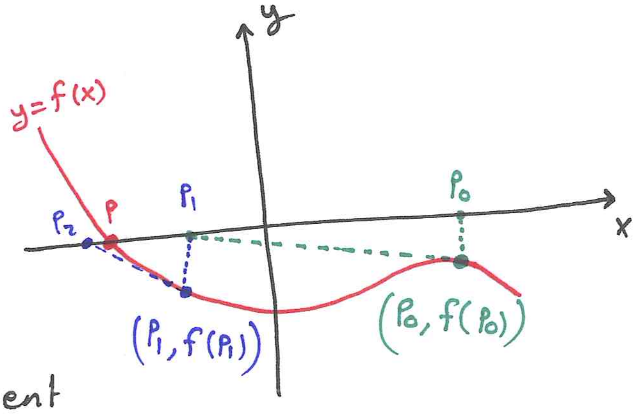
2.4 Newton – Raphson Method

- or simply " Newton's Method "
- This method develops algorithm that produces a sequence $\{p_n\}$ converges to the root p faster than Bisection and False Position methods (The best known method).
- Conditions to find the root of $f(x)=0$: $f \in C^2[a,b]$ with $f'(x) \neq 0$



- Assume the initial approximation p_0 is near the root p
- Next approximation p_1 is the point intersection between the x-axis and the line tangent to the curve at $(p_0, f(p_0))$:

$$f'(p_0) = m = \frac{0 - f(p_0)}{p_1 - p_0}$$

Hence,

$$\boxed{p_1 = p_0 - \frac{f(p_0)}{f'(p_0)}}$$

- The process is repeated to obtain a sequence $\{p_n\}$ that converges to p. That is Newton's method iteration:

$$\boxed{p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}} \qquad n = 0, 1, 2, \ldots$$

Exp Use Newton's Method to solve $x^2 = \sin x + 1$ using $P_0 = 1.5$ with accuracy 10^{-3}.

- $f(x) = x^2 - \sin x - 1 \Rightarrow f'(x) = 2x - \cos x$

- $$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)} = P_n - \frac{P_n^2 - \sin P_n - 1}{2P_n - \cos P_n}$$

n	P_n	$\lvert P_n - P_{n-1}\rvert$
0	1.5	—
1	1.413799126	0.0862 > 0.001
2	1.409633752	0.00416 > 0.001
3	1.409624004	0.00001 < 0.001

Stope

180 → 3.14
85.987 → 1.5
for sin and cos

Exp Use Newton's Method with $P_0 = 1$ estimate the root of $f(x) = e^x - \cos x - 1$ with error $< 10^{-4}$

- $P_0 = 1 \Rightarrow P_{n+1} = P_n - \dfrac{e^{P_n} - \cos P_n - 1}{e^{P_n} + \sin P_n}$

- $P_1 = 0.66908398 \Rightarrow |P_1 - P_0| > 10^{-4}$

- $P_2 = 0.603760843 \Rightarrow |P_2 - P_1| > 10^{-4}$

- $P_3 = 0.601349991 \Rightarrow |P_3 - P_2| > 10^{-4}$

- $P_4 = 0.601346767 \Rightarrow |P_4 - P_3| < 10^{-4}$ stope

180 → 3.14
57.325 → 1
for sin and cos ...

Exp Use Newton's method to estimate $\sqrt{5}$ starting with $P_0 = 2$

- Let $x = \sqrt{5} \Rightarrow x^2 = 5 \Rightarrow x^2 - 5 = 0 \Rightarrow f(x) = x^2 - 5$
  $\Rightarrow f'(x) = 2x$

- Hence, $P_{n+1} = P_n - \dfrac{f(P_n)}{f'(P_n)}$

$$= P_n - \frac{P_n^2 - 5}{2P_n}$$

$$= \frac{P_n - \frac{5}{P_n}}{2}$$

- $P_1 = \dfrac{P_0 - \frac{5}{P_0}}{2} = \dfrac{2 + \frac{5}{2}}{2} = 2.25$

- $P_2 = \dfrac{2.25 + \frac{5}{2.25}}{2} = 2.236111111$

- $P_3 = \dfrac{2.236111111 + 5/2.236111111}{2} = 2.236067978$

- $P_4 = \dfrac{2.236067978 + 5/2.236067978}{2} = 2.236067978$

- Note that all $\{P_n\}$ with $n > 4$ will give same result as in P_4, so we see the convergence accurate to 9 decimal places.

