrightarrow x=\cos x$ "harder"

Def (Fixed Point Iteration)

- The iteration $p_{n+1} = g(p_n)$, $n = 0, 1, 2, \ldots$ is called fixed point iteration.
- That is,

$$
\begin{aligned}
&p_0 \quad \text{starting value} \\
p_1 &= g(p_0) \\
p_2 &= g(p_1) \\
&\vdots \\
p_{k+1} &= g(p_k) \\
&\vdots
\end{aligned}
$$

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* which types of functions $g(x)$ that produce convergent sequence $\{p_k\}$ is our main aim to study in this section.

Th* • Assume g is continuous function.

- Let $\{p_n\}_{n=0}^{\infty}$ be a fixed point iteration.
- If $\{p_n\}_{n=0}^{\infty}$ converges to p "$\lim_{n\to\infty} p_n = p$", then p is fixed point of $g(x)$

Proof • Since $\{p_n\}_{n=0}^{\infty}$ is fixed point iteration $\Rightarrow p_{n+1} = g(p_n)$

- If $\lim_{n\to\infty} p_n = p$, then $\lim_{n\to\infty} p_{n+1} = p$.
- Hence,

$$g(p) = g\left(\lim_{n\to\infty} p_n\right) = \lim_{n\to\infty} g(p_n) = \lim_{n\to\infty} p_{n+1} = p$$

(g is cont.)

Remark • Let $f(x) = x - g(x)$

- To solve $f(x) = 0$, we solve $x = g(x)$ for fixed points.
- That is, to find the roots of $f \Rightarrow$ we find the fixed points of $g(x)$.

Exp $x^2 + 3x - 4 = 0$

- The fixed points are $(x-1)(x+4) = 0 \Leftrightarrow x = 1, -4$
- $g(x)$ can have one of the following forms:

1. $g_1(x) = \dfrac{4 - x^2}{3}$ • $\Rightarrow$ if $p_0 = 3$ then (3 digits)

$p_1 = -1.67 \Rightarrow p_2 = 0.403 \Rightarrow p_3 = 1.28 \cdots \Rightarrow p_{13} = 1.005$

which converges to the fixed point 1

• $\Rightarrow$ if $p_0 = -6$ then

$p_1 = -10.7 \Rightarrow p_2 = -36.7 \Rightarrow p_3 = -445$ .... diverges

• Hence, g_1 can find only the fixed point 1

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2. $g_2(x) = -\sqrt{4 - 3x}$ • $\Rightarrow$ if $p_0 = -6$ then

$p_1 = -4.69 \Rightarrow p_2 = -4.25 \Rightarrow p_3 = -4.10 \Rightarrow \cdots \Rightarrow p_{10} = -4.0000964$

which converges to the fixed point -4

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3. $g_3(x) = \dfrac{4}{x+3}$ • $\Rightarrow$ if $p_0 = 3$ then

$$\begin{cases} x(x+3) = 4 \\ x = \dfrac{4}{x+3} = g(x) \end{cases}$$

$p_1 = 0.667 \Rightarrow p_2 = 1.09 \Rightarrow p_3 = 0.978 \cdots \Rightarrow p_8 = 1.00002$

which converges to the fixed point 1

• $\Rightarrow$ if $p_0 = -6$ then

$p_0 = -1.33 \Rightarrow p_2 = 2.40$ $p_3 = 0.741$ $p_4 = 1.07$ ... 1.004 which converges to 1

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4. $g_4(x) = x^2 + 4x - 4$

$$\begin{cases} x^2 + 3x - 4 + x = x \\ x^2 + 4x - 4 = x = g(x) \end{cases}$$

it will diverge for both cases

Exp. Consider $f(x) = x^2 - 2x - 3$

- clearly the roots of $f(x) = 0 \iff (x-3)(x+1) = 0$ are $x = 3$ and $x = -1$
- Note that we can estimate the roots as follows:

$\boxed{1}$ $x^2 - 2x - 3 = 0 \iff x^2 = 2x - 3 \iff \boxed{x = \sqrt{2x+3} = g(x)}$

if $p_0 = 4$ then (If $p_0 = \frac{-3}{2}$ then $p_n \to 3$ and we could not find -1)

$$p_1 = g(p_0) = g(4) = \sqrt{11} \approx 3.31662$$

$$p_2 = g(p_1) = g(3.31662) = \sqrt{9.63324} \approx 3.10375$$

$$p_3 = g(p_2) = g(3.10375) = \sqrt{9.2075} \approx 3.03439$$

$$p_4 = g(p_3) = g(3.03439) = \sqrt{9.06878} \approx 3.01144$$

$\vdots$

$p_n \to 3$ (Note that 3 is fixed point of $g(x) = \sqrt{2x+3}$)

$\boxed{2}$ $x(x-2) = 3 \Rightarrow x = \dfrac{3}{x-2} = g(x)$

If $p_0 = 4$ then

$$p_1 = g(p_0) = g(4) = \frac{3}{2} = 1.5$$

$$p_2 = g(p_1) = g(1.5) = \frac{3}{-0.5} = -6$$

$$p_3 = g(p_2) = g(-6) = \frac{3}{-8} = -0.375$$

$$p_4 = g(p_3) = g(-0.375) = \frac{3}{-2.375} = -1.26316$$

$$p_5 = g(p_4) = g(-1.26316) = \frac{3}{-3.26316} = -0.91935$$

$$p_6 = g(p_5) = g(-0.91935) = -1.02763$$

$$p_7 = g(p_6) = g(-1.02763) = -0.99087$$

$\vdots$

$p_n \to -1$ (Note that -1 is fixed point of $g(x) = \frac{3}{x-2}$)

If $p_0 = -2$ then

$$p_1 = \frac{3}{-4} = -0.75$$

$$p_2 = \frac{3}{-2.75} = -1.091$$

$$p_3 = \frac{3}{-3.091} = -0.971$$

$$p_4 = -1.01$$

$$p_5 = -0.997$$

$$p_6 = -1.001$$

$$p_7 = -1$$

the fixed point

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3\) Note that if we choose

$2x = x^2 - 3 \Rightarrow x = \frac{x^2-3}{2} = g(x)$ then we get a divergence sequence and so it will not work.

$p_0 = 4$

$p_1 = g(p_0) = g(4) = 6.5$

$p_2 = g(p_1) = g(6.5) = 19.625$

$p_3 = g(p_2) = g(19.625) = 191.0703$

$\vdots$

This is something related to the slope of the function

so it depends on how to write $g(x)$

Th (Fixed Point Theorem I) - FPTI

Assume $g \in C[a,b]$.

- If $g(x) \in [a,b]$ for all $x \in [a,b]$, then g has a fixed point in $[a,b]$.
- Furthermore, if $|g'(x)| \leq K < 1$ for all $x \in (a,b)$, then g has a unique fixed point in $[a,b]$.

Proof • If $g(a) = a$ or $g(b) = b$, then we are done.

Otherwise, $g(a) \in (a,b]$ and $g(b) \in [a,b)$.

Now let $f(x) = x - g(x) \Rightarrow f$ is continuous and

$$f(a) = a - g(a) < 0 \text{ and}$$
$$f(b) = b - g(b) > 0$$

Hence, by Belzano Theorem $\exists\, p \in (a,b)$ s.t

$$f(p) = 0$$
$$p - g(p) = 0$$
$$p = g(p) \quad \Rightarrow p \text{ is a fixed point.}$$

- (Uniqueness)

Suppose ℓ_1 and ℓ_2 are two fixed points of g

$\Rightarrow \quad g(\ell_1) = \ell_1$ and $g(\ell_2) = \ell_2$

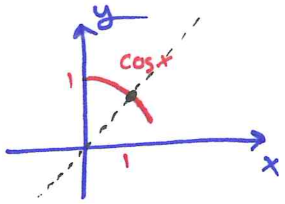
Apply Mean Value Theorem on $(\ell_1, \ell_2) \Rightarrow$

$\exists \quad c \in (\ell_1, \ell_2)$ s.t

$$g'(c) = \frac{g(\ell_2) - g(\ell_1)}{\ell_2 - \ell_1} = \frac{\ell_2 - \ell_1}{\ell_2 - \ell_1} = 1 \quad \times$$

since $|g'(x)| \leq K < 1$. Hence, $\ell_1 = \ell_2$

Exp show that $g(x) = \cos x$ has a unique fixed point in $[0,1]$



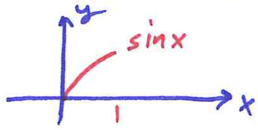
- g is continuous on $[0,1] \Rightarrow g \in C[0,1]$

g is decreasing on $[0,1]$ with

$$\cos 1 \leq \cos x \leq \cos 0$$

Hence, $g(x) \in [\underset{0.5402}{\cos 1}, 1] \subseteq [0,1]$ for all $x \in [0,1]$

