1 Gaussian Elimination (G.E.)

The Augmented matrix is denoted by

$$[A|b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right] \quad \cdots \quad *$$

Th (Elementary Row Operations ERO)

The following operations applied to the augmented matrix * yield an equivalent linear system:

Row Operation I: Interchange two rows

Row Operation II: Multiply a row by a nonzero constant

Row operation III: The row R_K can be replaced by the sum of R_K and a nonzero multiple of any other row R_p. That is, $R_K = R_K - m_{kp} R_p$

where $m_{kp} = \dfrac{a_{kp}}{a_{pp}}$ is called kp^{th} multiplier

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Exp Consider the following linear system:

$$\begin{aligned} x_1 + 2x_2 + x_3 + 4x_4 &= 13 \\ 2x_1 \qquad + 4x_3 + 3x_4 &= 28 \\ 4x_1 + 2x_2 + 2x_3 + x_4 &= 20 \\ -3x_1 + x_2 + 3x_3 + 2x_4 &= 6 \end{aligned}$$

$\boxed{\text{i}}$ Express this system in augmented matrix form.

Pivot $\qquad$ Pivot row

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 2 & 0 & 4 & 3 & 28 \\ 4 & 2 & 2 & 1 & 20 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right) = (A \mid b)$$

$m_{21} = \frac{a_{21}}{a_{11}} = 2$

$m_{31} = \frac{a_{31}}{a_{11}} = 4$

$m_{41} = \frac{a_{41}}{a_{11}} = -3$

are the multipliers

or $(12)(2) = 24 + 3$ multiplies

entry $\qquad$ operations

$+,- : 1$

$\times : 1$

$\boxed{\text{ii}}$ Find an equivalent upper-triagular system using G.E.

• Step 1 • Find multipliers m_{21}, m_{31}, m_{41} $\rightarrow$

• Apply them

$$\begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \\ R_4 - 3R_1 \end{array} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & -6 & -2 & -15 & -32 \\ 0 & 7 & 6 & 14 & 45 \end{array}\right)$$

Costs

$\div : 3$

$+,- : 3(4)$

$\times : 3(4)$

• Step 2 • Find multipliers $m_{32} = \frac{-6}{-4} = 1.5$ and $m_{42} = \frac{7}{-4} = -1.75$

• Apply them

$$\begin{array}{l} \\ \\ R_3 - 1.5R_2 \\ R_4 + 1.75R_2 \end{array} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & 0 & -5 & -7.5 & -35 \\ 0 & 0 & 9.5 & 5.25 & 48.5 \end{array}\right)$$

$\div : 2$

$+,- : 2(3)$

$\times : 2(3)$

• step 3 • Find the multiplier $m_{43} = \frac{9.5}{-5} = -1.9$

• Apply it

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & 0 & -5 & -7.5 & -35 \\ 0 & 0 & 0 & -9 & -18 \end{array}\right)$$

$\div$: 1

$+,-$: 1 (2)

$\times$: 1 (2)

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iii Use B.S to solve the resulting system

$$x_4 = \frac{-18}{-9} = 2$$

$$x_3 = \frac{7.5(2) - 35}{-5} = 4$$

$$x_2 = \frac{-2(4) + 5(2) + 2}{-4} = -1$$

$$x_1 = 13 - 2(-1) - 4 - 4(2) = 3$$

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iv Find the cost of solving this system using G.E.

• Cost 1 : $(A|b) \longrightarrow (U|c)$ for this 4x4 system

Step	$+,-$	$\div, \times$
1	3 (4)	3, 3(4)
2	2(3)	2, 2(3)
3	1 (2)	1, 1(2)
Total	20	26

cost 1 = 46

• Cost 2 : Using B.S. $\Rightarrow$ cost 2 $= n^2 = 4^2 = 16$

Hence, total cost for 4x4 system = cost1 + cost2 = 62

Result ① The cost of reducing $n\times n$ linear system to an upper-triangular system using G.E. is

$$\frac{4n^3+3n^2-7n}{6}$$

② The cost of solving $n\times n$ linear system using G.E is

$$\frac{4n^3+9n^2-7n}{6}$$

Proof

① Cost of $(A|b)\longrightarrow(U|c)$ as we have seen in $\boxed{\text{Exp - iv}}$ page 61 is as follow:

step	$+,-$	$\div$	$\times$
1	$(n-1)\,n$	$n-1$	$(n-1)\,n$
2	$(n-2)(n-1)$	$n-2$	$(n-2)(n-1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	$(n-k)(n-k+1)$	$n-k$	$(n-k)(n-k+1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
Total	$\sum_{k=1}^{n-1}(n-k)(n-k+1)$	$\sum_{k=1}^{n-1}(n-k)$	$\sum_{k=1}^{n-1}(n-k)(n-k+1)$