Exp Estimate $\sqrt[3]{15}$ $\Rightarrow x = \sqrt[3]{15} \Rightarrow x^3 - 15 = 0 \Rightarrow P_{n+1} = P_n - \dfrac{P_n^3 - 15}{3P_n^2}$

- $P_0 = 2$

$P_1 = 2.\overset{5}{\cancel{8}}\bar{3}$

$P_2 = 2.471441785$

$P_3 = 2.466223133$

$P_4 = 2.466212074$ as in calculator

طريقة نيوتن الأفضل والأسرع

Th (Newton-Raphson Theorem)

- Assume $f \in C^2[a,b]$ and $\exists$ a number $p \in [a,b]$ s.t $f(p)=0$.
- If $f'(p) \neq 0$, then $\exists$ $\delta > 0$ s.t the sequence $\{p_k\}_{k=0}^{\infty}$ defined by

$$* \quad \boxed{p_{k+1} = g(p_k) = p_k - \frac{f(p_k)}{f'(p_k)} \quad \text{for } k = 0,1,2,\ldots}$$

will converge to p for any initial approximation $p_0 \in [p-\delta, p+\delta]$, where $g(x) = x - \dfrac{f(x)}{f'(x)}$

Proof. Taylor polynomial of degree 1 about p_0 is

$$f(x) = f(p_0) + f'(p_0)(x-p_0)$$

- Substitute $x = p$ and note that $f(p)=0 \Rightarrow$

$$0 = f(p_0) + f'(p_0)(p-p_0)$$

- Solve for $p \Rightarrow p = p_0 - \dfrac{f(p_0)}{f'(p_0)} \approx p_1$
- This is used to define the next approximation p_1 and so $*$ is established.
- To prove the convergence: Note that $g(p) = p - \dfrac{f(p)}{f'(p)} = p$ so p is fixed point of g.
- $g'(x) = 1 - \dfrac{f'f' - ff''}{(f'(x))^2} = \dfrac{ff''}{(f')^2} \Rightarrow g'(p) = 0 < 1$ and $g'(x)$ is continuous

Hence, $\exists$ $\delta > 0$ s.t $|g'(x)| < 1$ on $(p-\delta, p+\delta)$ by Th FPIT II page 21.

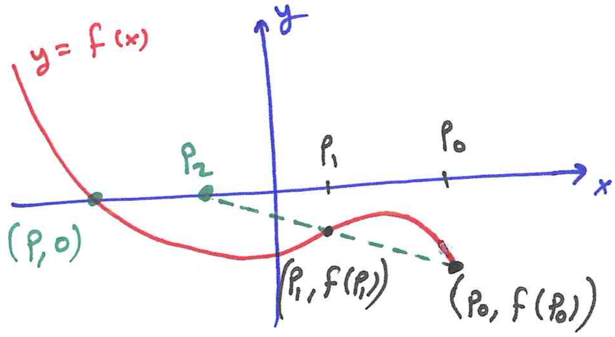
Secant Method

- Recall that in Newton-Raphson method, it is required the evaluation of $f(p_n)$ and $f'(p_n)$ per iteration since

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} \quad \text{for } n = 0, 1, 2, \dots$$

- It is desirable to have a method "secant method" that converges almost as fast as Newton's method and involves only evaluations of f and not f'.

$\Rightarrow$ Given $(p_0, f(p_0))$
$(p_1, f(p_1))$



$\Rightarrow$ To find p_2:

$$\frac{f(p_1) - f(p_0)}{p_1 - p_0} = m = \frac{0 - f(p_1)}{p_2 - p_1}$$

$\Rightarrow$ solve for $p_2 \Rightarrow \quad p_2 = p_1 - \left(\frac{p_1 - p_0}{f(p_1) - f(p_0)}\right) f(p_1)$

$\Rightarrow$ In general: $\boxed{p_{n+2} = p_{n+1} - \left(\frac{p_{n+1} - p_n}{f(p_{n+1}) - f(p_n)}\right) f(p_{n+1})}$

Exp Consider the equation $x = \cos x$. Take $p_0 = 0.5$ and $p_1 = \frac{\pi}{4}$ to Find the next iteration "p_2" using secant method to approximate the solution of $x = \cos x$.

