$\boxed{5}$ Inverse Method

To solve the linear system $AX=b$:

- Find A^{-1}: $(A \mid I) \longrightarrow (I \mid A^{-1})$
- Then $x = A^{-1}b$

Exp Find the cost of solving the following linear system using inverse method:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

- $(A \mid I) = \left(\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{31} & a_{33} & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 1 & \end{array}\right)$

- Cost of $A^{-1}_{3\times3}$:

Step	$+,-$	$\div, \times$
1	2(5)	5 , 2(5)
2	2(4)	4 , 2(4)
3	2(3)	3 , 2(3)
	24	12 , 24

Total cost of $A^{-1}_{3\times3} = 60 = \frac{n}{2}(2n-1)(3n-1)$

$= \frac{n(6n^2-5n+1)}{2}$ at $n=3$

- Cost of $x = A^{-1}b$

$$= \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$ is $(3+3+3)+(2+2+2) = 15 = n(2n-1)$

$= 2n^2 - n$ at $n=3$

- Hence, total cost of 3x3 linear system using inverse method is $60+15=75$

Exp show that the cost of finding $A^{-1}_{n\times n}$ is $\frac{n}{2}(2n-1)(3n-1)$

Step	$+ , -$	$\div$, $\times$
1	$(n-1)(2n-1)$	$(2n-1)$, $(n-1)(2n-1)$
2	$(n-1)(2n-2)$	$(2n-2)$, $(n-1)(2n-2)$
3	$(n-1)(2n-3)$	$(2n-3)$, $(n-1)(2n-3)$
$\vdots$	$\vdots$	$\vdots$
K	$(n-1)(2n-K)$	$(2n-K)$, $(n-1)(2n-K)$
$\vdots$	$\vdots$	$\vdots$
n	$(n-1)\,n$	n , $(n-1)\,n$

Total cost of A^{-1} $= \sum_{K=1}^{n}\left[(n-1)(2n-K)+(2n-K)+(n-1)(2n-K)\right]$

$$= \sum_{K=1}^{n}(2n-1)(2n-K)$$

$$= 2n^2(2n-1)-(2n-1)\sum_{K=1}^{n}K$$

$$= 2n^2(2n-1)-(2n-1)\frac{n(n+1)}{2}$$

$$= (2n-1)\left(2n^2-\frac{n^2+n}{2}\right)$$

$$= \frac{n}{2}(2n-1)(3n-1)$$

Hence total cost of solving $n\times n$ linear system by inverse method is

$$\frac{n}{2}(2n-1)(3n-1)+n(2n-1)=\frac{n}{2}(2n-1)(3n+1)$$