

4 Gauss - Jordan Reduction (G.J.R)

72

To solve the linear system $AX = b$:

$$(A|b) \rightarrow (I|X)$$

Exp Find the cost of solving the 4×4 linear system

$$2x_1 + 2x_2 + 2x_3 + 2x_4 = 4$$

$$2x_1 - x_2 + 3x_3 - 5x_4 = -7$$

$$3x_1 - 2x_2 - x_3 - 4x_4 = -2$$

$$-x_1 + 3x_2 - 2x_3 + 2x_4 = 0$$

using

G.J.R

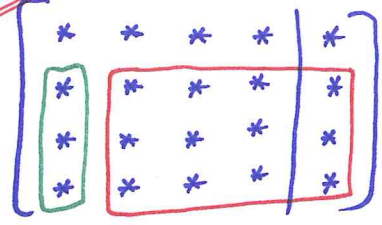
$$(A|b) = \left(\begin{array}{cccc|c} 2 & 2 & 2 & 2 & 4 \\ 2 & -1 & 3 & -5 & -7 \\ 3 & -2 & -1 & -4 & -2 \\ -1 & 3 & -2 & 2 & 0 \end{array} \right) \Rightarrow (I|X) = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right)$$

step	+, -	÷, ×
1	3(4)	4, 3(4)
2	3(3)	3, 3(3)
3	3(2)	2, 3(2)
4	3(1)	1, 3(1)

$$\text{Total cost} = 70 = \frac{2n^3 + n^2 - n}{2} \quad \Big|_{n=4}$$

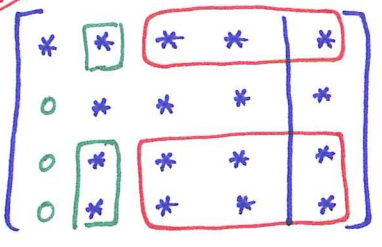
72.1

step 1



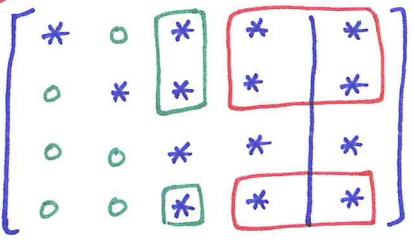
cost of $x = 3 \times 4 = 12$
 cost of $\pm = 3 \times 4 = 12$
 $\frac{12}{24}$

step 2



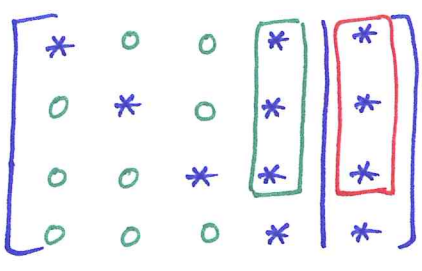
cost of $x = 3 \times 3 = 9$
 cost of $\pm = 3 \times 3 = 9$
 $\frac{9}{18}$

step 3

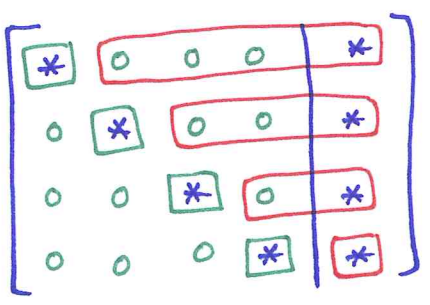


cost of $x = 3 \times 2 = 6$
 cost of $\pm = 3 \times 2 = 6$
 $\frac{6}{12}$

step 4



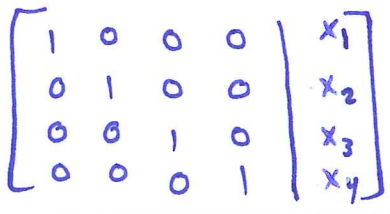
cost of $x = 3 \times 1 = 3$
 cost of $\pm = 3 \times 1 = 3$
 $\frac{3}{6}$



cost of $\div = 4$
 $\frac{4}{10}$

Total cost =
 $\frac{24}{18}{12}{6}{10}{70}$

↓



Exp Show that the cost of solving $n \times n$ linear system using G.J.R is $\frac{2n^3 + n^2 - n}{2}$

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step	+, -	x, ÷
1	$(n-1)n$	$n + (n-1)n$
2	$(n-1)(n-1)$	$(n-1) + (n-1)(n-1)$
3	$(n-1)(n-2)$	$(n-2) + (n-1)(n-2)$
⋮	⋮	⋮
k	$(n-1)(n-k+1)$	$(n-k+1) + (n-1)(n-k+1)$
⋮	⋮	⋮
n	$(n-1)(1)$	$1 + (n-1)(1)$

$$\begin{aligned}
 \text{Total cost} &= \sum_{k=1}^n [(n-1)(n-k+1) + (n-k+1) + (n-1)(n-k+1)] \\
 &= \sum_{k=1}^n (2n-1)(n-k+1) \\
 &= (2n-1) \left[n \sum_{k=1}^n 1 - \sum_{k=1}^n k + \sum_{k=1}^n 1 \right] \\
 &= (2n-1) \left[n^2 - \frac{n(n+1)}{2} + n \right] \\
 &= (2n-1) \left(\frac{n^2+n}{2} \right) \\
 &= \frac{2n^3 + n^2 - n}{2}
 \end{aligned}$$

The cost $\rightarrow n^3$ as $n \rightarrow \infty$