$3.14 \to \pi = 180$
$\frac{1}{2} \to ? = 28.7$

- $f(x) = x - \cos x$, $p_1 = \frac{\pi}{4} = 0.785$, $f(\frac{\pi}{4}) = \frac{\pi}{4} - \cos\frac{\pi}{4} = 0.785 - 0.707 = 0.078$

$f(\frac{1}{2}) = \frac{1}{2} - \cos(28.7) = {}^{-}0.377$

- $p_2 = p_1 - \left(\frac{p_1 - p_0}{f(p_1) - f(p_0)}\right) f(p_1) = \frac{\pi}{4} - \left(\frac{\frac{\pi}{4} - \frac{1}{2}}{f(\frac{\pi}{4}) - f(\frac{1}{2})}\right) f(\frac{\pi}{4}) = 0.73638414$

Def (Multiplicity of Roots)

- Assume $f, f', \dots, f^{(M)}$ are defined and continuous on interval about the root p, where $M \in \mathbb{Z}^+$.
- We say $f(x)=0$ has a root of order M at $x=p$ (or p has multiplicity M) iff

$$f(p)=0,\ f'(p)=0,\ f''(p)=0,\ \dots,\ f^{(M-1)}(p)=0,\ f^{(M)}(p)\neq 0$$

Def
- A root p of order $M=1$ is called simple root.
- A root p of order $M>1$ is called multiple root.
  - ↳ if $M=2$, then p is called double root.
  - ↳ if $M=3$, then p is called cubic root.
  - ⋮

Exp Find the roots of $f(x)$ and their multiplicity

1) $f(x) = x^3 - 3x + 2$

- one can write $f(x) = (x+2)(x-1)^2$ so $p=-2$, $p=1$ roots
- $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$

$\boxed{p=1}$ $\Rightarrow f(1)=0,\ f'(1)=0,\ f''(1)=6\neq 0$ so $M=2$ and $p=1$ is double root.

$\boxed{p=-2}$ $\Rightarrow f(-2)=0,\ f'(-2)=9\neq 0$ so $M=1$ and $p=-2$ is simple root.

$\boxed{2}$ $f(x) = (x-1)\ln x$

- $p = 1$ is the only root.
- $f'(x) = \dfrac{x-1}{x} + \ln x$ and $f''(x) = \dfrac{1}{x^2} + \dfrac{1}{x}$
- $f(1) = 0$, $f'(1) = 0$, $f''(1) = 2 \neq 0$

so The multiplicity of $p=1$ is 2 and its double root.

Lemma If $f(x)=0$ has a root p and $\exists$ a continuous function $h(x)$ s.t

$f(x) = (x-p)^M h(x)$ where $h(p) \neq 0$ then the root p has multiplicity M

Remark • In Exp $\boxed{1}$ page 41 $\Rightarrow$ $p_1 = 1$ has $M_1 = 2$ and $p_2 = -2$ has $M_2 = 1$ so

$$f(x) = (x+2)(x-1)^2$$ with $h_1(x) = x+2$, $h_1(1) \neq 0$

$h_2(x) = (x-1)^2$, $h_2(-2) \neq 0$

• In Exp $\boxed{2}$ page 42 $\Rightarrow$ $p=1$ has $M = 2$

$f(x) = (x-1)\ln x$ but $\ln x = (x-1) - \dfrac{(x-1)^2}{2} + \dfrac{(x-1)^3}{3} - \cdots$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n}$$ $$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^{n}}{n}$$

with $h(1) = 0$

$$= (x-1)^2 - \frac{(x-1)^3}{2} + \frac{(x-1)^4}{3} + \cdots$$

$\boxed{3}$ $f(x) = x^{101} - x^{100} + x^{30} - 1$ $\Rightarrow$ $p=1$ is root

$f'(x) = 101x^{100} - 100x^{99} + 30x^{29}$ $\Rightarrow$ $f'(1) = 31 \neq 0$

$\Rightarrow$ $p=1$ has $M=1$

Def (Speed of Convergence)

- Assume that the sequence $\{P_n\}_{n=0}^{\infty}$ converges to p.
- If there exist two positive constants $A \neq 0$ and $R > 0$ s.t

$$\lim_{n\to\infty} \frac{|p - P_{n+1}|}{|p - P_n|^R} = \lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^R} = A ,$$

Then we say that $\{P_n\}$ converges to p with order of convergence R.

