Ch 4

Interpolation and Polynomial Approximation

- Interpolation means polynomial approximation.
    - given a function $f(x)$ on $[a, b] = [x_0, x_n]$
    - given $n+1$ points

$$\underset{\downarrow \atop f(x_0)}{(x_0, y_0)}, \underset{\downarrow \atop f(x_1)}{(x_1, y_1)}, \cdots, \underset{\downarrow \atop f(x_n)}{(x_n, y_n)}$$

through the partition

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

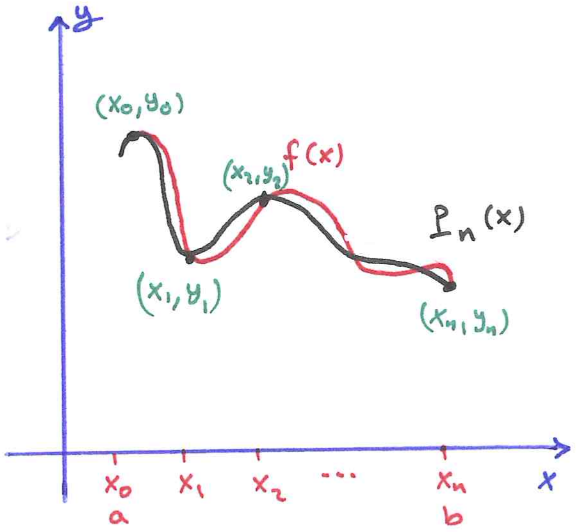
- We need to approximate $f(x)$ by a polynomial of order at most n passing through these $n+1$ points "nodes" on $[a, b]$.

- That is, $f(x) \approx P_n(x) + \underbrace{E_n(x)}_{\text{Truncation error}}$ on $[a, b]$

$P_n(x)$ is called interpolation polynomial

- $\text{degree}\,(P_n(x)) \leq n$

- $P_n(x) \approx f(x)$ on $[x_0, x_n] = [a, b]$



Th Given $n+1$ points: $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$.
Then $\exists$ a unique polynomial $P_n(x)$ with degree $\leq n$ that passes through these points.

Exp Find a linear interpolating polynomial passes through (x_0, y_0) and (x_1, y_1)

- $P_1(x)$ is the interpolating polynomial of order 1 "linear" given by $P_1(x) = ax + b$
- To find a and b $\Rightarrow$ $P_1(x_0) = y_0 = ax_0 + b$
$$P_1(x_1) = y_1 = ax_1 + b$$
- Hence, $a = \dfrac{y_1 - y_0}{x_1 - x_0}$ and $b = y_0 - \left(\dfrac{y_1 - y_0}{x_1 - x_0}\right) x_0$
- Thus, $P_1(x) = \left(\dfrac{y_1 - y_0}{x_1 - x_0}\right) x + y_0 - \left(\dfrac{y_1 - y_0}{x_1 - x_0}\right) x_0$
- Note that $f(x) = P_1(x)$ on $[x_0, x_1]$ with no errors.

- If $(x_0, y_0) = (1, 2)$ and $(x_1, y_1) = (3, 4)$ then

$$a = \frac{4-2}{3-1} = \frac{2}{2} = 1 \quad \text{and} \quad b = 2 - (1)(1) = 1$$

$$\Rightarrow P_1(x) = x + 1$$

- One can write $P_1(x) = m(x - x_0) + y_0$ where

$$m = \frac{y_1 - y_0}{x_1 - x_0} \text{ is the slope.}$$

Exp* Given $(-1,6), (2,9), (0,3)$.

1. Find the polynomial of degree ≤ 2 that passes through these points
2. Estimate $f(1)$
3. Estimate $f'(\frac{1}{4})$
4. Estimate $\int_1^2 f(x)\,dx$

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1 • $P_2(x) = ax^2 + bx + c$ $\qquad$ $P_2(0) = 3 \Rightarrow \boxed{c = 3}$

$P_2(-1) = a - b + 3 = 6 \Rightarrow \boxed{a - b = 3} \quad \boxed{a = 2}$

$P_2(2) = 4a + 2b + 3 = 9 \Rightarrow \boxed{2a + b = 3} \quad \boxed{b = -1}$

• Hence, $\boxed{P_2(x) = 2x^2 - x + 3}$

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2 $f(1) \approx P_2(1) = 2 - 1 + 3 = 4$

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3 $f'(x) \approx P_2'(x) = 4x - 1 \Rightarrow f'(\frac{1}{4}) \approx P_2'(\frac{1}{4}) = 1 - 1 = 0$

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4 $\int_1^2 f(x)\,dx \approx \int_1^2 P_2(x)\,dx = \frac{2}{3}x^3 - \frac{x^2}{2} + 3x \Big|_1^2$

$= \left(\frac{16}{3} - 2 + 6\right) - \left(\frac{2}{3} - \frac{1}{2} + 3\right)$

$= \frac{37}{6} \approx 6.17$

Remark: If $f(x)$ is given and analytic at x_0

" has continuous derivatives of all orders and can be represented as Taylor series in an interval about x_0 ", then we can use Taylor Polynomial Approximation to estimate $f(x)$ by a Taylor Polynomial

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Th (Taylor Polynomial Approximation)

- Assume $f \in C^{n+1}[a,b]$ and $x_0 \in [a,b]$ is fixed
- If $x \in [a,b]$, then $f(x) \simeq P_n(x) + E_n(x)$

where $P_n(x)$ is the Taylor polynomial of degree n that estimates $f(x)$ on $[a,b]$ given by

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$$

and $E_n(x)$ is the truncation error given by

$$E_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-x_0)^{n+1}$$

However, f is usually not known or hard to compute. So How to find $P_n(x)$

1. Lagrange Interpolation $\Rightarrow P_n(x)$ is the Lagrange Polynomial
2. Newton Interpolation $\Rightarrow P_n(x)$ is the Newton Polynomial

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