Thus, g has a fixed point in $[0,1]$



- for all $x \in (0,1) \Rightarrow |g'(x)| = |-\sin x| = \sin x \leq \sin 1 < 1$

Thus, $K = \sin 1 \approx 0.8415 < 1 \Rightarrow g$ has a unique fixed point in $[0,1]$.

Question: When the fixed point Iteration (FPI)

$$p_{k+1} = g(p_k) \quad , \quad p_0 \quad , \quad k = 0, 1, 2, \ldots$$

produce a convergence or divergence sequence?

Note FPIT II page $\boxed{21}$ does not apply when $g'(\ell) = 1$.

see Exp* page 24

Th (Fixed Point Iteration Theorem II) - FPIT II

Assume the following :

$g, g' \in C[a,b]$, K is positive constant, $p_0 \in (a,b)$ is the starting point, $g(x) \in [a,b]$ for all $x \in [a,b]$

*• If $|g'(x)| \leq K < 1$ for all $x \in [a,b]$, then the FPI converges to the unique fixed point $p \in [a,b]$.

(In this case p is called attractive fixed point)

• If $|g'(x)| > 1$ for all $x \in [a,b]$, then the FPI diverges.

(In this case p is called repelling fixed point)

Remark* If p is given, then we can replace the above two conditions by :

- If $|g'(p)| < 1$, then FPI converges and p is attracter
- If $|g'(p)| > 1$, then FPI diverges and p is repeller

Proof We will prove *• First we prove that the points $\{p_n\}_{n=0}^{\infty} \in (a,b)$.

• since $p_0 \in (a,b)$ $\Rightarrow$ Apply MVT on (p_0, p)

① ... $|p - p_1| = |g(p) - g(p_0)| = |g'(c_0)\,(p - p_0)| = |g'(c_0)|\,|p - p_0|$

$\leq K\,|p - p_0| < |p - p_0|$

• Note that ↳ the assumptions above implies that $\exists_1 p \in [a,b]$ by FPTI page 19
unique

↳ $K < 1$

↳ $c_0 \in (a,b)$

• We see from ① that p_1 is closer to p than $p_0 \Rightarrow p_1 \in (a,b)$

• Similar to ① we see that $|p - p_2| < |p - p_1|$ for some $c_1 \in (a,b)$ and so $p_2 \in (a,b)$

• In general, suppose that $p_{n-1} \in (a,b) \Rightarrow$

② ... $|p - p_n| = |g(p) - g(p_{n-1})| = |g'(c_{n-1})|\,|p - p_{n-1}| \leq K\,|p - p_{n-1}| < |p - p_{n-1}|$

Therefore, $p_n \in (a,b)$ and hence, by induction all the points $\{p_n\}_{n=0}^{\infty} \in (a,b)$.

- Now we need to prove $\lim\limits_{n\to\infty} |p - p_n| = 0$

Claim $|p - p_n| \leq K^n |p - p_0|$

Proof by induction when $n=1 \Rightarrow |p - p_1| \leq K\,|p - p_0|$ see ①

- Assume the claim holds for $n-1 \Rightarrow |p - p_{n-1}| \leq K^{n-1}|p - p_0|$
- From ② we have

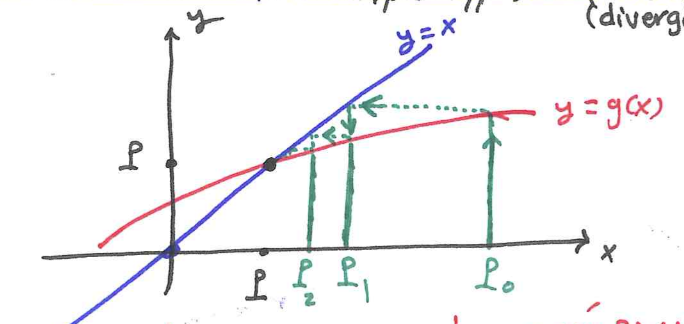
$$|p - p_n| \leq K\,|p - p_{n-1}| \leq K\,K^{n-1}\,|p - p_0| = K^n\,|p - p_0|$$