Remarks ① We use the order of convergence R to measure the speed of convergence of any method:

- if $R=1$, then the convergence of $\{P_n\}$ is linear
- if $R=\frac{3}{2}$, then the convergence of $\{P_n\}$ is super linear
- if $R=2$, then the convergence of $\{P_n\}$ is quadratic
- if $R=3$, then the convergence of $\{P_n\}$ is cubic

② when $R\uparrow \Rightarrow$ speed $\uparrow \Rightarrow$ error $\downarrow$

③ A is called the asymptotic error constant

This is because • as n gets Large $\Rightarrow$

$$|E_{n+1}| \approx A\,|E_n|^R$$

- if $|E_n| = 0.01$ then for

$$R=1 \Rightarrow |E_{n+1}| \approx A|E_n| = A\,(0.01)$$

$$R=2 \Rightarrow |E_{n+1}| \approx A|E_n|^2 = A(0.01)^2 = A(0.0001)$$

Exp Find A and R for the following sequences:

[1] $\left\{\frac{1}{10^n}\right\}_{n=0}^{\infty} = 1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots$

- $\lim_{n\to\infty} \frac{1}{10^n} = 0 = p$

- $$\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^R} = \lim_{n\to\infty} \frac{|p - p_{n+1}|}{|p - p_n|^R} = \lim_{n\to\infty} \frac{\left|0 - \frac{1}{10^{n+1}}\right|}{\left|0 - \frac{1}{10^n}\right|^R}$$

$$= \lim_{n\to\infty} \frac{10^{nR}}{10^{n+1}} = \begin{cases} \boxed{\frac{1}{10} \quad \text{if } R = 1} \\ \infty \quad \text{if } R > 1 \quad \times \\ 0 \quad \text{if } R < 1 \quad \times \end{cases}$$

$$= \frac{1}{10} \lim_{n\to\infty} 10^{n(R-1)}$$

- Hence, by definition $\Rightarrow$ $A = \frac{1}{10}$ and $R = 1$

$\Rightarrow$ The convergence is linear

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[2] $p_n = \left\{\frac{1}{2^n}\right\}_{n=0}^{\infty} \Rightarrow \lim_{n\to\infty} \frac{1}{2^n} = 0 = p$

- $$\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^R} = \lim_{n\to\infty} \frac{\left|0 - \frac{1}{2^{n+1}}\right|}{\left|0 - \frac{1}{2^n}\right|^R} = \lim_{n\to\infty} \frac{2^{nR}}{2^{n+1}}$$

$$= \frac{1}{2} \lim_{n\to\infty} 2^{n(R-1)} = \begin{cases} \boxed{\frac{1}{2} \quad \text{if } R = 1} \\ \infty \quad \text{if } R > 1 \\ 0 \quad \text{if } R < 1 \end{cases}$$

- Hence, $A = \frac{1}{2}$ and $R = 1$

and the convergence is linear

Th (Speed of Newton's Method)

Assume Newton-Raphson iteration

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)} \quad , \text{ given } P_0 \; , \; n = 0, 1, 2, \cdots$$

produces a sequence $\{P_n\}$ that converges to the root p of the function $f(x)$. Then,

1. if p is simple root $(M=1)$, then Newton's iteration $\{P_n\}$ converges to p quadratically $(R=2)$ with

$$A = \left| \frac{f''(p)}{2f'(p)} \right| \quad \text{and} \quad \lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^2} \approx A$$

2. if p is a multiple root (of order $M > 1$), then Newton's iteration $\{P_n\}$ converges to p linearly $(R=1)$ with

$$A = \frac{M-1}{M} \quad \text{and} \quad \lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|} \approx \frac{M-1}{M}$$

Exp Let $f(x) = x^3 - 3x + 2$

1. Find the order of convergence R and the asymptotic error constant A when Newton-Raphson iteration is used to find the roots of $f(x) = 0$

- Recall the roots of $f(x) = 0 \Rightarrow P = -2 \; , \; P = 1$

- Note that $P = -2$ is Simple root $(M=1) \Rightarrow$ so $R=2$ by Th above and hence, $\boxed{A = \left| \frac{f''(-2)}{2f'(-2)} \right| = \left| \frac{-12}{2(9)} \right| = \frac{2}{3}}$

- Note that $P = 1$ is multiple root $(M=2) \Rightarrow$ so $R = 1$ by Th above and hence, $\boxed{A = \frac{M-1}{M} = \frac{2-1}{2} = \frac{1}{2}}$

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$\boxed{2}$ start with $P_0 = -2.4$ and use Newton's – Raphso iteration to find the root $p = -2$. (Prove the quadratic convergence at simple root in $\boxed{1}$).

