2.2 Bracketing Methods for Locating a Root

Def.
- Assume $f(x)$ is continuous function.
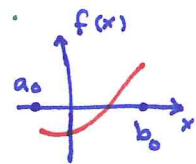- The number r is called a root of the equation $f(x) = 0$ iff $f(r) = 0$.   *or zero*

Exp $2x^2 + 3x - 2 = 0$ has two roots $r_1 = \frac{1}{2}$ and $r_2 = -2$

$x^2 + \frac{3}{2}x - 1 = 0 \iff (x - \frac{1}{2})(x + 2) = 0$

In this section we will learn two **Bracketing** methods for finding a zero of a continuous function $f(x) = 0$:
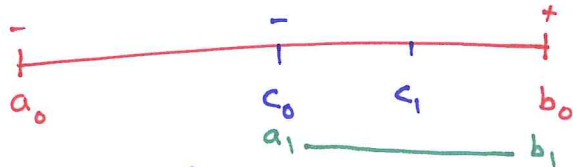
$\boxed{1}$ Bisection Method of Bolzano

$\boxed{2}$ False Position Method

Bisection Method of Bolzano

- This method used to solve $f(x) = 0$.
- This method depends on Bolzano Theorem.
- Conditions required
  - $f \in C[a, b]$ and
  - $f(a)\,f(b) < 0$

- Let $[a, b] = [a_0, b_0]$

- First Iteration $c_0 = \dfrac{a_0 + b_0}{2}$ is the midpoint

- Now find $f(c_0)$. We have three cases   (c_1 is 2nd iteration)

  If $f(c_0) = 0$, then c_0 is the root and we are done.

  If $f(c_0)\,f(a_0) < 0$, then $[a_1, b_1] = [a_0, c_0]$ and $c_1 = \dfrac{a_1 + b_1}{2} = \dfrac{a_0 + c_0}{2}$

  If $f(c_0)\,f(b_0) < 0$, then $[a_1, b_1] = [c_0, b_0]$ and $c_1 = \dfrac{a_1 + b_1}{2} = \dfrac{c_0 + b_0}{2}$

- In general, the $n+1^{th}$ iteration is $\boxed{c_n = \dfrac{a_n + b_n}{2}}$ 31

      in Bisection Method where $n = 0, 1, 2, \ldots$

Exp Use the Bisection Method to find the first five iterations c_0, c_1, c_2, c_3, c_4 that estimate the root of $X \sin X - 1 = 0$ on $[0, 2]$. "Use 4 digits"

- $f(x) = x \sin x - 1$ $[a_0, b_0] = [0, 2]$ 1st interval

- $f(0) = -1 < 0$ and $f(2) = 0.8180 > 0$

    1st iteration $c_0 = \dfrac{a_0 + b_0}{2} = \dfrac{0+2}{2} = 1$

- $f(c_0) = f(1) = -0.1585 < 0 \Rightarrow [a_1, b_1] = [1, 2]$ 2nd interval

    2nd iteration $c_1 = \dfrac{a_1 + b_1}{2} = \dfrac{1+2}{2} = 1.5$

- $f(c_1) = f(1.5) = 0.4962 > 0 \Rightarrow [a_2, b_2] = [1, 1.5]$ 3rd interval

    3rd iteration $c_2 = \dfrac{a_2 + b_2}{2} = \dfrac{1 + 1.5}{2} = 1.25$

- $f(c_2) = f(1.25) = 0.1862 > 0 \Rightarrow [a_3, b_3] = [1, 1.25]$ 4th interval

    4th iteration $c_3 = \dfrac{a_3 + b_3}{2} = \dfrac{1 + 1.25}{2} = 1.125$

- $f(c_3) = f(1.125) = 0.01505 > 0 \Rightarrow [a_4, b_4] = [1, 1.125]$ 5th interval

    5th iteration $c_4 = \dfrac{a_4 + b_4}{2} = \dfrac{1 + 1.125}{2} = 1.063$

- $f(c_4) = f(1.063) = -0.07183$

Note $\sin x$ is evaluated in radians $\Rightarrow$

    $f(2) = 2 \sin(114.6) - 1 = 2(0.9092) - 1$

      $= 1.818 - 1 = 0.8180$

    $\pi \longrightarrow 3.14$

    $? \rightarrow 2$ 180

    $? = 0.6369\pi$

      $= 114.6$

Th (Bisection Theorem)

- Assume that $f \in C[a,b]$,

  $\exists$ a number $r \in [a,b]$ s.t $f(r) = 0$,

  $f(a)f(b) < 0$ and $\{c_n\}_{n=0}^{\infty}$ represents the sequence

  of midpoints generated by the bisection method.