- Since $0<K<1 \Rightarrow \lim\limits_{n\to\infty} K^n = 0$. Hence,

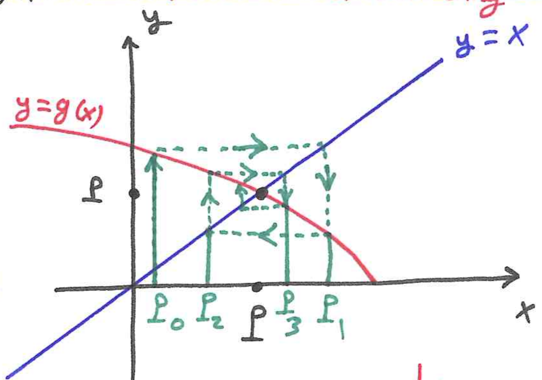
$$0 \leq \lim_{n\to\infty} |p - p_n| \leq \lim_{n\to\infty} K^n\,|p - p_0| = 0$$

- Thus, by Sandwich Theorem $\lim\limits_{n\to\infty} |p - p_n| = 0$. So $\lim\limits_{n\to\infty} p_n = p$. By Th* page 17 the iteration $p_n = g(p_{n-1})$ converges to the fixed point p (since it is unique).

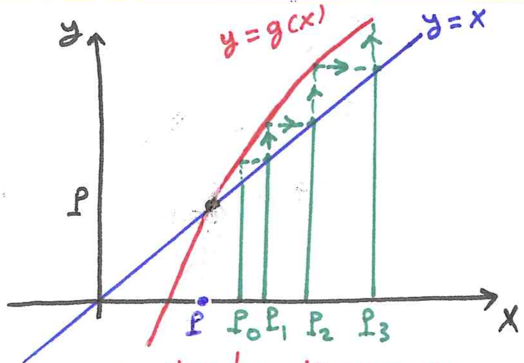
Two simple types of (convergent) (divergent) iteration: Monoton and oscillating.



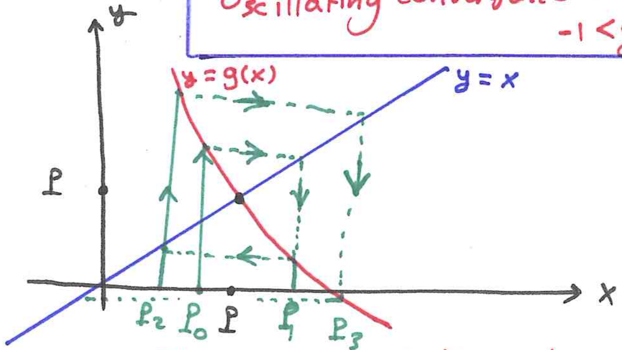
Monoton convergence when $0<g'(p)<1$



Oscillating convergence when $-1<g'(p)<0$.



Monoton divergence when $1<g'(p)$



Divergence oscillation when $g'(p)<-1$

Exp Given $g(x) = \sqrt{3x-2}$.
Find the fixed points of $g(x)$ and determine the nature of these fixed points.

- $x = g(x) \iff x = \sqrt{3x-2} \iff x^2 = 3x - 2$
  $\iff x^2 - 3x + 2 = 0 \iff (x-1)(x-2) = 0$
  $\iff x = 1, \ x = 2$ are fixed points

- $g'(x) = \dfrac{3}{2\sqrt{3x-2}}$ $\Rightarrow |g'(1)| = \dfrac{3}{2} > 1 \Rightarrow x = 1$ is repeller
  $\Rightarrow$ FPI diverges

  $\Rightarrow |g'(2)| = \dfrac{3}{4} < 1 \Rightarrow x = 2$ is attractor
  $\Rightarrow$ FPI converges to 2

