Solving n×n Linear and Nonlinear Systems

Linear System: $\underset{n\times n}{A}\ \underset{n\times 1}{X} = \underset{n\times 1}{b}$ which is equivalent to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

Cost (Complexity): is the number of operations $+, -, \times, \div$ required to complete a certain calculation.

Exp[1] Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Find the cost of finding $|A| = \det(A)$

$$|A| = a\times d - b\times c \quad \Rightarrow \quad \text{Cost} = 3$$

Exp[2] Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Find the cost of calculating $|A|$.

$$|A| = a\times \underbrace{\begin{vmatrix} e & f \\ h & i \end{vmatrix}}_{\text{cost}=3} - b\times \underbrace{\begin{vmatrix} d & f \\ g & i \end{vmatrix}}_{\text{cost}=3} + c\times \underbrace{\begin{vmatrix} d & e \\ g & h \end{vmatrix}}_{\text{cost}=3} \quad \text{by exp}^1$$

Cost $= 14$

Remark: My student proved that the cost of finding $|A_{n\times n}|$ for $n \geq 2$ is ① $\text{cost} = n!\sum_{k=2}^{n} \frac{2k-1}{k!}$ or

② $\text{cost} = [n!\, e - 2]$ where $[\ ]$ is the greatest integer function and $e \approx 2.718$

Exp[3] Let A be 3×3 matrix.
Find the cost of calculating A^2.

$$A^2 = AA = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$= \begin{bmatrix} a\times a + b\times d + c\times g & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & & \end{bmatrix}$$

Cost $=(9)(5) = 45$

Exp[4] Let A and B be 3×3 matrices.
Find the cost of $A + |B|\,B$

$|B|$ requires cost $= 14$ by Exp[2]

$|B|\times B$ requires cost $= 9$

$A + |B|B$ requires cost $= 9$

Total Cost $= 32$

Result: If A is $n\times n$ matrix, then the cost of calculating A^2 is $2n^3 - n^2$ Check!

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see Exp[3] $\Rightarrow n=3 \Rightarrow 2(3)^3 - (3)^2 = 2(27) - 9$

$= 54 - 9$

$= 45$

Exercise show that $n! \sum_{k=2}^{n} \frac{2k-1}{k!} = [n!\, e - 2]$ 53.1

where $[\,]$ is the greatest integer function

Proof:

$$n! \sum_{k=2}^{n} \frac{2k-1}{k!} = n! \sum_{k=2}^{n} \left(\frac{2k}{k!} - \frac{1}{k!} \right)$$

$k! = k(k-1)!$

$$= n! \left[2 \sum_{k=2}^{n} \frac{1}{(k-1)!} - \sum_{k=2}^{n} \frac{1}{k!} \right]$$

$$= n! \left[2 \left(1 + \sum_{k=3}^{n} \frac{1}{(k-1)!} \right) - \sum_{k=2}^{n} \frac{1}{k!} + \sum_{k=2}^{n} \frac{1}{k!} - \sum_{k=2}^{n} \frac{1}{k!} \right]$$

$$= n! \left[2 + \sum_{k=2}^{n} \frac{1}{k!} + 2 \left(\underbrace{\sum_{k=3}^{n} \frac{1}{(k-1)!}}_{\text{shifting index}} - \sum_{k=2}^{n} \frac{1}{k!} \right) \right]$$

Note that $\sum_{k=2}^{n-1} \frac{1}{k!} - \sum_{k=2}^{n} \frac{1}{k!} = 0 + 0 + \cdots + 0 - \frac{1}{n!} = \frac{-1}{n!}$

$$= n! \left[\underbrace{2 + \sum_{k=2}^{n} \frac{1}{k!}}_{\sum_{k=0}^{n} \frac{1}{k!}} - \frac{2}{n!} \right]$$

$$= n! \left[\sum_{k=0}^{n} \frac{1}{k!} - \frac{2}{n!} \right]$$

$$= n! \sum_{k=0}^{n} \frac{1}{k!} - 2$$

- But $e^x = \sum_{n=0}^{\infty} \frac{x^2}{n!}$ so $e = \sum_{k=0}^{\infty} \frac{1}{k!}$

- Hence, $n!\, e = n! \sum_{k=0}^{n} \frac{1}{k!} + \boxed{n! \sum_{k=n+1}^{\infty} \frac{1}{k!}}$

- That is, $n! \sum_{k=0}^{n} \frac{1}{k!} = n!\,e - n! \sum_{k=n+1}^{\infty} \frac{1}{k!}$

- Now $n! \sum_{k=2}^{n} \frac{2k-1}{k!} = n! \sum_{k=0}^{n} \frac{1}{k!} - 2$

$$= n!\,e - 2 - \boxed{n! \sum_{k=n+1}^{\infty} \frac{1}{k!}}$$

Assume this value is R

- Note that R represents the error of calculating $n!\,e$ which is always less than one " since k starts at $n+1$ and we have $n \geq 2$ ".

- So we can get rid of R by taking the floor function or greatest integer number.

$\lfloor 2.9 \rfloor = 2$
$\lfloor 2.5 \rfloor = 2$
$\lfloor 2.1 \rfloor = 2$
$\lfloor -2.9 \rfloor = -3$
$\lfloor -2.1 \rfloor = -3$

- That is, $n! \sum_{k=2}^{n} \frac{2k-1}{k!} = \lfloor n!\,e - 2 \rfloor$