Hence, total cost is $\sum_{k=1}^{n-1}\left[2(n-k)(n-k+1)+n-k\right]$

$$=\sum_{m=1}^{n-1}\left[2m(m+1)+m\right]=\sum_{m=1}^{n-1}2m^2+3m$$

$\boxed{m=n-k}$

$k=1\Rightarrow m=n-1$

$k=n-1\Rightarrow m=1$

$$=2\,\frac{n(n-1)(2n-1)}{6}+3\,\frac{n(n-1)}{2}=\frac{4n^3+3n^2-7n}{6}$$

② Cost of any $n\times n$ system using G.E is cost of $(A|b)\to(U|c)$ + cost of B.S.

$$=\frac{4n^3+3n^2-7n}{6}+n^2=\frac{4n^3+9n^2-7n}{6}$$

Gaussian Elimination and Pivoting

- If the pivot element $a_{pp} = 0$ in row p, then row p can not be used to eliminate the elements in column p below the main diagonal.
- The process of finding a row k with nonzero pivote element $a_{kp} \neq 0$, $k > p$ and interchang row p by row k is called pivoting.
- Pivoting is two types:

  1. Trivial pivoting:
     - if $a_{pp} \neq 0$, then do not switch rows
     - if $a_{pp} = 0$, then switch row p by the first $\underset{k}{\text{row}}$ below $\underset{p}{}$ s.t $a_{kp} \neq 0$.

Exp

$$\left[\begin{array}{ccc|c} 0 & -1 & 2 & 3 \\ 4 & 3 & -1 & 2 \\ 7 & 0 & -3 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 4 & 3 & -1 & 2 \\ 0 & -1 & 2 & 3 \\ 7 & 0 & -3 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & -1 & 2 & 3 \\ 0 & 2 & 4 & 7 \\ 1 & 2 & 3 & 4 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 7 \\ 0 & -1 & 2 & 3 \end{array}\right]$$

2 Partial Pivoting:

- if $a_{pp} = 0$ or $a_{pp} \neq 0$, choose the pivotal row k whose pivot a_{kp}, $k > p$ satisfy $|a_{kp}| = \max\{|a_{pp}|, |a_{p+1,p}|, \cdots, |a_{np}|\}$
- This will make all multipliers m_{rp}, $r = p+1, \cdots, n$ less than or equal to 1 in absolute value.
- And hence, reducing the error being propagated when using a finite-digit arithmetic

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Exp Consider the following linear system

$$1.133\,x_1 + 5.281\,x_2 = 6.414$$
$$24.14\,x_1 - 1.210\,x_2 = 22.93$$

whose solution is $(x_1, x_2) = (1, 1)$.

1 Use G.E. with trivial pivoting and use four-digit arithmetic to solve this system.

$$\begin{array}{r} \text{pivot} \rightarrow \\ R_2 - m_{21} R_1 \end{array} \left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 24.14 & -1.210 & 22.93 \end{array}\right]$$

since pivot $a_{11} \neq 0$ so we do not switch rows

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{24.14}{1.133} = 21.31$$

$$\left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 0 & -113.7 & -113.8 \end{array}\right]$$

$$x_2 = \frac{-113.8}{-113.7} = 1.001$$

$$x_1 = \frac{6.414 - 5.281(1.001)}{1.133} = 0.9956$$

Note that the error in the solution is due to the magnitude of the multiplier m_{21} which is $\gg 1$.

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2 Use G.E. with partial pivoting and use four-digit arithmetic to solve this system.

pivot
$$\left[\begin{array}{cc|c} 24.14 & -1.210 & 22.93 \\ 1.133 & 5.281 & 6.414 \end{array}\right]$$

- $\max\{1.133, 24.14\} = 24.14$
- Hence, pivot $= 24.14$

$$m_{21} = \frac{1.133}{24.14} = 0.04693 < 1$$

$$\left[\begin{array}{cc|c} 24.14 & -1.210 & 22.93 \\ 0 & 5.338 & 5.338 \end{array}\right]$$

$$x_2 = \frac{5.338}{5.338} = 1$$

$$x_1 = \frac{22.93 + 1.210(1)}{24.14} = 1$$

No Error

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