Exp $x^2 + 3x - 4 = 0$ has fixed points $x = 1, -4$

1. $g_1(x) = \dfrac{4 - x^2}{3}$

$g_1'(x) = -\frac{2}{3}x \Rightarrow$

- $|g_1'(1)| = |-\frac{2}{3}| < 1 \Rightarrow x = 1$ is attractor

$\Rightarrow$ FPI converges to 1. That is we can use the FPI to find the solution of $x = g(x)$. (see page 17.1)

- $|g_1'(-4)| = |\frac{8}{3}| > 1 \Rightarrow x = -4$ is repeller

$\Rightarrow$ FPI diverges (see page 17.1)

2. $g_2(x) = -\sqrt{4 - 3x} \Rightarrow g_2'(x) = \dfrac{3}{2\sqrt{4-3x}}$

- $g_2'(1) = |\frac{3}{2}| > 1 \Rightarrow x = 1$ is repeller $(p_0 \not\geq 2)$

$\Rightarrow$ FPI does not work (see page 17.1)

see also Expy page 17.1

- $g_2'(-4) = |\frac{3}{8}| < 1 \Rightarrow x = -4$ is attractor

$\Rightarrow$ FPI Converges to -4 (see page 17.1)

Exp Test the convergence of FPI for 1. $g(x) = -4 + 4x - \frac{x^2}{2}$

- $g(x) = x \Leftrightarrow -4 + 3x - \frac{x^2}{2} = 0 \Leftrightarrow x^2 - 6x + 8 = 0 \Leftrightarrow x = 2, 4$ fixed points
- $g'(x) = 4 - x \Rightarrow g'(2) = |4-2| = 2 > 1 \Rightarrow x = 2$ is repeller $\Rightarrow$ FPI div.

$\Rightarrow g'(4) = |4-4| = 0 < 1 \Rightarrow x = 4$ is attractor $\Rightarrow$ FPI conv.

age 24

2. $g(x) = 2\sqrt{x-1} \Leftrightarrow x = g(x) \Leftrightarrow x^2 = 4(x-1) \Leftrightarrow x^2 - 4x + 4 = 0 \Leftrightarrow$

$x = -2$ fixed point

$g'(x) = \dfrac{1}{\sqrt{x-1}} \Rightarrow g'(2) = |1| = 1$ Test Fails so we build FPI to check (see Exp* page 24).

Exp Let $g(x) = 1 + x - \frac{x^2}{4}$. Can we use the FPI to find the solution of the equation $x = g(x)$? Why?

Solution • $x = g(x) \Leftrightarrow x = 1 + x - \frac{x^2}{4} \Leftrightarrow x^2 = 4 \Leftrightarrow \boxed{x = \pm 2}$ Fixed Points

• $\boxed{x=2}$ $\Rightarrow g'(x) = 1 - \frac{x}{2} \Rightarrow |g'(2)| = \overset{K \text{ for } x=2}{0} < 1$

Hence, by Remark* page 21 $\Rightarrow$ the FPI converges to 2

• So we can use the FPI to find the solution of $x = g(x)$:

$p_0 = 1.6$

$p_1 = g(p_0) = g(1.6) = 1 + 1.6 - \frac{(1.6)^2}{4} = 1.96$

$p_2 = g(p_1) = g(1.96) = 1 + 1.96 - \frac{(1.96)^2}{4} = 1.9996$

$p_3 = g(p_2) = g(1.9996) = 1 + 1.9996 - \frac{(1.9996)^2}{4} = 2$

$\boxed{p_n \to 2}$ ↑ attracter fixed point

$p_0 = 2.5$

$p_1 = g(p_0) = g(2.5) = 1 + 2.5 - \frac{(2.5)^2}{4} = 1.9375$

$p_2 = g(p_1) = g(1.9375) = 1.999$

$p_3 = g(p_2) = g(1.999) = 2$

$\boxed{p_n \to 2}$

• $\boxed{x=-2}$ $\Rightarrow |g'(-2)| = \left|1 - \frac{-2}{2}\right| = 2 > 1$. Hence, by Remark*

FPI diverges and $p = -2$ is repeller (repulsive) fixed point.

To see that:

$p_0 = -2.05$

$p_1 = g(p_0) = g(-2.05) = 1 - 2.05 - \frac{(-2.05)^2}{4} = -2.1$

$p_2 = g(p_1) = g(-2.1) = -2.2025$

$p_3 = g(p_2) = g(-2.2025) = -2.4153$

$\vdots$

p_n diverges

Exp* Consider the iteration $p_{n+1} = g(p_n)$ where $g(x) = 2\sqrt{x-1}$ $\boxed{24}$
for $x \geq 1$. Can FPI be used to find the solution of $x = g(x)$?

