4 Gauss - Jordan Reduction (G.J.R)

To solve the linear system $AX = b$:

$$(A|b) \longrightarrow (I|X)$$

Exp Find the cost of solving the 4×4 linear system

$$\begin{aligned}
2x_1 + 2x_2 + 2x_3 + 2x_4 &= 4 \\
2x_1 - x_2 + 3x_3 - 5x_4 &= -7 \\
3x_1 - 2x_2 - x_3 - 4x_4 &= -2 \\
-x_1 + 3x_2 - 2x_3 + 2x_4 &= 0
\end{aligned}$$

using G.J.R

$$(A|b) = \left(\begin{array}{cccc|c} 2 & 2 & 2 & 2 & 4 \\ 2 & -1 & 3 & -5 & -7 \\ 3 & -2 & -1 & -4 & -2 \\ -1 & 3 & -2 & 2 & 0 \end{array}\right) \Rightarrow (I|X) = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

step	$+, -$	$\div$, $\times$
1	3(4)	4 , 3(4)
2	3(3)	3 , 3(3)
3	3(2)	2 , 3(2)
4	3(1)	1 , 3(1)

$$\text{Total cost} = 70 = \left.\frac{2n^3 + n^2 - n}{2}\right|_{n=4}$$

step 1

$$\left[\begin{array}{cccc|c} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{array}\right]$$

cost of $\times$ = 3×4 = 12
cost of $\pm$ = 3×4 = 12
$\overline{24}$

step 2

$$\left[\begin{array}{cccc|c} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \end{array}\right]$$

cost of $\times$ = 3×3 = 9
cost of $\pm$ = 3×3 = 9
$\overline{18}$

step 3

$$\left[\begin{array}{cccc|c} * & 0 & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{array}\right]$$

cost of $\times$ = 3×2 = 6
cost of $\pm$ = 3×2 = 6
$\overline{12}$

step 4

$$\left[\begin{array}{cccc|c} * & 0 & 0 & * & * \\ 0 & * & 0 & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \end{array}\right]$$

cost of $\times$ = 3×1 = 3
cost of $\pm$ = 3×1 = 3
$\overline{6}$

$$\left[\begin{array}{cccc|c} * & 0 & 0 & 0 & * \\ 0 & * & 0 & 0 & * \\ 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & * & * \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & x_1 \\ 0 & 1 & 0 & 0 & x_2 \\ 0 & 0 & 1 & 0 & x_3 \\ 0 & 0 & 0 & 1 & x_4 \end{array}\right]$$

cost of $\div$ = 4
3
2
1
$\overline{10}$

Total cost =
24 +
18 +
12 +
6 +
10
$\overline{70}$

Exp show that the cost of solving $n \times n$ linear system using G.J.R is $\frac{2n^3+n^2-n}{2}$

step	$+ , -$	$\times , \div$
1	$(n-1)\,n$	$n + (n-1)\,n$
2	$(n-1)(n-1)$	$(n-1) + (n-1)(n-1)$
3	$(n-1)(n-2)$	$(n-2) + (n-1)(n-2)$
$\vdots$	$\vdots$	$\vdots$
k	$(n-1)(n-k+1)$	$(n-k+1) + (n-1)(n-k+1)$
$\vdots$	$\vdots$	$\vdots$
n	$(n-1)(1)$	$1 + (n-1)(1)$

$$\text{Total cost} = \sum_{k=1}^{n}\Big[(n-1)(n-k+1) + (n-k+1) + (n-1)(n-k+1)\Big]$$

$$= \sum_{k=1}^{n}(2n-1)(n-k+1)$$

$$= (2n-1)\left[n\sum_{k=1}^{n}1 - \sum_{k=1}^{n}k + \sum_{k=1}^{n}1\right]$$

$$= (2n-1)\left[n^2 - \frac{n(n+1)}{2} + n\right]$$

$$= (2n-1)\left(\frac{n^2+n}{2}\right)$$

$$= \frac{2n^3+n^2-n}{2}$$

The cost $\to n^3$ as $n \to \infty$