$$P = -2 \quad , \quad P_0 = -2.4$$

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)} = P_n - \frac{P_n^3 - 3P_n + 2}{3P_n^2 - 3}$$

$$= \frac{2P_n^3 - 2}{3P_n^2 - 3}$$

n	P_n	$P_{n+1} - P_n$ (no need)	$E_n = \|P - P_n\|$	$\|E_{n+1}\| / \|E_n\|^2$
0	-2.4	0.323809524	0.4	0.476190475
1	-2.076190476	0.072594465	0.076190476	0.619469086
2	-2.003596011	0.003587422	0.003596011	0.664202613 $\approx \frac{2}{3}$
3	-2.000008589	0.000008589	0.000008589	
4	-2	0	0	

- Note that $|E_{n+1}| \approx A\,|E_n|^2$ for larg n
- To check this $\Rightarrow$

$$|E_3| = |P - P_3| = 0.000008589$$

$$|E_2| = |P - P_2| = 0.003596011 \Rightarrow |E_2|^2 = 0.000012931$$

- Now it is easy to see that

$$|E_3| \approx A\,|E_2|^2 \iff 0.000008589 \approx \frac{2}{3}(0.000012931)$$

$$= 0.000008621$$

3 Start with $p_0 = 1.2$ and use Newton's Method to prove the linear convergence at the double root $p = 1$.

$$p = 1 \quad , \quad p_0 = 1.2$$

$$p_{n+1} = \frac{2p_n^3 - 2}{3p_n^2 - 3}$$

n	p_n	$E_n = \lvert p - p_n \rvert$	$\lvert E_{n+1}\rvert / \lvert E_n \rvert$
0	1.2	0.2	0.515151515
1	1.103030303	0.103030303	0.508165253
2	1.052356420	0.052356420	0.496751115
3	1.026400811	0.026400811	0.509753688
4	1.013257730	0.013257730	0.501097775
5	1.006643419	0.006643419	0.500550093 ≈ 0.5
⋮	1.003325375	0.003325375	
20	1.000000409		

- Note that $|E_{n+1}| \approx A\,|E_n|$ for larg n
- To check this $\Rightarrow$

$$|E_5| = |p - p_5| = 0.006643419$$

$$|E_4| = |p - p_4| = 0.1325 7730$$

- Now it is easy to see that

$$|E_5| \approx A\,|E_4| \iff 0.006643419 \approx (0.5)(0.13257730)$$

$$= 0.06628865$$

Remark. In the previous Exp the root p was know.

- However, sometimes p is unknow (see next Exp).

Exp (P is unknown)

Consider the equation $x^2 - \sin x - 1 = 0$

① Use Newton's method with $P_0 = 1.5$ to estimate the solution of this equation with error less than 10^{-3}.

$3.14 \to 180$
$1.5 \to 85.987$

n	P_n	$P_{n+1} - P_n$
0	1.5	—
1	1.413799126	0.086
2	1.409633752	0.004
3	1.409624004	0.000009748

$f(x) = x^2 - \sin x - 1$

$f'(x) = 2x - \cos x$

$P_{n+1} = P_n - \dfrac{f(P_n)}{f'(P_n)}$

$\boxed{P = 1.409624004}$

② Find the order of convergence and the asymptotic error constant

- We find the multiplicity of the root p
- $f'(p) = 2(1.409624004) - \cos(80.81)$

$= 2.819248008 - 0.1597088975$

$= 2.6595391033 \neq 0$

$\Rightarrow M = 1$ and $p = 1.409624004$ is simple root.

- Hence, by Th above $R = 2$ and $A = \left| \dfrac{f''(p)}{2f'(p)} \right| = 0.56173286$

$\boxed{3}$ Prove part $\boxed{2}$ Numerically

n	P_n	$E_n = \lvert P - P_n \rvert$	$\lvert E_{n+1} \rvert / \lvert E_n \rvert^2$
0	1.5	0.090375	0.5111162
1	1.413799126	0.004175	0.559212 $\approx A$
2	1.409624004	0.0000009748	

- $|E_2| = 0.0000009748$

  $|E_1| = 0.004175 \Rightarrow |E_1|^2 = 0.0000174306$

- Note that $|E_2| \approx A\,|E_1|^2 \Leftrightarrow$

$$0.0000009748 \approx (0.56173286)(0.0000174306)$$

$$= 0.0000097913$$

Th (Accelerated Newton-Raphson Iteration)

- Assume Newton - Raphson iteration

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}, \text{ given } P_0, \ n=0,1,2,\cdots$$

produces a sequence $\{P_n\}$ that converges to the root p.