Fixed point

$$x = g(x) \iff x = 2\sqrt{x-1} \iff x^2 = 4(x-1)$$

$$\iff x^2 - 4x + 4 = 0 \iff (x-2)(x-2) = 0 \iff \boxed{x=2}$$

FPI

$$g'(x) = \frac{1}{\sqrt{x-1}} \Rightarrow g'(2) = 1$$

$\Rightarrow$ FPIT II does not apply

There are two cases to consider:

$\boxed{\text{Case 1}}$ start with $p_0 = 1.5$

$$p_1 = g(p_0) = g(1.5) = 2\sqrt{1.5-1} = 1.4142$$

$$p_2 = g(p_1) = g(1.4142) = 1.2872$$

$$p_3 = g(p_2) = g(1.2872) = 1.0718$$

$$p_4 = g(p_3) = g(1.0718) = 0.5359$$ outside the domain of g

p_5 can not be computed

$\boxed{\text{case 2}}$ start with $p_0 = 2.5$

$$p_1 = g(p_0) = g(2.5) = 2\sqrt{2.5-1} = 2.4495$$

$$p_2 = g(p_1) = g(2.4495) = 2.4079$$

$$p_3 = g(p_2) = g(2.4079) = 2.3731$$

$$p_4 = g(p_3) = g(2.3731) = 2.3436$$

$\vdots$

$$\lim_{n \to \infty} p_n = 2$$ "slowly" $\Rightarrow p_{1000} = 2.004$

Hence, the FPI converges in this exp for every $p_0 > 2$
and diverges for every $1 \leq p_0 < 2$

Corollary Assume g satisfies conditions of FPIT II. If p_n is used to approximate the fixed point p, then an upper bounds for the error are

$$|p - p_n| \le K^n |p - p_0| \quad \text{for all } n \ge 1 \quad \cdots (A)$$

Error

and

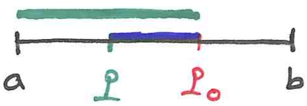
$$|p - p_n| \le \frac{K^n}{1-K} |p_1 - p_0| \quad \text{for all } n \ge 1 \quad \cdots (B)$$

Proof (HW) see page 25.1

- (A) was the claim in the proof of FPIT II and has been proven. (K is the upper bound of $g'(p)$)

- To prove (B) we use (A) as follows:

$$|p - p_n| \le K^n |p - p_0| \le K^n \max\{p_0 - a,\ b - p_0\}$$



- For $n \ge 1$ we have

$$* \quad |p_{n+1} - p_n| = |g(p_n) - g(p_{n-1})| \le K |p_n - p_{n-1}| \le K^n |p_1 - p_0|$$

- Therefore, for $m > n \ge 1$ we have

$$|p_m - p_n| = |p_m - p_{m-1} + p_{m-1} + \cdots + p_{n+1} - p_n|$$

$$\le |p_m - p_{m-1}| + |p_{m-1} - p_{m-2}| + \cdots + |p_{n+1} - p_n|$$

$$\le K^{m-1} |p_1 - p_0| + K^{m-2} |p_1 - p_0| + \cdots + K^n |p_1 - p_0| \quad \text{by } *$$

$$= (1 + K + K^2 + \cdots + K^{m-n-1}) K^n |p_1 - p_0| = K^n |p_1 - p_0| \sum_{i=0}^{m-n-1} K^i$$

- Since $\lim_{m \to \infty} p_m = p$ it follows that

$$|p - p_n| = \lim_{m \to \infty} |p_m - p_n| \le K^n |p_1 - p_0| \sum_{i=0}^{\infty} (K)^i = \frac{K^n}{1-K} |p_1 - p_0|$$

$m > n \Rightarrow K^m < K^n$ since $K < 1$

Proof of (B) in different way :

