Solving Nonlinear Systems of Equations

We will study the following three Methods:

1. Newton's Method → for 2x2 nonlinear system
2. Fixed Point Iteration
3. Gauss-Seidel Iteration

(2 and 3) → For 2x2 or 3x3 nonlinear systems

1 Newton's Method

- Given 2x2 non linear system

$$f(x,y) = 0$$
$$g(x,y) = 0$$

with initial point (x_0, y_0)

- This method find a sequence of points

$$(x_1, y_1), (x_2, y_2), \ldots \quad \text{that approximates } (x,y)$$

- We will only find (x_1, y_1) as follow:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - J^{-1}(x_0,y_0) \begin{pmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{pmatrix} \quad \cdots\cdots *$$

where J is the Jacobian matrix given by

$$J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

Exp Solve the following nonlinear system using Newton's method with initial guess $(1, \frac{1}{2})$ $\boxed{77}$

$$x^2 + y^2 = 10$$
$$xy = 5$$

use three significant digits and find the first iteration

- $x^2 + y^2 - 10 = 0 \rightarrow f(x,y) = x^2 + y^2 - 10$

  $xy - 5 = 0 \rightarrow g(x,y) = xy - 5$

- $(x_0, y_0) = (1, \frac{1}{2}) \rightarrow f(1, \frac{1}{2}) = 1 + 0.25 - 10 = -8.75$

  $g(1, \frac{1}{2}) = 1 - 5 = -4.50$

- $J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 2x & 2y \\ y & x \end{pmatrix} \rightarrow J(1, \frac{1}{2}) = \begin{pmatrix} 2 & 1 \\ 0.5 & 1 \end{pmatrix}$

- $J^{-1}(1, \frac{1}{2}) = \frac{1}{1.5} \begin{pmatrix} 1 & -1 \\ -0.5 & 2 \end{pmatrix} = \begin{pmatrix} 0.667 & -0.667 \\ -0.33\cancel{3}4 & 1.33 \end{pmatrix}$

- $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - J^{-1}(x_0, y_0) \begin{pmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{pmatrix}$

$$= \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} - \begin{pmatrix} 0.667 & -0.667 \\ -0.33\cancel{3}4 & 1.33 \end{pmatrix} \begin{pmatrix} -8.75 \\ -4.50 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} - \begin{pmatrix} -2.84 \\ -3.0\cancel{8}7 \end{pmatrix} = \begin{pmatrix} 3.84 \\ 3.5\cancel{8}7 \end{pmatrix}$$

To find $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - J^{-1}(x_1, y_1) \begin{pmatrix} f(x_1, y_1) \\ g(x_1, y_1) \end{pmatrix}$

Exp Find the first iteration that approximates the solution of the following non linear system

$$x^2 - 2x - y + 0.5 = 0$$
$$x^2 + 4y^2 - 4 = 0$$

using Newton's method and starting with $(2, 0.25)$.

- $f(x,y) = x^2 - 2x - y + 0.5 \Rightarrow f(2, 0.25) = 0.25$

  $g(x,y) = x^2 + 4y^2 - 4 \Rightarrow g(2, 0.25) = 0.25$

- $(x_0, y_0) = (2, 0.25)$

- $J(x,y) = \begin{pmatrix} 2x-2 & -1 \\ 2x & 8y \end{pmatrix} \Rightarrow J(2, 0.25) = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}$

- $J^{-1}(2, 0.25) = \frac{1}{8}\begin{pmatrix} 2 & 1 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} 0.25 & 0.125 \\ -0.5 & 0.25 \end{pmatrix}$

- $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0.25 \end{pmatrix} - \begin{pmatrix} 0.25 & 0.125 \\ -0.5 & 0.25 \end{pmatrix}\begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix}$

$$= \begin{pmatrix} 2 \\ 0.25 \end{pmatrix} - \begin{pmatrix} 0.0938 \\ -0.0625 \end{pmatrix}$$

$$= \begin{pmatrix} 1.91 \\ 0.313 \end{pmatrix}$$

<u>Exp</u> show the formula * page 76

- Given the following nonlinear system

$$f(x,y)=0 \qquad \text{starting at } (x_0,y_0)$$
$$g(x,y)=0$$

- The Taylor series expansions of f and g at (x_0,y_0) are

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$g(x,y) \approx g(x_0,y_0) + g_x(x_0,y_0)(x-x_0) + g_y(x_0,y_0)(y-y_0)$$

- But $f(x,y) = g(x,y) = 0 \Rightarrow$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{pmatrix} + \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$

- Multiply both sides by $J^{-1}_{(x_0,y_0)} \Rightarrow$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = J^{-1} \begin{pmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

- Rearrange the last equation $\Rightarrow$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - J^{-1}(x_0,y_0) \begin{pmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{pmatrix}$$