3.3 Backward Substitution

Backward substitution method used to solve a linear system of equations that has an upper-triangular coefficient matrix:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
\ddots \qquad & \vdots \\
a_{nn}x_n &= b_n
\end{aligned}
$$

$$
\left[\begin{array}{ccccc|c}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\
0 & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\
0 & 0 & a_{33} & \cdots & a_{3n} & b_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a_{nn} & b_n
\end{array}\right] = [U \mid b]
$$

U: coefficient matrix

Exp Solve the following linear system using Backward Substitution and find the cost.

$$
\begin{aligned}
4x_1 - x_2 + 2x_3 + 3x_4 &= 20 \\
-2x_2 + 7x_3 - 4x_4 &= -7 \\
6x_3 + 5x_4 &= 4 \\
3x_4 &= 6
\end{aligned}
$$

step 1 : $x_4 = \frac{6}{3} = 2$ $\Rightarrow$ one operation

$-,+$	0
$\div,\times$	1

step 2 : $x_3 = \frac{4 - 5\times 2}{6} = -1$ $\Rightarrow$ Three operations

$-,+$	1
$\div,\times$	2

step 3 : $x_2 = \frac{-7\times -1 + 4\times 2 - 7}{-2} = -4$ $\Rightarrow$ Five operations

$-,+$	2
$\div,\times$	3

step 4 : $x_1 = \frac{1\times -4 - 2\times -1 - 3\times 2 + 20}{4} = 3$ $\Rightarrow$ Seven operations

$-,+$	3
$\div,\times$	4

Hence, total cost $= 16 = (4)^2 = n^2$

Cost of Backward Substitution (B.S.) for solving $n\times n$ linear system

step	$+,-$	$\times,\div$
1	0	1
2	1	2
3	2	3
4	3	4
$\vdots$	$\vdots$	$\vdots$
n	$n-1$	n

Total $+,-$ is $0 + 1 + 2 + 3 + \cdots + n-1 = \frac{n(n-1)}{2} = \frac{n^2-n}{2}$

Total $\times,\div$ is $1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$

Hence, total cost $= \frac{n^2-n}{2} + \frac{n^2+n}{2} = n^2$

Remark

$$1+2+3+\cdots+n=\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$1+2^2+3^2+\cdots+n^2=\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Five Methods to solve the linear system $AX=b$

1 Gaussian Elimination:

$$\underset{\text{Augmented matrix}}{(A|b)} \rightarrow \underset{\text{Upper matrix}}{(U|c)}$$ solve by B.S.

2 Gauss - Jordan Reduction:

$$(A|b) \longrightarrow (I|X)$$

3 Inverse Method:

- Find A^{-1}:

$$(A|I) \rightarrow (I|A^{-1})$$

- Then $X = A^{-1}b$

4 Cramer's Rule:

- Find $|A| \neq 0$
- Then $x_1 = \frac{|A_1|}{|A|}$, $x_2 = \frac{|A_2|}{|A|}$, $\ldots$, $x_n = \frac{|A_n|}{|A|}$

5 LU Factorization :

- Write $A = LU$ where L is lower triangle matrix
  U is upper triangle matrix

- Let $Y = UX \Rightarrow$

$$AX = b \quad \text{becomes} \quad LUX = b$$
$$LY = b$$

- Now solve $LY = b$ by F.S and find Y

- Then solve $UX = Y$ by B.S and find X

Remark : The speed of these methods is like this

$$\boxed{4} < \boxed{3} < \boxed{2} < \boxed{1} = \boxed{5}$$