- $|p - p_0| = |p - p_1 + p_1 - p_0|$

$$\le |p - p_1| + |p_1 - p_0| \quad \text{by Triangle Inequality}$$

$$\le K\,|p - p_0| + |p_1 - p_0| \quad \text{by (A)}$$

- Hence, $|p - p_0| - K|p - p_0| \le |p_1 - p_0|$

$$(1 - K)\,|p - p_0| \le |p_1 - p_0|$$

$$|p - p_0| \le \frac{1}{1-K}\,|p_1 - p_0| \quad \cdots \; *$$

- From (A) we have

$$|p - p_n| \le K^n\,|p - p_0|$$

$$\le K^n \frac{1}{1-K}\,|p_1 - p_0|$$

$$= \frac{K^n}{1-K}\,|p_1 - p_0|$$

Remark
- This Corollary provides stopping crecteria to the FPI
- That is, it tells us the number of iterations n for a given upper bound of the error.

Exp Let $x^3 - x - 5 = 0$.

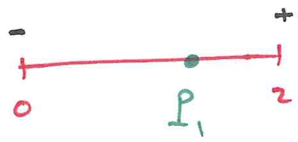
Use fixed point iteration to find all the roots.

Find K for each case. (up to 5 decimal places)

- Let $f(x) = x^3 - x - 5$

  $f(0) = -5$

  $f(2) = 1$



$p_1 \in [0,2]$

- $x^3 = x + 5 \quad \Leftrightarrow \quad x = \sqrt[3]{x+5} = g(x)$

- $g'(x) = \dfrac{1}{3\sqrt[3]{(x+5)^2}} \Rightarrow |g'(x)| < 1$ for all $x \in (0,2)$

  so the FPI converges to p_1.

- To find $p_1 \Rightarrow$ start with $p_0 = 1.5$

$$p_1 = g(p_0) = g(1.5) = \sqrt[3]{6.5} = 1.8663$$

$$p_2 = g(p_1) = g(1.8663) = \sqrt[3]{6.8663} = 1.9007$$

$$p_3 = g(p_2) = g(1.9007) = \sqrt[3]{6.9007} = 1.9038$$

$$p_4 = g(p_3) = g(1.9038) = \sqrt[3]{6.9038} = 1.9041$$

$$p_5 = g(p_4) = g(1.9041) = \sqrt[3]{6.9041} = 1.9042 = p_6$$

$$\boxed{p_1 = 1.9042}$$

$p_{2,3} = -0.9521 \pm 1.3113\, i$

- To find K $\Rightarrow$ Recall that K is an upper bound of $g'(x)$

$\Rightarrow |g'(x)| = \left| \dfrac{1}{3\sqrt[3]{(x+5)^2}} \right|$ is decreasing for all $x \in [0,2]$

$$< \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6} < 1$$

$$\Rightarrow \boxed{K = \frac{1}{6}}$$

That is $g'(x) = \dfrac{1}{3\sqrt[3]{(x+5)^2}} < \dfrac{1}{3} \cdot \dfrac{1}{\sqrt[3]{25}} < \dfrac{1}{3} \cdot \dfrac{1}{2}$

$x > 0 \Rightarrow x + 5 > 5$

$(x+5)^2 > 25$

$\sqrt[3]{(x+5)^2} > (25)^{\frac{1}{3}} > 2$

$\dfrac{1}{\sqrt[3]{(x+5)^2}} < \dfrac{1}{2}$

Note

$|g'(0)| = \dfrac{1}{3\sqrt[3]{25}} = K_{Max} < 1$

$|g'(2)| = \dfrac{1}{3\sqrt[3]{49}} = K_{min} < 1$

$K_{max} = 0.114$

$K_{min} = 0.0911$

Exp Consider $g(x) = \frac{1}{x^3} + 2$ on $[2,3]$

1. show that $g(x)$ has fixed points in $[2,3]$
   - Note that g is continuous on $[2,3]$
   - Note also g is decreasing on $[2,3]$

     $2 < 2.037 = g(3) \le g(x) \le g(2) = 2.125 < 3 \quad \forall x \in [2,3]$
   - Hence, $g(x) \in [2,3] \quad \forall x \in [2,3]$
   - Therefore by FPTI page 19 g has a fixed point in $[2,3]$

2. Is it unique?
   - $|g'(x)| = \left| -\frac{3}{x^4} \right| = \frac{3}{x^4}$ decreasing on $(2,3)$
   - $|g'(x)| \le \frac{3}{(2)^4} = 0.1875 = K_{max} < 1$ for all $x \in (2,3)$
   - Hence, g has a unique fixed point in $[2,3]$