- Assume p is a multiple root (of order $M>1$). Then

✓ $\boxed{1}$ " by Th page 45 $\Rightarrow$ Newton's iteration $\{P_n\}$ converges linearly ($R=1$)
Done
with $A = \frac{M-1}{M}$ and $\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|} = A$ "

$\boxed{2}$ the modification of Newton's iteration

$$P_{n+1} = P_n - \frac{M \, f(P_n)}{f'(P_n)}, \text{ given } P_0, \ n=0,1,2,\cdots$$

converges quadratically ($R=2$) to p and $A = \lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^2}$

Exp $f(x) = x^3 - 3x + 2$. Estimate $p=1$ using accelerated newton method with $P_0 = 1.2$

- Note that $p=1$ has multiplicity $M=2$ since $f(1) = f'(1) = 0$ but $f''(1) = 6 \neq 0$
- Acceleration formular: $P_{n+1} = P_n - \frac{2 f(P_n)}{f'(P_n)} = \frac{P_n^3 + 3P_n - 4}{3P_n^2 - 3}$

n	P_n	$E_n = \lvert p - P_n \rvert$	$\lvert E_{n+1}\rvert / \lvert E_n\rvert^2$
0	1.2	0.2	0.151515150
1	1.006060606	0.006060606	0.165718578 $\approx A$
2	1.000006087	0.000006087	
3	1	0	

Speed of Convergence for Secant Method

$$P_{n+2} = P_{n+1} - \left(\frac{P_{n+1} - P_n}{f(P_{n+1}) - f(P_n)} \right) f(P_{n+1}) \qquad \text{given } P_0, P_1$$

1. if p is simple root $(M=1)$, then secant's iteration $\{P_n\}$ converges to p with

$$R = 1.618 \quad \text{and} \quad \lim_{n \to \infty} \frac{|E_{n+1}|}{|E_n|^{1.618}} = A = \left| \frac{f''(p)}{2 f'(p)} \right|^{0.618}$$

2. if p is multiple root (of order $M>1$), then secant's iteration converges to p with

$$R = 1 \quad \text{and} \quad A \text{ depends on } f(x)$$

Exp Start with $P_0 = -2.6$ and $P_1 = -2.4$ and use the secant method to

1. find the root $p=-2$ of $f(x) = x^3 - 3x + 2$
2. find the order of convergence R for $p=-2$
3. find the asymptotic error constant A for $p=-2$
4. Prove part 3 numerically.

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2. Recall that $p=-2$ is simple root since $f(-2)=0$ but $f'(-2) = 9 \neq 0$. Hence, $R = 1.618$

3. $$A = \left| \frac{f''(-2)}{2 f'(-2)} \right|^{0.618} = \left(\frac{2}{3}\right)^{0.618} = 0.778351205$$

n	p_n	$E_n = \|p - p_n\|$	$\|E_{n+1}\| / \|E_n\|^{1.618}$
0	-2.6	0.6	0.914152831
1	-2.4	0.4	0.469497765
2	-2.106598985	0.106598985	0.847290012
3	-2.022641412	0.022641412	0.693608922
4	-2.001511098	0.001511098	0.825841116
5	-2.000022537	0.000022537	0.727100987 $\approx A$
6	-2.000000022	0.000000022	
7	-2	0	

$1.618 \approx \frac{1+\sqrt{5}}{2}$

- This exp shows the convergence of the secant method at simple root $p = -2$

- Note that $E_5 = |p - p_5| = 0.000022537$

$$E_4^{1.618} = |p - p_4|^{1.618} = (0.001511098)^{1.618} = 0.000027296$$

- It is easy to check that $|E_5| \approx A\, |E_4|^{1.618}$ $\Leftarrow\!\!\Rightarrow$

$$0.000022537 \approx (0.778351205)(0.000027296)$$

$$= 0.0000212459$$

- Speed of Convergence for Bisection Method: $R=1$ and $A = \frac{1}{2}$ $\quad \frac{|E_{n+1}|}{|E_n|} \approx \frac{1}{2}$

- Speed of Convergence for False Position Method: $R=1$ and A depends on $f(x)$ $\Rightarrow \frac{|E_{n+1}|}{|E_n|} \approx A$