LU - Factorization

To solve the linear system $AX = b$:

- Write $A = LU$ where L is lower triangular matrix

  and U is upper triangular matrix

- Let $Y = UX$ $\Rightarrow$

  $AX = b$ becomes $LUX = b$

  $LY = b$

- Now solve $LY = b$ by F.S and find Y

- Then solve $UX = Y$ by B.S and find X

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* We will see that:

  cost of LU = cost of G.E.

Exp ① Use LU factorization to solve the linear system:

$$4X_1 + 3X_2 - X_3 = 1$$
$$-2X_1 - 4X_2 + 5X_3 = 6$$
$$X_1 + 2X_2 + 6X_3 = 14$$

② find the cost of this method

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①

$$A = \begin{pmatrix} 4 & 3 & -1 \\ -2 & -4 & 5 \\ 1 & 2 & 6 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 6 \\ 14 \end{pmatrix}$$

- We need to write $A = LU$
- First we find U using row operations:

step 1

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{-2}{4} = -0.5 \qquad \Rightarrow \qquad R_2 + 0.5\,R_1$$

$$m_{31} = \frac{a_{31}}{a_{11}} = \frac{1}{4} = 0.25 \qquad \Rightarrow \qquad R_3 - 0.25\,R_1$$

$$\begin{pmatrix} 4 & 3 & -1 \\ 0 & -2.5 & 4.5 \\ 0 & 1.25 & 6.25 \end{pmatrix}$$

Cost $\div : 2$

$+,- : 2(2)$

$\times : 2(2)$

step 2 $\quad m_{32} = \dfrac{a_{32}}{a_{22}} = \dfrac{1.25}{-2.5} = -0.5 \Rightarrow R_3 + 0.5\,R_2$

$$U = \begin{pmatrix} 4 & 3 & -1 \\ 0 & -2.5 & 4.5 \\ 0 & 0 & 8.5 \end{pmatrix}$$

Cost $\div : 1$

$+,- : 1(1)$

$\times : 1(1)$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0.25 & -0.5 & 1 \end{pmatrix}$$

Note that cost of $A = LU$ is $13 = \dfrac{4n^3 - 3n^2 - n}{6} \Big|_{n=3}$

Now solve $LY = b$ using F.S. where $Y = UX$

$$\begin{pmatrix} 1 & 0 & 0 & | & 1 \\ -0.5 & 1 & 0 & | & 6 \\ 0.25 & -0.5 & 1 & | & 14 \end{pmatrix}$$

$y_1 = 1$

$y_2 = 6 + 0.5(1) = 6.5$

$y_3 = 14 - 0.25(1) + 0.5(6.5) = 17$

cost
$\div : 0$
$+,- : 3$
$\times : 3$

Note that cost of F.S. is $n^2 - n = 3^2 - 3 = 9 - 3 = 6$

Now solve $UX = Y$ using B.S.

$$\begin{pmatrix} 4 & 3 & -1 & | & 1 \\ 0 & -2.5 & 4.5 & | & 6.5 \\ 0 & 0 & 8.5 & | & 17 \end{pmatrix}$$

$X_3 = \dfrac{17}{8.5} = 2$

$X_2 = \dfrac{6.5 - 4.5(2)}{-2.5} = \dfrac{-2.5}{-2.5} = 1$

$X_1 = \dfrac{1 + (2)(1) - 3(1)}{4} = \dfrac{0}{4} = 0$

cost
$\div : 3$
$+,- : 3$
$\times : 3$

Note that cost of B.S. is $n^2 = 3^2 = 9$

② Hence, total cost is $13 + 6 + 9 = 28$ 　　69

Remark The total cost of solving any linear $n \times n$ system $AX = b$ using LU factorization

is 　cost of LU + cost of F.S. + cost of B.S.

$$= \frac{4n^3 - 3n^2 - n}{6} + n^2 - n + n^2$$

$$= \frac{4n^3 + 9n^2 - 7n}{6} \longrightarrow \frac{2}{3}n^3 \quad \text{as } n \to \infty$$

$$= \text{cost of G.E.}$$