3. show that the FPI converges for every $P_0 \in (2,3)$
   - Note that $g, g' \in C[2,3]$ and $g(x) \in [2,3] \ \forall x \in [2,3]$
   - Note also $|g'(x)| \le 0.1875 = K < 1 \quad \forall x \in [2,3]$
   - Hence, by FPIT II page 21 the FPI converges $\forall P_0 \in (2,3)$

4. Using 4 digits rounding, estimate the fixed point of $g(x)$ using $P_0 = 1.5$ and with error less than 0.001.

n	p_n	Upper bound of the error
0	1.500	—
1	2.296	0.1838 > 0.001
2	2.083	0.03445 > 0.001
3	2.111	0.006458 > 0.001
4	2.106	0.001211 > 0.001
5	2.107	0.0002270 < 0.001

$$p_{n+1} = \frac{1}{p_n^3} + 2$$

$$|p - p_n| \le \frac{k^n}{1-k} |p_1 - p_0|$$

we can use the idea of $\boxed{5}$ below here too, by finding n first

see page 36

we can use $|p_n - p_{n-1}|$ as stopping criteria

- $p_1 = \frac{1}{p_0^3} + 2 = \frac{1}{(1.5)^3} + 2 = 2.296$

$p_2 = \frac{1}{p_1^3} + 2 = \frac{1}{(2.296)^3} + 2 = 2.083$

$\vdots$

- $E_1 \le \frac{k}{1-k} |p_1 - p_0| = \frac{0.1875}{1 - 0.1875} |2.296 - 1.500| = 0.1838$

$E_2 \le \frac{k^2}{1-k} |p_1 - p_0| = \frac{(0.1875)^2}{1 - 0.1875} |2.296 - 1.500| = 0.03445$

$\vdots$

$\boxed{5}$ Find number of iterations required to estimate the fixed point of $g(x)$ with accuracy of 10^{-5}.

$$|p - p_n| \le \frac{k^n}{1-k} |p_1 - p_0| < 10^{-5}$$

$$\frac{(0.1875)^n}{1 - 0.1875} |2.296 - 1.500| < 10^{-5}$$

$$\frac{(0.1875)^n}{0.8125} (0.7960) < 10^{-5}$$

$$(0.1875)^n (0.9797) < 10^{-5}$$

$$(0.1875)^n < 1.021 \times 10^{-5} = 0.00001021 \iff \boxed{n = 7}$$

① Show $\exists$ unique FP.

Exp ② Find the number of iterations needed to estimate the fixed point of $g(x) = \cos x$ with error of magnitude less than 10^{-3} (use $P_0 = 0.5$ and 4 digits)

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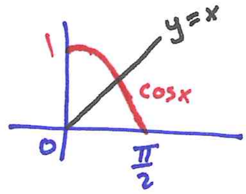
①

- g is continuous on $[0,1]$

  g is decreasing on $[0,1]$



$0.9998 = g(1) \leq g(x) \leq g(0) = 1 \leq 1 \quad \forall x \in [0,1]$

Hence, $g(x) \in [0,1] \quad \forall x \in [0,1]$

Therefore, by FPTI g has a fixed point in $[0,1]$.

- $|g'(x)| = |-\sin x| = \sin x$ which is increasing on $(0,1)$

$$|g'(x)| \leq \sin 1 = 0.0175 = K_{max} < 1 \quad \forall x \in (0,1)$$

Hence, g has a unique fixed point in $[0,1]$.

- $P_1 = \cos P_0 = \cos(0.5) = 1 \Rightarrow |P_1 - P_0| = 0.5$

$1 - K = 1 - 0.0175 = 0.9825$

$$|P - P_n| \leq \frac{K^n}{1-K} |P_1 - P_0| < 10^{-3}$$

$$\frac{(0.0175)^n}{0.9825} (0.5) < 10^{-3} \Rightarrow$$

$$(0.0175)^n (0.5089) < 10^{-2} \Rightarrow (0.0175)^n < 1.965 \times 10^{-3}$$

$\Rightarrow (0.0175)^n < 0.001965 \Rightarrow \boxed{n = 2}$ since it gives 0.0003063