- Then **an upper bound of the error** is

$$|r - c_n| \leq \frac{b-a}{2^{n+1}} \quad \text{for} \quad n = 0, 1, 2, \ldots$$

- **Furthermore**, the sequence $\{c_n\}_{n=0}^{\infty}$ converges to the

  zero r. That is, $\displaystyle\lim_{n \to \infty} c_n = r$.

Proof

- Let $[a_0, b_0] = [a, b]$

  - Note that $b_1 - a_1 = \dfrac{b_0 - a_0}{2} = \dfrac{b-a}{2}$

$$b_2 - a_2 = \frac{b_1 - a_1}{2} = \frac{b-a}{2^2} \qquad \text{see Exp in page 31}$$
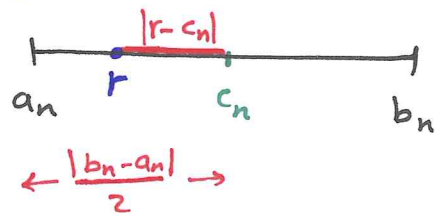
$$b_3 - a_3 = \frac{b_2 - a_2}{2} = \frac{b-a}{2^3}$$

$$\vdots$$

$$\boxed{b_n - a_n = \frac{b-a}{2^n}} \longrightarrow *$$

- Now since both the zero r and the midpoint $c_n = \dfrac{b_n - a_n}{2}$

  lie in the interval $[a_n, b_n]$ $\Rightarrow$

$$|r - c_n| \leq \frac{b_n - a_n}{2} \quad \text{for all } n$$

- **Using** $*$ we obtain

$$|r - c_n| \leq \frac{b-a}{2^{n+1}} \quad \text{for all } n$$

• To prove the second part, note that

$$0 \leq |r - c_n| \leq \frac{b-a}{2^{n+1}}$$

and $\displaystyle\lim_{n \to \infty} \frac{b-a}{2^{n+1}} = 0$. Hence, by Sandwich Theorem

$$\lim_{n \to \infty} |r - c_n| = 0 \iff \lim_{n \to \infty} (r - c_n) = 0$$

$$\iff r = \lim_{n \to \infty} c_n$$

Remark : The Bisection Theorem provides a strategy to find the number of iteration for a given accuracy δ :

$$\frac{b-a}{2^{n+1}} < \delta \implies \ln(b-a) - \ln 2^{n+1} < \ln \delta$$

$$\implies \ln(b-a) - \ln \delta < (n+1) \ln 2$$

$$\implies \frac{\ln(b-a) - \ln \delta}{\ln 2} < n+1$$

$$\implies \boxed{n > \frac{\ln\left(\frac{b-a}{\delta}\right)}{\ln 2} - 1}$$

Exp $e^x - \cos x - 1 = 0$ on $[0,1]$

n	a_n	c_n	b_n	$f(c_n)$
0	0 −	0.5 −	1 +	−0.3512
1	0.5 −	0.75 +	1 +	0.1171
2	0.5 −	0.625 −	0.75 +	−0.1317
3	0.625 −	0.6875 −	0.75 +	−0.01119
4	0.6875 −	0.7186 +	0.75 +	0.05164

- $f(x) = e^x - \cos x - 1$
- $f(0) = 1 - 1 - 1 = -1$

$f(1) = e - \cos 1 - 1 = 0.7184$

$f(0.5) = e^{0.5} - \cos(0.5) - 1 = -0.3512$

> To see the speed:
> $f(c_n) \longrightarrow 0$ or
> $\{a_n - b_n\} \longrightarrow 0$
> Here speed is slow

Exp Suppose that the Bisection method is used to find a zero of $f(x)$ in the interval $[2,7]$. How many times must this interval be bisected to guarantee that the approximation c_n has an accuracy of 5×10^{-9}.

- $[a,b] = [2,7]$ and $\delta = 5 \times 10^{-9}$

- Apply the derived inequality:

$$n > \frac{\ln\left(\frac{b-a}{\delta}\right)}{\ln 2} - 1 = \frac{\ln\left(\frac{7-2}{5\times10^{-9}}\right)}{\ln 2} - 1 = \frac{9\ln 10}{\ln 2} - 1$$

$$= \frac{20.72}{0.6931} - 1 = 29.89 - 1$$

$$= 28.89$$

- Hence $n \in \{29, 30, 31, \cdots\}$